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Certain inequalities for (α, m) - convex functions

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ABSTRACT

In this paper two inequalities of Hadamard type involving two (α, m)-convex functions are proved.

Key words: *(α, m)-convex functions, Hadamard type inequality, $(\alpha, m)-$convex function.*

AMS Subject Classification: *26D15, 26A51.*

INTRODUCTION:

We say that the function f is convex on $[0,b]$ if

$$f(tx+(1-t)y) \le tf(x)+(1-t)f(y)$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$.

In the paper [1], G. Toader defined $m-$convexity, an intermediate between the usual convexity and starshaped property, as the following:

Definition: 1.1 The function $f:[0,b] \to \mathbb{R}$ is said to be $m-$convex, where $m \in [0,1]$, if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx+m(1-t)y) \le tf(x)+m(1-t)f(y).$$

Definition: 1.2 The function $f:[0,b] \to \mathbb{R}$, $b>0$, is said to be starshaped if for every $x \in [0,b]$ and $t \in [0,1]$ we have

$$f(tx) \le tf(x).$$

Obviously, for $m=1$, we recapture the concept of convex functions defined on $[0,b]$ and for $m=0$ we get the concept of starshaped functions on $[0,b]$.

It is interesting to emphasize here that for any $m \in (0,1)$ there are continuous and differentiable functions which are $m-$convex, but which are not convex in the standard sense. Moreover, in [3], the following fact was proved.

Theorem: 1.3 For $m \in (0,1)$ there is an $m-$convex polynomial f such that f is not $n-$convex for any $m<n\le 1$.

For example, the function $f:[0,\infty] \to \mathbb{R}$ defined as

$$f(x)=\frac{1}{12}\left(x^4-5x^3+9x^2-5x\right)$$

is $16/17-$convex, but it is not $n-$convex for any $n \in (16/17,1]$ (see [4]).

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The notion of $m-$convexity can be further generalized through introduction of another parameter $\alpha \in [0,1]$ in the definition of $m-$convexity. The class of $(\alpha, m)-$ convex functions was first introduced in [5] as follows.

Definition: 1.4 The function $f:[0,b] \to \mathbb{R}$, $b>0$, is said to be $(\alpha, m)-$ convex, where $(\alpha, m) \in [0,1]^2$, if

$$f(tx + m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y)$$

for all $x, y \in [0,b]$ and $t \in [0,1]$.

It is clear that for $(\alpha, m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ we obtain the following classes of functions: increasing, $\alpha-$starshaped, starshaped, $m-$convex, convex, and $\alpha-$convex functions.

The reader could find some properties of convex and starshaped functions in the paper of S. S. Dragomir [2] and papers cited therein. Also, in the same paper are proved some new inequalities of Hermite-Hadamard type for $m-$convex functions. The main goal of this paper is to establish two inequalities of Hadamard type involving two $(\alpha, m)-$ convex functions.

Now we pass to the main results.

1. MAIN RESULTS:

Theorem: 2.1 Let $f, g:[0,b] \to \mathbb{R}$ be two $(\alpha, m)-$ convex functions with $(\alpha, m) \in [0,1]^2$ and $0 \leq a < b$. If $f, g \in L_1[a,b]$, then the following inequality holds

$$\begin{aligned}
&\frac{g(a)}{(mb-a)^{1+\alpha}} \int_a^{mb} (mb-x)^{\alpha} f(x)dx \\
&+\frac{mg(b)}{(mb-a)^{1+\alpha}} \int_a^{mb} [(mb-a)^{\alpha} - (mb-x)^{\alpha}] f(x)dx \\
&+\frac{f(a)}{(mb-a)^{1+\alpha}} \int_a^{mb} (mb-x)^{\alpha} g(x)dx \\
&+\frac{mf(b)}{(mb-a)^{1+\alpha}} \int_a^{mb} [(mb-a)^{\alpha} - (mb-x)^{\alpha}] g(x)dx \\
\leq &\frac{M(a,b;m)}{3} + \frac{N(a,b;m)}{6} + \frac{1}{\alpha(mb-a)^{\frac{1}{\alpha}}} \int_a^{mb} (mb-x)^{\frac{1}{\alpha}-1} f(x)g(x)dx,
\end{aligned} \tag{2.1}$$

where $M(a,b;m) := f(a)g(a) + m^2 f(b)g(b)$, $N(a,b;m) := m[f(a)g(b) + f(b)g(a)]$.

