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On strongly rg-regular and strongly rg-normal spaces

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ABSTRACT

In 1997, Arockiarani and Balachandran introduced and studied the concepts of rg-regular and rg-normal spaces. In this paper strongly rg-regular and strongly rg-normal spaces are introduced and their basic properties are studied.

Keywords and Phrases: *regular closed, regular open, rg-open, rg-closed, pgpr-closed, pgpr-open.*

MSC 2010: *54D10.*

1. Introduction:

After the works of Levine [6] on generalized open sets, various mathematicians turned their attention to the generalizations of some concepts in topology. Palaniappan and Chandrasekhara Rao [9] introduced the concept of regular-generalized closed sets. Arockiarani and Balachandran [3] introduced and investigated rg-closed and rg-open maps. They also defined the concepts of rg-regular and rg-normal spaces. In this paper strongly rg-regular and strongly rg-normal spaces are introduced and their properties are discussed. Further, some of the existing lower separation axioms are characterized by using strongly rg-regular and strongly rg-normal separation axioms.

2. Preliminaries:

Throughout this paper (X,τ) and (Y,σ) denote the topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of X, $cl(A)$ and $int(A)$ respectively denote the closure of A and the interior of A in (X,τ). Suppose $B\subseteq A\subseteq X$, the interior and the closure of B relative to A is respectively written as $int_A(B)$ and $cl_A(B)$. We recall the following definitions and results that are utilized in this paper.

Definition: 2.1 Let (X,τ) be a topological space and $A \subseteq X$. Then A is regular open [14] if $A=int(cl(A))$; pre-open[7] if $A\subseteq int(cl(A))$; generalized closed(briefly g-closed)[6] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open in X; and regular generalized closed (briefly rg-closed)[9] if $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is regular open in X.

The complement of a regular open(resp. pre-open) set is regular closed(resp.pre-closed). Analogously the complement of a g-closed (resp.rg-closed) set is g-open(resp.rg-open). The pre-closure of a subset A of X is the intersection of all pre-closed sets containing A and is denoted by $pcl(A)$. The pre-interior of a subset A of X is the union of all pre-open sets contained in A and it is denoted by $pint(A)$. In a similar manner the rg-closure of A is defined and is denoted by $cl_r^*(A)$.

Definition: 2.2 The set A is pre-generalized pre-regular-closed (briefly pgpr-closed)[2] if $pcl(B)\subseteq U$ whenever $B\subseteq U$ and U is rg-open. The complement of a pgpr-closed set is
pgpr-open.

Lemma: 2.3 A subset A of X is pgpr-open if and only if $F\subseteq pint(A)$ whenever $F\subseteq A$, F is rg-closed. [2]

Lemma: 2.4 Let $B \subseteq A \subseteq X$ and A be g-closed and open in X. Then if B is rg-closed relative to A, then B is rg-closed relative to X. [9]

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Definition: 2.5 A subset A of a topological space X is said to be quasi H-closed [15] relative to X if for every cover $\{V\alpha : \alpha \in \nabla\}$ of A by open sets in X, there exists a finite subset ∇_0 of ∇ such that $A \subseteq \bigcup \{cl(V\alpha): \alpha \in \nabla_0\}$.

Definition: 2.6 A subset B of a topological space X is said to be rg-compact[1] relative to X if for every collection $\{A\alpha : \alpha \in \nabla\}$ of rg-open subsets of X such that $B \subseteq \bigcup \{A\alpha : \alpha \in \nabla\}$, there exists a finite subset ∇_0 of ∇ such that $B \subseteq \bigcup \{A\alpha : \alpha \in \nabla_0\}$.

Definition: 2.7 Let $f :(X,\tau) \to (Y,\sigma)$. Then f is rg-continuous (resp. rg-irresolute) [9] if $f^{-1}(V)$ is rg-open in X for every open (resp. rg-open) set V in Y.

Definition: 2.8 Let $f :(X,\tau) \to (Y,\sigma)$. Then f is perfectly rg-continuous [3] if the inverse image of every rg-open set in Y is both open and closed in X.

