EXISTENCE OF POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH FORCING TERM

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ABSTRACT

This paper deals with the existence of positive solutions of nonlinear neutral delay difference equations with positive and negative coefficients of the form

$$\Delta \left( r(n)\Delta [x(n) + p(n)x(n-\tau)] + q_1(n)x(n-\sigma_1) - q_2(n)x(n-\sigma_2) = e(n), \right.$$ \n
under the following assumption that

$$\sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_i(j) < \infty, \ i = 1, 2$$

for various ranges of $p(n)$. By using Banach’s contraction mapping principle, some sufficient conditions are established for the existence of nonoscillatory solutions.

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1. INTRODUCTION:

In this paper, we consider the second order nonlinear neutral delay difference equation with forcing term of the form

$$\Delta \left( r(n)\Delta [x(n) + p(n)x(n-\tau)] + q_1(n)x(n-\sigma_1) - q_2(n)x(n-\sigma_2) = e(n), \right.$$ \n
where $n \geq n_0$, $\tau > 0$, $\sigma_1, \sigma_2 \geq 0$ are integers and $r(n), q_i(n), i = 1, 2$ are sequences of nonnegative real numbers.

(A1) \( f, g \in C(R, R) \) satisfy Lipschitz condition and \( xf(x) > 0, xg(x) > 0, \) for \( x \neq 0, \)

(A2) \( \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_i(j) < \infty, \ i = 1, 2 \)

(A3) \( \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) < \infty. \)

The nonoscillatory behaviour of linear and nonlinear neutral delay difference and differential equations with positive and negative coefficients have been investigated by several authors, see, for example [3], [4], [5], [6], [7], [8] and [9], the references cited therein. We refer monographs [1] and [2] for a good amount of discussion concerning existence of

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solutions of delay difference equations. Our aim in this paper is to establish nonoscillation criteria for the second order nonlinear neutral delay difference equation (1.1) for various ranges of \( p(n) \).

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. MAIN RESULTS:

**Theorem 2.1** Suppose that conditions \((A1)\) to \((A3)\) hold and that there exists a constant \( p_1 \) such that

\[
0 < p(n) \leq p_1 < 1.
\]

Then equation (1.1) has a nonoscillatory solution.

**Proof:** Suppose that (2.1) holds. Choose a positive constant \( n \geq n_1 \) and such that

\[
\sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} e(j) < \frac{1 - p_1}{10}, \tag{2.2}
\]

\[
M_1 \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_1(j) < \frac{1 - p_1}{10}, \tag{2.3}
\]

\[
M_2 \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_2(j) < \frac{1 - p_1}{5}, \tag{2.4}
\]

where \( M_1 = \max \{L_1, f(1)\} \), \( M_2 = \max \{L_2, g(1)\} \) and \( L_1, L_2 \) are Lipschitz constants of \( f \) and \( g \) respectively on \([\frac{1 - p_1}{10}, 1]\).

Consider the Banach space \( X = \ell^\infty \) of all real sequences \( x = x(n) \) with the norm \( \|x\| = \sup_{n \geq n_1} |x(n)| \). Set

\[
S = \left\{ x \in X : \frac{1 - p_1}{10} \leq x(n) \leq 1, \ n \geq n_1 \right\}.
\]

Define a mapping \( T_1 : S \rightarrow X \) as follows:

\[
(T_1 x)(n) = \begin{cases}
\frac{1 + 4p_1}{5} - p(n)x(n - \tau) - \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_1(j)f(x(j - \sigma_1)) \\
+ \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_2(j)g(x(j - \sigma_2)) + \sum_{s=n}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} e(j), \ n \geq n_1 \\
(T_1 x)(n_1), \quad n_0 \leq n \leq n_1.
\end{cases}
\]

Now, we show that \( T_1 x \) is continuous. Let \( \{x_k(n)\} \) be a sequence in \( X \) such that \( \|x_k - x\| = 0 \) as \( k \rightarrow \infty \). Since \( X \) is a closed set, we obtain

\[
\left\| (T_1 x_k)(n) - (T_1 x)(n) \right\| \leq \left\| x_k - x \right\| \left\{ p_1 + L_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_1(j) + L_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=x}^{\infty} q_2(j) \right\} = 0 \quad \text{as} \quad k \rightarrow \infty.
\]
which proves that $T_n x$ is continuous on $X$. For every $x \in S$ and $n \geq n_1$,

$$
(T_n x)(n) \leq \frac{1+4p_1}{5} + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} q_1(j) g(x(j-\sigma_1)) \right) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} e(j) \right)
$$

$$
< \frac{1+4p_1}{5} + \frac{1-p_1}{5} + \frac{1-p_1}{10}
$$

$$
< \frac{1+p_1}{2} < 1
$$

and

$$
(T_n x)(n) \geq -p_1 + \frac{1+4p_1}{5} - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} q_1(j) f(x(j-\sigma_1)) \right)
$$

