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ON THE "θ" PARTIAL ORDERING OF s-UNITARY MATRICES

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ABSTRACT

The concept of 'θ' partial ordering on s-unitary matrices is introduced. Some results relating to 'θ' partial ordering on s-unitary matrices are given.

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Keywords: *s-unitary matrix, Partial ordering, Lowener partial ordering, Star partial ordering, 'θ' partial ordering.*

1. INTRODUCTION:

Some characterizations of the star partial ordering and rank subtractivity for matrices was discussed by R. E. Hartwig and G. P. H. Styan in [3]. Jurgen Grob observed some remarks on partial ordering of Hermitian Matrices in[5]. Xifu Liu and Hu yang discussed some results on the partial ordering of block matrices[7], In [4], Jorma K. Merikoski and Xiaogi Liu have developed star partial ordering on Normal Matrices. In this paper we introduced the concept of 'θ'partial ordering on s-unitary matrices.

1.1. PRELIMINARIES AND NOTATIONS:

Let C_{nxn} be the space of n x n complex matrices of order n. For $A \in C_{nxn}$,
let A^T $\bar{A}$, A^*, A^s, $\bar{A}^s$ $(= A^\theta)$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively. Anna Lee[1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^s are related as $A^s = VA^TV$ where 'V' is the associated permutation matrix whose elements on the secondary diagonal are 1 and other elements are zero. .Also $\bar{A}^s$ denotes the conjugate secondary transpose of A.
i.e $\bar{A}^s = (c_{ij})$ where $c_{ij} = \overline{a_{n-j+1,\ n-i+1}}$ [2] and $\bar{A}^s = VA^*V = A^\theta$. Also 'V' satisfies the following properties. $V^T = \bar{V} = V^* = V^\theta = V$ and $V^2 = I$.

A matrix $A \in C_{nxn}$ is said to be s-hermitian if $A = A^\theta$ where $A^\theta = \bar{A}^s$ or $A^\theta = VA^*V$

A matrix $A \in C_{nxn}$ is said to be s-unitary if $AA^\theta = A^\theta A = I$ or $A^\theta = A^{-1}$ [6].

i.e. $AVA^*V = VA^*VA = I$ i.e. $VA^*V = A^{-1}$.

In other words $VAA^* = A^*AV = V$.

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2. MATRIX PARTIAL ORDERING ON MATRICES:

Definition: 2.1 [Lowener Partial Ordering]

For $A, B \in C_{nxn}$ $\quad A \underset{L}{\leq} B$ iff $B - A \geq 0$

Definition: 2.2 [Star Partial Ordering]

For $A, B \in C_{nxn}$, $A \underset{*}{\leq} B$ iff $A^*A = A^*B$ and $AA^* = BA^*$

Definition: 2.3 [Minus Partial Ordering]

For $A, B \in C_{nxn}$, $A \underset{rs}{\leq} B$ iff rank $(B - A)$ = rankB − rankA

Remark: 2.4 If $A \underset{L}{\leq} B$ for matrices A and B then B– A is s- hermitian.

3. "θ" PARTIAL ORDERING OF S-UNITARY MATRICES:

Definition: 3.1 ['θ' Partial Ordering]

For $A, B \in C_{nxn}$, $A \underset{\theta}{\leq} B$ iff $A^{\theta}A = A^{\theta}B$ and $AA^{\theta} = BA^{\theta}$

Theorem: 3.2 Let A and B be s-hermitian matrices and nonnegative definite. Then $A^2 \underset{\theta}{\leq} B^2$

iff $A \underset{\theta}{\leq} B$

Proof: Assuming $A^2 = C$, $B^2 = D$

$C \underset{\theta}{\leq} D \Rightarrow C^{\theta}C = C^{\theta}D$

$(A^2)^{\theta} A^2 = (A^2)^{\theta} B^2$

$(A^{\theta})^2 A^2 = (A^{\theta})^2 B^2$

$A^{\theta}A\,A = A^{\theta}\,BB$

$A^{\theta}A\,A = A^{\theta}\,BA \Rightarrow A^{\theta}A = A^{\theta}B$

Therefore $A \underset{\theta}{\leq} B$

Conversely, Assume, $A \underset{\theta}{\leq} B \Rightarrow A^{\theta}A = A^{\theta}B$