Definition: 2.9 A function $f:(X,\tau)\to(Y,\sigma)$ is said to be rg-closed[3] if for every open set U in (X, τ), the set f(U) is rg-closed in (Y,σ).

Definition: 2.10 A topological space (X,τ) is almost regular [11](resp. rg-regular[4]) if for every regular closed set F and a point $x \notin F$, there exist disjoint open(resp. rg-open) sets U and V such that $x \in U$ and $F \subseteq V$.

Definition: 2.11 A topological space (X,τ) is said to be g-normal [8](resp. mildly normal [13], resp. rg-normal[3]) if for each pair A, B of disjoint g-closed (resp. regular closed, resp. regular closed) sets in X, there exist disjoint open (resp. open, rg-open) sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition: 2.12 A topological space (X,τ) is almost normal[12] if for every pair of disjoint sets A and B, one of which is closed and the other is regularly closed, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Definition: 2.13 A topological space (X,τ) is said to be regular-$T_{1/2}$[9] (resp. $T^*_{1/2}$ [3], resp. T_{rg}[3]) if every rg-closed set is regular closed(resp. closed, resp. g-closed).

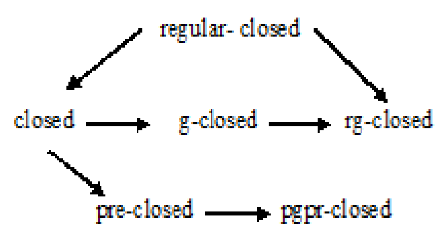
Definition: 2.14 A topological space (X,τ) is said to symmetric [10] if for any two distinct points x and y of X, $x \in cl(\{y\})$ implies that $y \in cl(\{x\})$.

Definition: 2.15 A topological space (X,τ) is said to Urysohn[10] if and only if whenever $x \neq y$ in X, there are neighborhods of U of x and V of y with $cl(U) \cap cl(V) = \varnothing$.

Lemma: 2.16 A topological space (X,τ) is symmetric if and only if $\{x\}$ is g-closed in X for each x in X. [6]

The following diagram is useful later.

Diagram: 2.17



3. Strongly rg-regular spaces:

In this section, the strongly rg-regularity axiom is introduced and its analog properties with other separation axioms are studied..

Definition: 3.1 A space (X,τ) is said to be strongly rg-regular if for every rg-closed set F and a point $x \notin F$, there exist disjoint open sets U and V such that $x \in U$ and $F \subseteq V$.

It is obvious from the Diagram 2.17, that every strongly rg-regular space is regular, almost-regular and rg-regular. But the reverse implications are not true as shown in the following examples.

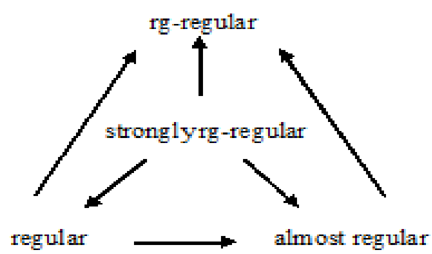
Example: 3.2 Let X= {a, b, c} with topology τ= {∅, {a}, {b, c}, X}. Clearly (X,τ) is regular. Let F= {b, c}. Then F is rg-closed. As c and F are not separated by disjoint open sets in X, it follows that (X,τ) is not strongly rg-regular. This shows that a regular space need not be strongly rg-regular.

Example: 3.3 Let X = {a, b, c, d} with topology τ = {∅, {a}, {a, b},{a, b, c},{a, b, d}, X}. Then (X,τ) is almost-regular but not strongly rg-regular. Since the rg-closed set {a} and the point b cannot be separated by disjoint open sets, (X,τ) is not strongly rg-regular.

Example: 3.4 Let X = {a, b, c, d} with topology τ = {∅, {a}, {a, b}, {a, c}, {a, b, c}, X}. Clearly (X,τ) is rg-regular but not strongly rg-regular.

Thus we have the following implication diagram.

Diagram: 3.5



Proposition: 3.6

(i) Let (X,τ) be a regular space. If (X,τ) is $T^*_{1/2}$ or regular-$T_{1/2}$, then it is strongly rg-regular.

