

EXISTENCE OF POSITIVE SOLUTIONS OF SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENCE EQUATIONS WITH FORCING TERM

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ABSTRACT

This paper deals with the existence of positive solutions of nonlinear neutral delay difference equations with positive and negative coefficients of the form

$$\Delta(r(n)\Delta[x(n) + p(n)x(n-\tau)]) + q_1(n)x(n-\sigma_1) - q_2(n)x(n-\sigma_2) = e(n),$$

under the following assumption that

$$\sum_{s=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_i(j) < \infty, \quad i = 1, 2$$

for various ranges of $p(n)$. By using Banach's contraction mapping principle, some sufficient conditions are established for the existence of nonoscillatory solutions.

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1. INTRODUCTION:

In this paper, we consider the second order nonlinear neutral delay difference equation with forcing term of the form

$$\Delta(r(n)\Delta[x(n) + p(n)x(n-\tau)]) + q_1(n)x(n-\sigma_1) - q_2(n)x(n-\sigma_2) = e(n),$$

where $n \geq n_0$, $\tau > 0$, $\sigma_1, \sigma_2 \geq 0$ are integers and $r(n), q_i(n), i = 1, 2$ are sequences of nonnegative real numbers.

(A1) $f, g \in C(R, R)$ satisfy Lipschitz condition and $xf(x) > 0, xg(x) > 0$, for $x \neq 0$,

(A2) $\sum_{s=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_i(j) < \infty, \quad i = 1, 2$

(A3) $\sum_{s=0}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) < \infty.$

The nonoscillatory behaviour of linear and nonlinear neutral delay difference and differential equations with positive and negative coefficients have been investigated by several authors, see, for example [3], [4], [5], [6], [7], [8] and [9], the references cited therein. We refer monographs [1] and [2] for a good amount of discussion concerning existence of

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solutions of delay difference equations. Our aim in this paper is to establish nonoscillation criteria for the second order nonlinear neutral delay difference equation (1.1) for various ranges of $p(n)$.

As is customary, a solution of (1.1) is said to be nonoscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. MAIN RESULTS:

Theorem: 2.1 Suppose that conditions (A1) to (A3) hold and that there exists a constant p_1 such that

$$0 < p(n) \leq p_1 < 1. \quad (2.1)$$

Then equation (1.1) has a nonoscillatory solution.

Proof: Suppose that (2.1) holds. Choose a positive constant $n \geq n_1$ and such that

$$\sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) < \frac{1-p_1}{10}, \quad (2.2)$$

$$M_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) < \frac{1-p_1}{10}, \quad (2.3)$$

$$M_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) < \frac{1-p_1}{5}, \quad (2.4)$$

where $M_1 = \max\{L_1, f(1)\}$, $M_2 = \max\{L_2, g(1)\}$ and L_1, L_2 are Lipschitz constants of f and g respectively on $\left[\frac{1-p_1}{10}, 1\right]$.

Consider the Banach space $X = \ell_{\infty}^{n_0}$ of all real sequences $x = x(n)$ with the norm $\|x\| = \sup_{n \geq n_1} |x(n)|$. Set

$$S = \left\{ x \in X : \frac{1-p_1}{10} \leq x(n) \leq 1, n \geq n_1 \right\}.$$

Define a mapping $T_1 : S \rightarrow X$ as follows:

$$(T_1 x)(n) = \begin{cases} \frac{1+4p_1}{5} - p(n)x(n-\tau) - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) \\ + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j), & n \geq n_1 \\ (T_1 x)(n_1), & n_0 \leq n \leq n_1. \end{cases}$$

Now, we show that $T_1 x$ is continuous. Let $\{x_k(n)\}$ be a sequence in X such that $\|x_k - x\| = 0$ as $k \rightarrow \infty$. Since

X is a closed set, we obtain

$$\begin{aligned} |(T_1 x_k)(n) - (T_1 x)(n)| &\leq \|x_k - x\| \left\{ p_1 + L_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) + L_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) \right\} \\ &= 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which proves that T_1x is continuous on X . For every $x \in S$ and $n \geq n_1$,

$$\begin{aligned}(T_1x)(n) &\leq \frac{1+4p_1}{5} + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) \\ &< \frac{1+4p_1}{5} + \frac{1-p_1}{5} + \frac{1-p_1}{10} \\ &< \frac{1+p_1}{2} < 1\end{aligned}$$

and

$$\begin{aligned}(T_1x)(n) &\geq -p_1 + \frac{1+4p_1}{5} - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) \\ &\geq -p_1 + \frac{1+4p_1}{5} - \frac{1-p_1}{10} \\ &\geq \frac{1-p_1}{10}.\end{aligned}$$