Premultiplying bothsides by A

$A^{\theta}\ A^2 = A^{\theta}BA$

$A^{\theta}\ A^2 = A^{\theta}AB$ (since BA=AB)

$A^{\theta}\ A^2 = A^{\theta}BA$ (since $A^{\theta}A = A^{\theta}B$)

$A^{\theta}\ A = A^{\theta}B$

Premultiplying bothsides by A^{θ}

$(A^{\theta})^2 A^2 = (A^{\theta})^2 B^2 \Rightarrow (A^2)^{\theta} A^2 = (A^2)^{\theta} B^2$

Therefore $A^2 \underset{\theta}{\leq} B^2$

Theorem: 3.3 Let $AV \underset{\theta}{\leq} VA$ .If A is s-unitary then A is unitary.

Proof: $AV \underset{\theta}{\leq} VA$

$\Rightarrow (AV)^{\theta}AV = (AV)^{\theta}VA$

$VA^{\theta}AV = VA^{\theta}VA$

$VIV = VA^{\theta}VA$

$I = VA^{\theta}VA$

$A^{-1} = VA^{\theta}V$

$A^{-1} = VVA^*VV$
$A^{-1} = A^*$

Therefore $AA^* = I$ (3.1)

$AV \underset{\theta}{\leq} VA \Rightarrow AV\,(AV)^{\theta} = VA\,(AV)^{\theta}$

$AVVA^{\theta}=VAVA^{\theta}$
$I=VAVVA^*V$
$VIV=AV^2A^*$

Therefore $I=AA^*$ (3.2)

From (3.1) and (3.2) $AA^* = A^*A = I$

Therefore A is unitary

Theorem: 3.4 If $A \underset{L}{\leq} B$ and $A \underset{\theta}{\leq} B$ then $A = B$

Proof: Given, $A \underset{\theta}{\leq} B \Rightarrow A^{\theta}A = A^{\theta}B$ and $AA^{\theta} = BA^{\theta}$

$\Rightarrow A^{\theta}(B - A) = 0$

Taking secondary conjugate transpose on bothsides we get
$(B - A)^{\theta} A = 0$ (3.3)

Since $A \underset{L}{\leq} B$ we have $(B - A)^* = V (B - A) V$

$(3.3) \Rightarrow V (B - A)^* VA = 0$
$\Rightarrow VV (B - A) VVA = 0$
$\Rightarrow I (B - A) IA = 0$
$\Rightarrow (B - A) A = 0$
$\Rightarrow BA - A^2 = 0$
$\Rightarrow BA = A^2$
$\Rightarrow B = A$

Theorem: 3.5 If A and B are s-unitary matrices. Then and $A \underset{\theta}{\leq} B \Rightarrow A^{-1} \underset{\theta}{\leq} B^{-1}$

Proof: Given, $A \underset{\theta}{\leq} B \Rightarrow A^{\theta}A = A^{\theta}B$ and $AA^{\theta}= BA^{\theta}$

$A^{\theta}A = A^{\theta}B$
$A^{-1}A = A^{-1}B$

Taking secondary conjugate transpose on bothsides we get
$(A^{-1}A)^{\theta} = (A^{-1}B)^{\theta}$
$A^{\theta} (A^{-1})^{\theta}=B^{\theta} (A^{-1})^{\theta}$
$A^{-1} (A^{-1})^{\theta}=B^{-1} (A^{-1})^{\theta}$ (3.4)
$AA^{\theta}=BA^{\theta} \Rightarrow AA^{-1}=BA^{-1}$ [Since A and B are s-unitary matrices]