(ii) Let (X,τ) be an almost regular space. If (X,τ) is regular-$T_{1/2}$, then it is strongly rg-regular.

Proof: Suppose (X,τ) is regular and $T^*_{1/2}$. Let F be a rg-closed set of (X,τ) and x∉ F. Since (X,τ) is $T^*_{1/2}$, by Definition 2.13, F is closed. Since (X,τ) is regular, there exist disjoint open sets U and V such that x∈U and F⊆V. Therefore (X,τ) is strongly rg-regular. The proof for the rest is similar.

Theorem: 3.7 In a topological space (X,τ), the following are equivalent:

(a) (X,τ) is strongly rg-regular
(b) For every x∈X and every rg-open set U containing x, there exists an open set V such that x∈V⊆*cl*(V)⊆U.
(c) For every rg-closed set A, the intersection of all the closed neighborhoods of A is A.
(d) For every set A and a rg-open set B such that A∩B≠∅, there exists an open set F such that A∩F≠∅ and *cl*(F)⊆B.
(e) For every non-empty set A and rg-closed set B such that A∩B=∅, there exists disjoint open sets L and M such that A∩L≠∅ and B⊆M.

Proof: (a)⇒(b): Suppose (X,τ) is strongly rg-regular. Let x ∈ X and U be a rg-open set containing x so that X\U is rg-closed. Since (X,τ) is strongly rg-regular, there exist open sets V_1 and V_2 such that $V_1 \cap V_2 = \emptyset$ and x ∈ V_1, X\U ⊆ V_2. Take V = V_1. Since $V_1 \cap V_2 = \emptyset$, V⊆X\$V_2$ ⊆ U that implies *cl*(V) ⊆ *cl*(X\$V_2$) = X\$V_2$ ⊆ U. Therefore x∈V⊆ *cl* (V) ⊆ U.

(b)⇒ (c): Let A be rg-closed and x∉A. Since A is rg-closed, X\A is rg-open and x∈X\A.
Therefore by (b), there exists an open set V such that x ∈ V ⊆ *cl* (V) ⊆X\A. Thus
A⊆X\ *cl*(V)⊆X\V, and x∉ X\V. Consequently, X\V is a closed neighborhood of A.

(c) ⇒ (d): Let A∩B≠∅ and B be rg-open. Let x∈ A∩B. Since B is rg-open, X\B is rg-closed and x∉X\B. By our assumption, there exists a closed neighborhood V of X\B such that x∉V. Let X\B⊆U⊆V, where U is open. Then F=X\V is open such that x∈F and A∩F≠∅. Also X\U is closed and *cl*(F)=*cl*(X\V)⊆X\U⊆B. This shows that *cl(F)⊆B*.

(d)⇒(e): Suppose A∩B=∅, where A is non-empty and B is rg-closed. Then X\B is rg-open and A∩(X\B)≠∅. By (d), there exists an open set L such that A∩L≠∅, and L⊆*cl*(L)⊆X\B. Put M=X*cl*(L). Then B⊆M and L, M are open sets such that M=X*cl*(L)⊆(X\L).

(e)⇒(a): Let B be rg-closed and x∉B. Then B∩{x} =∅. By (e), there exist disjoint open sets L and M such that L∩{x} ≠∅ and B⊆M. Since L∩{x} ≠∅ , x∈L. This proves that (X,τ) is strongly rg-regular.

Corollary: 3.8 If a space (X,τ) is strongly rg-regular and a subset A of X is quasi H-closed relative to X, then A is pgpr-closed.

Proof: Suppose (X,τ) is strongly rg-regular and a subset A of X is quasi H-closed relative to X. Let U be any rg-open set of X containing A. Since X is strongly rg-regular, by using Theorem 3.7(b), for each $x\in A$, there exists an open set V_x such that $x\in V_x\subseteq cl(V_x)\subseteq U$. Clearly $\{V_x: x\in A\}$ is an open cover of A. Since A is quasi H-closed relative to X, by Definition 2.5, there exists a finite subset A_0 of A such that $A\subseteq\bigcup\{cl(V_x): x\in A_0\}$. Therefore we obtain $A\subseteq cl(A)\subseteq\bigcup\{cl(V_x): x\in A_0\}\subseteq U$ and hence $A\subseteq pcl(A)\subseteq cl(A)\subseteq U$. This shows that A is pgpr-closed.

