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EXISTENCE OF POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH FORCING TERM

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ABSTRACT

This paper deals with the existence of positive solutions of nonlinear neutral delay difference equations with positive and negative coefficients of the form

$$\Delta\left(r(n)\Delta[x(n)+p(n)x(n-\tau)]\right)+q_1(n)x(n-\sigma_1)-q_2(n)x(n-\sigma_2)=e(n),$$

under the following assumption that

$$\sum_{s=0}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_i(j)<\infty,\ i=1,2$$

for various ranges of $p(n)$. *By using Banach's contraction mapping principle, some sufficient conditions are established for the existence of nonoscillatory solutions.*

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1. INTRODUCTION:

In this paper, we consider the second order nonlinear neutral delay difference equation with forcing term of the form

$$\Delta\left(r(n)\Delta[x(n)+p(n)x(n-\tau)]\right)+q_1(n)x(n-\sigma_1)-q_2(n)x(n-\sigma_2)=e(n)\ ,$$

where $n\geq n_0$, $\tau>0$, $\sigma_1,\sigma_2\geq 0$ are integers and $r(n),q_i(n),i=1,2$ are sequences of nonnegative real numbers.

$(A1)$ $f,g\in C(R,R)$ satisfy Lipschitz condition and $xf(x)>0, xg(x)>0,$ for $x\neq 0,$

$(A2)$ $\sum_{s=0}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_i(j)<\infty,\ i=1,2$

$(A3)$ $\sum_{s=0}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}e(j)<\infty.$

The nonoscillatory behaviour of linear and nonlinear neutral delay difference and differential equations with positive and negative coefficients have been investigated by several authors, see, for example [3], [4], [5], [6], [7], [8] and [9], the references cited therein. We refer monographs [1] and [2] for a good amount of discussion concerning existence of

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solutions of delay difference equations. Our aim in this paper is to establish nonoscillation criteria for the second order nonlinear neutral delay difference equation (1.1) for various ranges of $p(n)$.

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. MAIN RESULTS:

Theorem: 2.1 Suppose that conditions $(A1)$ to $(A3)$ hold and that there exists a constant p_1 such that

$$0 < p(n) \le p_1 < 1. \tag{2.1}$$

Then equation (1.1) has a nonoscillatory solution.

Proof: Suppose that (2.1) holds. Choose a positive constant $n \ge n_1$ and such that

$$\sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) \quad < \frac{1-p_1}{10}, \tag{2.2}$$

$$M_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) < \frac{1-p_1}{10}, \tag{2.3}$$

$$M_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) < \frac{1-p_1}{5}, \tag{2.4}$$

where $M_1 = \max\{L_1, f(1)\}$, $M_2 = \max\{L_2, g(1)\}$ and L_1, L_2 are Lipschitz constants of f and g respectively on $\left[\frac{1-p_1}{10}, 1\right]$.

Consider the Banach space $X = \ell_\infty^{n_0}$ of all real sequences $x = x(n)$ with the norm $\|x\| = \sup_{n \ge n_1} |x(n)|$. Set

$$S = \left\{ x \in X : \frac{1-p_1}{10} \le x(n) \le 1,\ n \ge n_1 \right\}.$$

Define a mapping $T_1 : S \to X$ as follows:

$$(T_1 x)(n) = \begin{cases} \dfrac{1+4p_1}{5} - p(n)x(n-\tau) - \displaystyle\sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) f(x(j-\sigma_1)) \\ + \displaystyle\sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j), & n \ge n_1 \\ (T_1 x)(n_1), & n_0 \le n \le n_1. \end{cases}$$

Now, we show that $T_1 x$ is continuous. Let $\{x_k(n)\}$ be a sequence in X such that $\|x_k - x\| = 0$ as $k \to \infty$. Since X is a closed set, we obtain

$$\left|(T_1 x_k)(n) - (T_1 x)(n)\right| \le \|x_k - x\| \left\{ p_1 + L_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) + L_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) \right\}$$
$$= 0 \quad as \quad k \to \infty,$$

which proves that $T_1 x$ is continuous on X. For every $x \in S$ and $n \geq n_1$,

$$\begin{aligned}(T_1 x)(n) &\leq \frac{1+4p_1}{5} + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) \\ &< \frac{1+4p_1}{5} + \frac{1-p_1}{5} + \frac{1-p_1}{10} \\ &< \frac{1+p_1}{2} < 1\end{aligned}$$

and

$$\begin{aligned}(T_1 x)(n) &\geq -p_1 + \frac{1+4p_1}{5} - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) f(x(j-\sigma_1)) \\ &\geq -p_1 + \frac{1+4p_1}{5} - \frac{1-p_1}{10} \\ &\geq \frac{1-p_1}{10}.\end{aligned}$$