Taking secondary conjugate transpose on bothsides we get
$(AA^{-1})^{\theta} = (BA^{-1})^{\theta}$
$(A^{-1})^{\theta} A^{\theta} = (A^{-1})^{\theta} B^{\theta} \Rightarrow (A^{-1})^{\theta} A^{-1} = (A^{-1})^{\theta} B^{-1}$ (3.5)

From (3.4) and (3.5) $A^{-1} \underset{\theta}{\leq} B^{-1}$

$A \underset{\theta}{\leq} B \Rightarrow A^{-1} \underset{\theta}{\leq} B^{-1}$

Theorem: 3.6 If A and B are s-unitary matrices, then and $A \underset{\theta}{\leq} B \Rightarrow VA \underset{*}{\leq} VB$

Proof: Given, $A \underset{\theta}{\leq} B \Rightarrow A^{\theta}A = A^{\theta}B$ and $AA^{\theta} = BA^{\theta}$

$A^{\theta}A = A^{\theta}B \Rightarrow VA^*VA = VA^*VB$

$(VA)^*(VA) = (VA)^*(VB)$

$(AV)^*(VA) = (AV)^*(VB)$ Since $(VA)^* = (AV)^*$

$(VA)^*(VA) = (VA)^*(VB)$ (3.6)

$AA^{\theta} = BA^{\theta}$

$AVA^*V = BVA^*V$

$VAVA^*VV = VBVA^*VV$

$VAVA^* = VBVA^*$

$(VA)(AV)^* = VB(VA)^*$

$(VA)(VA)^* = VB(VA)^*$ (3.7)

From (3.6) and (3.7) we get $VA \underset{\theta}{\leq} VB$

$A \underset{\theta}{\leq} B \Rightarrow VA \underset{*}{\leq} VB$

Theorem: 3.7 If A and B are s-unitary matrices.Then and $A \underset{*}{\leq} B \Rightarrow VA \underset{\theta}{\leq} VB$

Proof: Given, $A \underset{*}{\leq} B \Rightarrow A^*A = A^*B$ and $AA^* = BA^*$

$A^*A = A^*B$

$(VA^*V)VA = (VA^*V)VB$

$(A^{\theta})VA = (A^{\theta})VB$

$V(A^{\theta})VA = V(A^{\theta})VB$

$(AV)^{\theta}(VA) = (AV)^{\theta}(VB)$

$(VA)^{\theta}(VA) = (VA)^{\theta}(VB)$ (3.8)

$AA^* = BA^*$

$AVVA^* = BVVA^*$

$AV(VA^*V) = BV(VA^*V)$

$AV(A^{\theta}) = BV(A^{\theta})$

$AV(A^{\theta}V) = BV(A^{\theta}V)$

$(AV)(VA)^{\theta} = (BV)(VA)^{\theta}$

$(VA)(VA)^{\theta} = (VB)(VA)^{\theta}$ (3.9)

From (3.8) and (3.9) we get $VA \underset{\theta}{\leq} VB$

REFERENCES:

[1]. Anna Lee., 1976, Secondary symmetric, secondary skew symmetric, secondary orthogonal matrices, Period. Math. Hungary, Vol.7 63-70.

[2].Anna Lee. 1976, On s -symmetric, s-skew symmetric and s-orthogonal matrices, Period Math. Hungary Vol 7, 71–76.

[3].Hartwig. R. E and Styan. G.P.H., 1986, On some characterization of the star partial ordering for matrices and rank subtractivity, Lin. Alg. Appl., Vol. 82,145-161.

[4].Jorma.K. Merikoski and Xiaoji Liu., 2006 ,On the star partial ordering of normal matrices, Journal of inequalities in Pure and Applied Mathematics, Vol. 7, issue 7, Art 17.

[5].Jurgen Grob., 1997 ,Some remarks on partial ordering on hermitian matrices, Linear and Multilinear algebra, 42, 53-60.

[6] Krishnamoorthy.S and Govindarasu. A., 2010, On secondary unitary Matrices, International Journal of computational science and Mathematics. Vol. 2 number 3, 247-253.

[7] Liu and yang., 2011 "Some results on the partial ordering of block matrices" Vol. 54 1-9

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