Theorem: 3.9 If A is a rg-compact subset relative to X, then A is quasi H-closed relative to X.

Proof: Let A be rg-compact relative to X. Let $\{V_\alpha: \alpha\in\nabla\}$ be covering for A where each V_α is open in X. Since every open set is rg-open, $\{V\alpha:\alpha\in\nabla\}$ is covering of A by rg-open sets in X. Since A is rg-compact relative to X, by using Definition 2.6, there exists a finite subset ∇_0 of ∇ such that $A\subseteq\bigcup\{V\alpha:\alpha\in\nabla_0\}$. Clearly $A\subseteq\bigcup\{V\alpha:\alpha\in\nabla_0\}\subseteq\bigcup\{cl(V\alpha): \alpha\in\nabla_0\}$. Therefore by Definition 2.5, A is quasi H-closed relative to X.

Examples can be constructed to show that the converse of Theorem 3.9 is not true. However the converse is true provided the space (X,τ) is strongly rg-regular.

Theorem: 3.10 If (X,τ) is strongly rg-regular and a subset A of X is quasi H-closed relative to X, then A is rg-compact relative to X.

Proof: Suppose (X,τ) is strongly rg-regular and a subset A of X is quasi H-closed relative to X. Let $\{V\alpha:\alpha\in\nabla\}$ be an rg-open cover of A. That is $A\subseteq\bigcup\{V\alpha: \alpha\in\nabla\}$. Let $x\in A$. Then $x\in V\alpha$ for some α. For each $x\in A$, take $V_x= V\alpha$, where $V\alpha$ is any one of the rg-open sets in X containing x. Since X is strongly rg-regular, and V_x is rg-open, by Theorem 3.7(b), for each $x\in A$ there exists an open set U_x such that $x\in U_x\subseteq cl(U_x)\subseteq V$. Clearly $\{U_x: x\in A\}$ is an open cover of A. Since A is quasi H-closed relative to X, there exists a finite subset A_0 of A such that $A\subseteq\bigcup\{cl(U_x): x\in A_0\}\subseteq\bigcup\{V_x: x\in A_0\}$. This shows that A is rg-compact relative to X.

Theorem: 3.11 In a strongly rg-regular space X, every pair consisting of a compact set A and a disjoint rg-closed set B can be separated by open sets.

Proof: Let (X,τ) be strongly rg-regular and let A be a compact set, B a rg-closed set with $A\cap B=\varnothing$. Since (X,τ) is strongly rg-regular, for each $x\in A$, there exist disjoint open sets U_x and V_x such that $x\in U_x$, $B\subseteq V_x$. Clearly $\{U_x: x\in A\}$ is an open covering of the compact set A. Since A is compact, there exists a finite subfamily $\{Ux_i:i=1,2,..n\}$ which covers A. It follows that $A\subseteq\bigcup\{Ux_i: i=1,2,\dots,n\}$ and $B\subseteq\bigcap\{Vx_i: i=1,2,\dots,n\}$. Put $U=\bigcup\{Ux_i: i=1,2,\dots,n\}$ and $V=\bigcap\{Vx_i: i=1,2,\dots,n\}$, then $U\cap V=\varnothing$. For, if $x\in U\cap V\Rightarrow x\in Ux_j$ for some j and $x\in Vx_i$ for every i. This implies that $x\in Ux_j\cap Ux_j$, which is a contradiction to $Ux_j\cap Vx_j=\varnothing$. Thus U and V are disjoint open sets containing A and B respectively.

Corollary: 3.12 If X is a strongly rg-regular space, A is a compact subset of X and B is a rg-open set containing A, then there exists an rg-open set V such that $A\subseteq V\subseteq cl_r^*(V)\subseteq B$.