Hence $\frac{1-p_1}{10} \leq T_1(x(n)) \leq 1$. Now, for $x_1, x_2 \in S$ and $n \geq n_1$, we have

$$\begin{aligned}\left|(T_1 x_1)(n) - (T_1 x_2)(n)\right| &\leq p_1 \|x_1 - x_2\| + \frac{1-p_1}{10}\|x_1 - x_2\| + \frac{1-p_1}{5}\|x_1 - x_2\| \\ &\leq \left(p_1 + \frac{1-p_1}{10} + \frac{1-p_1}{5}\right)\|x_1 - x_2\| \\ &\leq \frac{7p_1+3}{10}\|x_1 - x_2\|.\end{aligned}$$

This proves that T_1 is a contraction mapping. T_1 has the unique fixed point $x \in X$. Thus $x(n)$ is a positive solution of equations (1.1),

$$\begin{aligned}x(n) = \frac{1+4p_1}{5} - p(n)x(n-\tau) - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) f(x(j-\sigma_1)) \\ + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j).\ n \geq n_1\end{aligned}$$

Therefore

$$\Delta\Big(r(n)\Delta\big[x(n) + p(n)x(n-\tau)\big]\Big) + q_1(n) f(x(n-\sigma_1)) - q_2(n) g(x(n-\sigma_2)) = e(n),$$

and so $x(n)$ is obviously a positive solution of equation (1.1). This completes the proof of the theorem.

By using similar arguments as in Theorem 2.1, we can prove the following theorem.

Theorem: 2.2 Suppose that conditions (A1) to (A3) hold and that there exists a constant p_2 such that

$$-1 < -p_2 \leq p(n) < 0. \tag{2.5}$$

Then equation (1.1) has a nonoscillatory solution.

Now we prove the following theorem.

Theorem: 2.3 Suppose that conditions (A1) to (A3) hold and that there exists constants p_3 and p_4 such that

$$-1 < p_3 \le p(n) \le p_4 < 1, \tag{2.6}$$

and $p(n)$ changes sign such that $p_4 < 1 + 5p_3$. Then equation (1.1) has a positive solution.

Proof: As in the proof of theorem (2.1), it is possible to choose a positive number $n \ge n_1$,

$$\sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}e(j) < \frac{1-p_4}{50}, \tag{2.7}$$

$$M_1\sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_1(j) < p_3 + \frac{1-p_4}{5}, \tag{2.8}$$

$$M_2\sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_2(j) < \frac{1-p_4}{50}, \tag{2.9}$$

where $M_1 = \max\{L_1, f(1)\}$, $M_2 = \max\{L_2, g(1)\}$ and L_1, L_2 are Lipschitz constants of f and g respectively on $\left[\frac{1-p_4}{10}, 1\right]$.

Set $S = \left\{x \in X : \frac{1-p_4}{10} \le x(n) \le 1,\ n \ge n_1\right\}$.

Define a mapping $T_3 : S \to X$ as follows:

$$(T_3x)(n) = \begin{cases} \dfrac{1+4p_4}{5} - p(n)x(n-\tau) - \displaystyle\sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_1(j)f(x(j-\sigma_1)) \\ +\displaystyle\sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}e(j), & n \ge n_1 \\ (T_3x)(n_1), & n_0 \le n \le n_1. \end{cases}$$

Clearly, T_3x is continuous. For every $x \in S$ and $n \ge n_1$,

$$\begin{aligned}(T_3x)(n) &\le \frac{1+4p_4}{5} + \sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}e(j) \\ &< \frac{1+4p_4}{5} + \frac{1-p_4}{50} + \frac{1-p_4}{50} \\ &< \frac{12+38p_4}{50} < 1\end{aligned}$$

and

$$\begin{aligned}(T_3x)(n) &\ge -p_4 + \frac{1+4p_4}{5} - \sum_{s=n_1}^{\infty}\frac{1}{r(s)}\sum_{j=s}^{\infty}q_1(j)f(x(j-\sigma_1)) \\ &\ge -p_4 + \frac{1+4p_4}{5} - p_3 - \frac{1-p_4}{10} \\ &\ge \frac{1-p_4}{10}.\end{aligned}$$

Hence $\frac{1-p_4}{10} \le T_3\left(x(n)\right) \le 1$. Now, for $x_1, x_2 \in S$ and $n \ge n_1$, we have

$$\left|\left(T_3 x_1\right)(n) - \left(T_3 x_2\right)(n)\right| \le p_4 \left\|x_1 - x_2\right\| + \frac{1-p_4}{5}\left\|x_1 - x_2\right\| + \frac{1-p_4}{50}\left\|x_1 - x_2\right\|$$
$$\le \frac{39 p_4 + 11}{50}\left\|x_1 - x_2\right\|.$$

This proves that T_3 is a contraction mapping. T_3 has the unique fixed point $x \in X$. Thus $x(n)$ is a positive solution of equations (1.1).

Remark: 2.4 Similarly we can study the existence of positive solution of equation (1.1) for the other ranges of $p(n)$ by using suitable transformations. Hence the details are omitted.

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