$$
\geq -p_1 + \frac{1+4p_1}{5} - \frac{1-p_1}{10}
$$

$$
\geq \frac{1-p_1}{10}.
$$

Hence \( \frac{1-p_1}{10} \leq T_1 (x(n)) \leq 1 \). Now, for $x_1, x_2 \in S$ and $n \geq n_1$, we have

$$
\left| (T_n x_1)(n) - (T_n x_2)(n) \right| \leq p_1 \left\| x_1 - x_2 \right\| + \frac{1-p_1}{10} \left\| x_1 - x_2 \right\| + \frac{1-p_1}{5} \left\| x_1 - x_2 \right\|
$$

$$
\leq \left( p_1 + \frac{1-p_1}{10} + \frac{1-p_1}{5} \right) \left\| x_1 - x_2 \right\|
$$

$$
\leq \frac{7p_1+3}{10} \left\| x_1 - x_2 \right\|.
$$

This proves that $T_1$ is a contraction mapping. $T_1$ has the unique fixed point $x \in X$. Thus $x(n)$ is a positive solution of equations (1.1),

$$
x(n) = \frac{1+4p_1}{5} - p(n)x(n-\tau) - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} q_1(j) f(x(j-\sigma_1)) \right)
$$

$$
+ \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} q_2(j) g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \left( \sum_{j=s}^{\infty} e(j) \right) \right). \quad n \geq n_1
$$

Therefore

$$
\Delta \left( r(n) \Delta \left[ x(n) + p(n)x(n-\tau) \right] \right) + q_1(n) f(x(n-\sigma_1)) - q_2(n) g(x(n-\sigma_2)) = e(n),
$$

and so $x(n)$ is obviously a positive solution of equation (1.1). This completes the proof of the theorem. 

By using similar arguments as in Theorem 2.1, we can prove the following theorem.

**Theorem: 2.2** Suppose that conditions (A1) to (A3) hold and that there exists a constant $p_2$ such that

$$
-1 < -p_2 \leq p(n) < 0.
$$

Then equation (1.1) has a nonoscillatory solution.
Now we prove the following theorem.

**Theorem 2.3** Suppose that conditions (A1) to (A3) hold and that there exists constants \( p_3 \) and \( p_4 \) such that

\[
-1 < p_3 \leq p(n) \leq p_4 < 1,
\]

and \( p(n) \) changes sign such that \( p_4 < 1 + 5p_3 \). Then equation (1.1) has a positive solution.

**Proof:** As in the proof of theorem (2.1), it is possible to choose a positive number \( n \geq n_1 \),

\[
\sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) < \frac{1-p_4}{50},
\]

\[
M_1 \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) < p_3 + \frac{1-p_4}{5},
\]

\[
M_2 \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) < \frac{1-p_4}{50},
\]

where \( M_1 = \max \{ L_1, f(1) \} \), \( M_2 = \max \{ L_2, g(1) \} \) and \( L_1, L_2 \) are Lipschitz constants of \( f \) and \( g \) respectively on \( \left[ \frac{1-p_4}{10}, 1 \right] \).

Set \( S = \left\{ x \in X : \frac{1-p_4}{10} \leq x(n) \leq 1, n \geq n_1 \right\} \).

Define a mapping \( T_3 : S \to X \) as follows:

\[
(T_3 x)(n) = \frac{1+4p_4}{5} - p(n)x(n-\tau) - \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) + \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j), \quad n \geq n_1
\]

\[
(T_3 x)(n_0) \leq n \leq n_1,
\]

Clearly, \( T_3 x \) is continuous. For every \( x S \) and \( n \geq n_1 \),

\[
(T_3 x)(n) \leq \frac{1+4p_4}{5} + \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j)
\]

\[
< \frac{1+4p_4}{5} + \frac{1-p_4}{50} + \frac{1-p_4}{50}
\]

\[
< \frac{12+38p_4}{50} < 1
\]

and

\[
(T_3 x)(n) \geq -p_4 + \frac{1+4p_4}{5} - \sum_{j=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1))
\]

\[
\geq -p_4 + \frac{1+4p_4}{5} - p_3 - \frac{1-p_4}{10}
\]

\[
\geq \frac{1-p_4}{10}.
\]
Hence \( \frac{1-p^4}{10} \leq T_1(x(n)) \leq 1 \). Now, for \( x_1, x_2 \in S \) and \( n \geq n_1 \), we have

\[
\left\| (T_1 x_1)(n) - (T_1 x_2)(n) \right\| \leq p_4 \left\| x_1 - x_2 \right\| + \frac{1-p_4}{5} \left\| x_1 - x_2 \right\| + \frac{1-p_4}{50} \left\| x_1 - x_2 \right\|
\]

\[
\leq \frac{39p_4 + 11}{50} \left\| x_1 - x_2 \right\|
\]

This proves that \( T_1 \) is a contraction mapping. \( T_1 \) has the unique fixed point \( x \in X \). Thus \( x(n) \) is a positive solution of equations (1.1).

**Remark: 2.4** Similarly we can study the existence of positive solution of equation (1.1) for the other ranges of \( p(n) \) by using suitable transformations. Hence the details are omitted.

**REFERENCES:**


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