Proof: Let (X,τ) be strongly rg-regular and let A be a compact set, B a rg-closed set with $A\subseteq B$. Since B is rg-closed, $X\backslash B$ is rg-open such that $(X\backslash B)\cap A=\varnothing$. Since (X,τ) is strongly rg-regular, by Theorem 3.11, there exist disjoint open sets U_1 and U_2 such that $A\subseteq U_1$ and $X\backslash B\subseteq U_2$.Then$V=int(cl(U_1))$ is open and hence rg-open. Now $A\subseteq U_1=intU_1\subseteq int(cl(U_1))=V$ which implies $A\subseteq V$ and $cl_r^*(V)\subseteq cl(V)\subseteq cl(U_1)\subseteq X\backslash U_2\subseteq B$.

Corollary: 3.13 If X is a strongly rg-regular space and A, B are subsets of X such that A is compact and B is rg-open with $A\cap B=\varnothing$, then there exist rg-open sets U and V such that $A\subseteq U$, $B\subseteq V$ and $U\cap V=\varnothing$.

Proof: Suppose X is strongly rg-regular and A, B are subsets of X such that A is compact and B is rg-open with $A\cap B=\varnothing$. By Theorem 3.11, there exist open sets U and V such that $A\subseteq U$, $B\subseteq V$ and $U\cap V=\varnothing$. Since every open set is rg-open, the result follows.

Theorem: 3.14 A topological space (X,τ) is strongly rg-regular if and only if for each rg-closed set F of (X,τ) and each $x\in X\backslash F$, there exist open sets U and V of (X,τ) such that $x\in U$ and $F\subseteq V$ and $cl(U)\cap cl(V)=\varnothing$.

Proof: Let F be a rg-closed set in (X,τ) and x∉ F. Then there exist open sets U_x and V such that x∈ U_x, F⊆V and U_x ∩ V=∅. This implies that U_x ∩ *cl*(V)=∅. Since *cl*(V) is closed and every closed set is rg-closed, *cl*(V) is rg-closed and x∉ *cl*(V). Since (X,τ) is strongly rg-regular, there exist open sets G and H of (X,τ) such that x∈G, *cl*(V)⊆H and G∩H=∅. This implies *cl*(G)∩H=∅. Take U=Ux∩G. Then U and V are open sets of (X,τ) such that x∈U and F⊆V and *cl*(U)∩*cl*(V)=∅ since *cl*(U)∩*cl*(V) ⊆ *cl*(G)∩H=∅. Conversely, suppose for each rg-closed set F of (X,τ) and each x∈X\F, there exist open sets U and V of (X,τ) such that x∈U and F⊆V and *cl*(U)∩*cl*(V)= ∅.
Now U∩V⊆ *cl*(U)∩*cl*(V)= ∅. Therefore U∩V=∅. Thus (X,τ) is strongly rg-regular.

Corollary: 3.15 If a space X is strongly rg-regular and symmetric, then it is Urysohn.

Proof: Let x and y be any two distinct points of (X,τ). Since (X,τ) is symmetric, by using Lemma 2.16, {x} is g-closed. Since every g-closed set is rg-closed, {x} is rg-closed. Using Theorem 3.14, there exist open sets U and V such that x∈U and y∈V and *cl*(U)∩*cl*(V)= ∅. By Definition 2.15, (X,τ) is Urysohn.

Now we show that a g-closed, open subspace of a strongly rg-regular space is again strongly rg-regular.

Theorem: 3.16 If X is a strongly rg-regular space and Y is a g-closed, open subset of X, then the subspace Y is strongly rg-regular.

Proof: Suppose (X,τ) is strongly rg-regular. Let Y be g-closed, open in (X,τ). Let F be any rg-closed subset of Y such that x∈Y\F. Since F⊆Y⊆X and x∈Y\F, x∈X\F. Since Y is g-closed, open and F is rg-closed in Y, by Lemma 2.4, F is rg-closed in (X,τ). Since (X,τ) is strongly rg-regular, by Definition 3.1, there exist disjoint open sets U_x and V_x such that x∈U_x and F⊆ V_x. Put U=U_x ∩Y and V= V_x ∩ Y. Clearly U and V are disjoint open sets in Y containing x and F respectively. Thus Y is strongly rg-regular.