Hence $\frac{1-p_1}{10} \leq T_1(x(n)) \leq 1$. Now, for $x_1, x_2 \in S$ and $n \geq n_1$, we have

$$\begin{aligned} |(T_1x_1)(n) - (T_1x_2)(n)| &\leq p_1\|x_1 - x_2\| + \frac{1-p_1}{10}\|x_1 - x_2\| + \frac{1-p_1}{5}\|x_1 - x_2\| \\ &\leq \left(p_1 + \frac{1-p_1}{10} + \frac{1-p_1}{5}\right)\|x_1 - x_2\| \\ &\leq \frac{7p_1+3}{10}\|x_1 - x_2\|.\end{aligned}$$

This proves that T_1 is a contraction mapping. T_1 has the unique fixed point $x \in X$. Thus $x(n)$ is a positive solution of equations (1.1),

$$\begin{aligned}x(n) &= \frac{1+4p_1}{5} - p(n)x(n-\tau) - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) \\ &\quad + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j), \quad n \geq n_1\end{aligned}$$

Therefore

$$\Delta(r(n)\Delta[x(n) + p(n)x(n-\tau)]) + q_1(n)f(x(n-\sigma_1)) - q_2(n)g(x(n-\sigma_2)) = e(n),$$

and so $x(n)$ is obviously a positive solution of equation (1.1). This completes the proof of the theorem.

By using similar arguments as in Theorem 2.1, we can prove the following theorem.

Theorem: 2.2 Suppose that conditions (A1) to (A3) hold and that there exists a constant p_2 such that

$$-1 < -p_2 \leq p(n) < 0. \quad (2.5)$$

Then equation (1.1) has a nonoscillatory solution.

Now we prove the following theorem.

Theorem: 2.3 Suppose that conditions (A1) to (A3) hold and that there exists constants p_3 and p_4 such that

$$-1 < p_3 \leq p(n) \leq p_4 < 1, \quad (2.6)$$

and $p(n)$ changes sign such that $p_4 < 1 + 5p_3$. Then equation (1.1) has a positive solution.

Proof: As in the proof of theorem (2.1), it is possible to choose a positive number $n \geq n_1$,

$$\sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) < \frac{1-p_4}{50}, \quad (2.7)$$

$$M_1 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j) < p_3 + \frac{1-p_4}{5}, \quad (2.8)$$

$$M_2 \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j) < \frac{1-p_4}{50}, \quad (2.9)$$

where $M_1 = \max\{L_1, f(1)\}$, $M_2 = \max\{L_2, g(1)\}$ and L_1, L_2 are Lipschitz constants of f and g respectively

on $\left[\frac{1-p_4}{10}, 1\right]$.

Set $S = \left\{x \in X : \frac{1-p_4}{10} \leq x(n) \leq 1, n \geq n_1\right\}$.

Define a mapping $T_3 : S \rightarrow X$ as follows:

$$(T_3x)(n) = \begin{cases} \frac{1+4p_4}{5} - p(n)x(n-\tau) - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) \\ + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j), & n \geq n_1 \\ (T_3x)(n_1), & n_0 \leq n \leq n_1. \end{cases}$$

Clearly, T_3x is continuous. For every $x \in S$ and $n \geq n_1$,

$$\begin{aligned} (T_3x)(n) &\leq \frac{1+4p_4}{5} + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_2(j)g(x(j-\sigma_2)) + \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} e(j) \\ &< \frac{1+4p_4}{5} + \frac{1-p_4}{50} + \frac{1-p_4}{50} \\ &< \frac{12+38p_4}{50} < 1 \end{aligned}$$

and

$$\begin{aligned} (T_3x)(n) &\geq -p_4 + \frac{1+4p_4}{5} - \sum_{s=n_1}^{\infty} \frac{1}{r(s)} \sum_{j=s}^{\infty} q_1(j)f(x(j-\sigma_1)) \\ &\geq -p_4 + \frac{1+4p_4}{5} - p_3 - \frac{1-p_4}{10} \\ &\geq \frac{1-p_4}{10}. \end{aligned}$$

Hence $\frac{1-p_4}{10} \leq T_3(x(n)) \leq 1$. Now, for $x_1, x_2 \in S$ and $n \geq n_1$, we have

$$\begin{aligned} |(T_3 x_1)(n) - (T_3 x_2)(n)| &\leq p_4 \|x_1 - x_2\| + \frac{1-p_4}{5} \|x_1 - x_2\| + \frac{1-p_4}{50} \|x_1 - x_2\| \\ &\leq \frac{39p_4 + 11}{50} \|x_1 - x_2\|. \end{aligned}$$

This proves that T_3 is a contraction mapping. T_3 has the unique fixed point $x \in X$. Thus $x(n)$ is a positive solution of equations (1.1).

Remark: 2.4 Similarly we can study the existence of positive solution of equation (1.1) for the other ranges of $p(n)$ by using suitable transformations. Hence the details are omitted.

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