4. Strongly rg-normal spaces:

The natural extension of strongly rg-regular axiom is strongly rg-normal axiom.

Definition: 4.1 A space (X,τ) is said to be strongly rg-normal if for any pair of disjoint rg-closed sets A and B, there exist disjoint open sets U and V such that A ⊆ U and B ⊆ V.

It is obvious that every strongly rg-normal space is normal, g-normal, mildly normal, almost-normal and rg-normal. But the reverse implications are not true as shown in the following examples.

Example: 4.2 Let X={a, b, c} with topology τ= ∅, {b}, {a, c}, X}. Clearly (X,τ) is normal. As {c} and {a} are disjoint rg-closed sets and are not separated by disjoint open sets in X it follows that (X,τ) is not rg-normal.

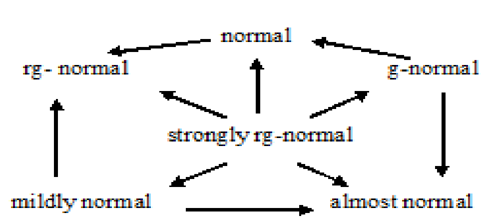
Example: 4.3 Let X = {a, b, c} with topology τ = {∅, {a}, {b}, {a, b}, X}. Clearly (X,τ) is g-normal. But it is not strongly rg-normal, because there are no disjoint open sets which contain the disjoint rg-closed sets {a, b} and {c}.

Example: 4.4 Let X = {a, b, c, d} with topology τ = {∅, {a, b}, {a, b, c}, X}. Then (X,τ) is mildly-normal but not strongly rg-normal. Since the disjoint rg-closed sets {b} and {c} cannot be separated by disjoint open sets. This shows that a mildly-normal space is strongly not rg-normal.

Example: 4.5 Let X = {a, b, c} with topology τ={∅, {a},{b},{a, b},{a, c}, X}. (X,τ) is almost-normal but not strongly rg-normal.

Example: 4.6 Let X = {a, b, c, d} with topology τ = {∅, {a, b}{c, d}, X}. (X,τ) is rg-normal but not strongly rg-normal.

From the Definition 4.1 and above examples, the following diagram is obtained.
Diagram: 4.7



Theorem: 4.8

(i) If a space (X,τ) is $T^*_{1/2}$ and if it is normal (resp. g-normal) then it is strongly rg-normal.

(ii) If a space (X,τ) is regular-$T_{1/2}$ and if it is normal (resp.almost normal, resp. g-normal) then it is strongly rg-normal.

(iii) If a space (X,τ) is T_{rg} and if it is g-normal then it is strongly rg-normal.

Proof: Suppose (X,τ) is normal and $T^*_{1/2}$. Let A and B be disjoint rg-closed sets in (X,τ). Since (X,τ) is $T^*_{1/2}$, by Definition 2.13, A and B are disjoint closed sets in X. Since (X,τ) is normal, there exist disjoint open sets U and V such that A⊆U and B⊆V. Therefore (X,τ) is strongly rg-normal. The proof for the rest is analog.

Theorem: 4.9 A topological space (X,τ) is strongly rg-normal if and only if for any disjoint rg-closed sets A and B of (X,τ), there exist open sets U and V of X such that A⊆U and B⊆V and *cl*(U)∩*cl*(V)= ∅.

Proof: Let A and B be any two disjoint rg-closed sets in (X,τ). Then there exist open sets U_x and V such that A⊆U_x, B⊆V and U_x ∩ V=∅. This implies that U_x ∩ *cl*(V)= ∅. Since (X,τ) is strongly rg-normal, by Definition 4.1, there exist open sets G and H of (X,τ) such that A⊆G, *cl*(V)⊆H and G∩H=∅. This implies *cl*(G)∩H=∅. Now Put U=Ux∩G, then U and V are open sets of (X,τ) such that A⊆U and B⊆V and *cl*(U)∩*cl*(V)=∅ since *cl*(U)∩*cl*(V)⊆*cl*(G)∩H=∅. It is easy to prove the converse part.

Theorem: 4.10 In a topological space (X,τ), the following are equivalent:

(a) (X,τ) is strongly rg-normal

(b) For any two disjoint rg-closed sets H and K of X, there exist disjoint pgpr-open sets U and V such that H⊆U and K ⊆V.

(c) For any rg-closed set H of X and any rg-open set V of X containing H, there exists a pgpr-open set U of X such that H⊆U⊆*cl*(U)⊆V.

Proof: (a)⇒(b): Suppose (X,τ) is strongly rg-normal. Let H and K be disjoint rg-closed sets in X. Since (X,τ) is strongly rg-normal, by Definition 4.1, there exist disjoint open sets U and V such that H⊆U and K ⊆V. The result follows from the fact that every open set is pgpr-open.

(b)⇒(c): Let H be a rg-closed set of X and V be a rg-open set of X such that H⊆V. Then H and X\V are disjoint rg-closed sets of X. By assumption, there exist disjoint pgpr-open sets U and W such that H⊆U and X\V⊆W. Since H and X\V are rg-closed in X and U, V are pgpr-open, by Lemma 2.3, we have X\V⊆*pint*(W) and U ∩ *pint*(W)=∅.

Therefore *cl*(U) ∩ *pint*(W)=∅ and hence H⊆U⊆*cl*(U)⊆X\ *pint*(W)⊆V.

(c)⇒(a): Let H and K be two disjoint rg-closed sets of X. Then H⊆ X\K, and X\K is rg-open in X. By (c), there exists a pgpr-open set G of X such that H⊆G⊆*cl*(G)⊆X\K. Put U=*int*(*cl*(*pint*(G)) and V=X*cl*(G). Since G is pgpr-open, H⊆G and H is rg-closed, by using Lemma 2.3, H⊆*pint* (G). Now K⊆X*cl*(G)=V and H⊆*pint*(G)⊆*int*(*cl*(*pint*(G))=U. Also U ∩ V= *int*(*cl*(*pint*(G))∩X*cl*(G)⊆ *int*(*cl(G)*∩X*cl*(G)⊆*cl*(*G*)∩X*cl*(G)=∅. Therefore U and V are disjoint open sets of X such that H⊆U and K⊆V. Therefore, (X,τ) is strongly rg-normal.

The next theorem shows that the g-closed, open subspace of a strongly rg-normal space is strongly rg-normal.

Theorem: 4.12 Let Y ⊆X be g-closed, open in (X,τ). If (X,τ) is strongly rg-normal, then (Y,τ_Y) is also strongly rg-normal.

Proof: Suppose (X, τ) is strongly rg-normal. Let Y be g-closed, open in (X,τ) and A and B any disjoint rg-closed subsets of Y. Since Y is g-closed, open and F is rg-closed in Y, by Lemma 2.4, A and B are rg-closed in (X, τ). Since (X, τ) is strongly rg-normal, by using Definition 4.1, there exist disjoint open sets U and V such A⊆U and B⊆ V. Therefore U∩Y and V∩Y are disjoint open sets of Y containing A and B respectively. This proves that Y is strongly rg-normal.

5. Applications:

In this section the applications of strongly rg-regular and strongly rg-normal spaces to functions are discussed.

Theorem: 5.1 Let f: X→Y be a surjective map such that f is rg-continuous and open.

If X is strongly rg-regular (resp. strongly rg-normal) then Y is regular (resp. normal).

Proof: Suppose X is strongly rg-regular(resp. strongly rg-normal). Let $y \in Y$ and F be closed in Y such that $y \notin F$ (resp. let A and B be .the disjoint closed sets in Y). Since f is rg-continuous, by Definition 2.7, $f^{-1}(F)$ is rg-closed (resp. $f^{-1}(A)$ and $f^{-1}(B)$ are rg-closed) in X. Since f is surjective, there exists $x \in X$ such that $y=f(x)$ and $x \notin f^{-1}(F)$ (resp. $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$ whenever $A \neq \emptyset$ and $B \neq \emptyset$). Since (X,τ) is strongly rg- regular(resp. strongly rg-normal), there exist disjoint open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq V$ (resp. $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$). This shows that Y is regular (resp. normal).

Theorem: 5.2 Let f: X$\rightarrow$Y be a continuous, injective and rg-closed map. If Y is strongly rg-regular (resp. strongly rg-normal) then X is regular (resp.normal).

Proof: Let f: X$\rightarrow$Y be a continuous function. Suppose Y is strongly rg-regular.

Let $x \in X$ and $A \subseteq X$ be closed such that $x \notin A$. Since f is a rg-closed, by Definition 2.9, f(A) is rg-closed in Y. Since f is injective, $f(x) \notin f(A)$. Since Y is strongly rg-regular, there exist disjoint open sets U and V of Y such that $f(x) \in U$ and $f(A) \subseteq f(V)$. That is $f^{-1}f(x) \in f^{-1}(U)$ and $f^{-1}f(A) \subseteq f^{-1}(V)$. That is $x \in f^{-1}(U)$ and $A \subseteq f^{-1}(V)$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets in X such that $x \in f^{-1}(U)$ and $A \subseteq f^{-1}(V)$. Thus X is regular. The proof for the rest is analog.

Theorem: 5.3 Let f: X$\rightarrow$Y be a perfectly rg-continuous, open, closed and surjective map. If X is regular then Y is strongly rg-regular.

Proof: Suppose X is regular. Let U be an rg-open set containing a point y in Y. Since f is onto, there exists $x \in X$ such that $y=f(x)$. Since f is perfectly rg-continuous, by Definition 2.8, $f^{-1}(U)$ is both open and closed in X. Since X is regular, and $f^{-1}(U)$ is open in X, there is a open set V such that $x \in V \subseteq cl(V) \subseteq f^{-1}(U)$. Then $f(x) \in f(V) \subseteq f(cl(V)) \subseteq f\, f^{-1}(V)$ and hence $y \in f(V) \subseteq U$. Since f is open, f(V) is open in Y. Since f is a closed map and since $f^{-1}(U)$ is closed, $ff^{-1}(U)=U$ is closed in Y. This implies $y \in f(V) \subseteq cl(f(V) \subseteq U$. Therefore by using Theorem 3.7, Y is strongly rg-regular.

Theorem: 5.4 If there is a function f: X$\rightarrow$Y such that f is perfectly rg-continuous, open and surjective then Y is strongly rg-normal.

Proof: Let A and B be any two disjoint rg-closed sets of Y. Since f is perfectly continuous, by Definition 2.8, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint sets that are both closed and open in X. Let $f^{-1}(A)=U$ and $f^{-1}(B)=V$. Since f is open, U and V are open sets in Y. Since f is onto, $A = ff^{-1}(A) = f(U)$ and $B = ff^{-1}(B) = f(V)$. This proves that A and B are separated by disjoint pen sets in Y that implies Y is strongly rg-normal.

Theorem: 5.5 Let f: X$\rightarrow$Y be a bijective map such that f is rg-irresolute and open. If X is strongly rg-regular (resp. strongly rg-normal) then Y is regular (resp. normal).

Proof: Suppose X is strongly rg-regular(resp. strongly rg-normal).Let F be rg-closed in Y and $y \in Y \backslash F$(resp. let A and B be the disjoint closed sets in Y). Since f is rg- irresolute, by Definition 2.7, $f^{-1}(F)$ is rg-closed (resp. $f^{-1}(A)$ and $f^{-1}(B)$ are rg-closed) in X. Since f is surjective, there exists $x \in X$ such that $y=f(x)$ and $x \notin f^{-1}(F)$ (resp. $f^{-1}(A) \neq \emptyset$ and $f^{-1}(B) \neq \emptyset$ whenever $A \neq \emptyset$ and $B \neq \emptyset$). Now $y \in Y \backslash F \Rightarrow y=f(x) \notin F \Rightarrow x \notin f^{-1}(F)$. Since X is strongly rg-regular (resp. strongly rg-normal) there exist open sets U and V in X such that $U \cap V = \emptyset$ and $x \in U$, $f^{-1}(F) \subseteq V$(resp.$f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$). Since $U \cap V = \emptyset$ and since f is bijective $f(U) \cap f(V) = \emptyset$. This proves the theorem.

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