Proof: From the fact that functions f and g are $(\alpha, m)-$ convex on $[a,b]$, then for $t \in [0,1]$ we have

$$\begin{aligned}
f[ta + m(1-t)b] &\leq t^{\alpha} f(a) + m(1-t^{\alpha}) f(b) \\
g[ta + m(1-t)b] &\leq t^{\alpha} g(a) + m(1-t^{\alpha}) g(b).
\end{aligned}$$

Using the inequality $ER + FP \leq EP + FR$, which holds for $E \leq F$ and $P \leq R$, we obtain

$$\begin{aligned}
&f[ta + m(1-t)b][t^{\alpha} g(a) + m(1-t^{\alpha}) g(b)] \\
&+g[ta + m(1-t)b][t^{\alpha} f(a) + m(1-t^{\alpha}) f(b)] \leq [tf(a) + m(1-t) f(b)][tg(a) + m(1-t) g(b)] \\
&\qquad + f[t^{\alpha} a + m(1-t^{\alpha}) b] g[t^{\alpha} a + m(1-t^{\alpha}) b]
\end{aligned}$$

or

$$\begin{aligned}
&g(a)t^{\alpha} f[ta + m(1-t)b] + mg(b)(1-t^{\alpha}) f[ta + m(1-t)b] \\
&+ f(a)t^{\alpha} g[ta + m(1-t)b] + mf(b)(1-t^{\alpha}) g[ta + m(1-t)b] \\
\leq &t^2 f(a)g(a) + m^2 (1-t)^2 f(b)g(b) + m(1-t)tf(b)g(a) \\
&+ m(1-t)tf(a)g(b) + f[t^{\alpha} a + m(1-t^{\alpha}) b] g[t^{\alpha} a + m(1-t^{\alpha}) b].
\end{aligned}$$

Now we integrate both sides of the latter inequality over $[0,1]$, so that we shall obtain

$$\begin{aligned}
& g(a)\int_0^1 t^\alpha f[ta+m(1-t)b]dt + mg(b)\int_0^1 (1-t^\alpha) f[ta+m(1-t)b]dt \\
& + f(a)\int_0^1 t^\alpha g[ta+m(1-t)b]dt + mf(b)\int_0^1 (1-t^\alpha) g[ta+m(1-t)b]dt \\
\le\ & f(a)g(a)\int_0^1 t^2 dt + m^2 f(b)g(b)\int_0^1 (1-t)^2 dt + mf(b)g(a)\int_0^1 (1-t)tdt \\
& + mf(a)g(b)\int_0^1 (1-t)tdt + \int_0^1 f[t^\alpha a+m(1-t^\alpha)b]g[t^\alpha a+m(1-t^\alpha)b]dt.
\end{aligned} \tag{2.2}$$

Substituting $ta+m(1-t)b=x,\ \ t^\alpha a+m(1-t^\alpha)b=x$ we easy find

$$\int_0^1 t^\alpha f[ta+m(1-t)b]dt = \frac{1}{(mb-a)^{1+\alpha}}\int_a^{mb}(mb-x)^\alpha f(x)dx$$

$$\int_0^1 (1-t^\alpha) f[ta+m(1-t)b]dt = \frac{1}{(mb-a)^{1+\alpha}}\int_a^{mb}[(mb-a)^\alpha-(mb-x)^\alpha]f(x)dx$$

$$\int_0^1 t^\alpha g[ta+m(1-t)b]dt = \frac{1}{(mb-a)^{1+\alpha}}\int_a^{mb}(mb-x)^\alpha g(x)dx$$

$$\int_0^1 (1-t^\alpha) g[ta+m(1-t)b]dt = \frac{1}{(mb-a)^{1+\alpha}}\int_a^{mb}[(mb-a)^\alpha-(mb-x)^\alpha]g(x)dx$$

$$\int_0^1 f[t^\alpha a+m(1-t^\alpha)b]g[t^\alpha a+m(1-t^\alpha)b]dt = \frac{1}{\alpha(mb-a)^{\frac{1}{\alpha}}}\int_a^{mb}(mb-x)^{\frac{1}{\alpha}-1} f(x)g(x)dx$$

and

$$\int_0^1 t^2 dt = \int_0^1 (1-t)^2 dt = \frac{1}{3}, \quad \int_0^1 t(1-t)dt = \frac{1}{6}.$$

Now inserting above equalities to the inequality (2.2), we immediately obtain (2.1).

Theorem: 2.2 Let $f, g : [0,b] \to \mathbb{R}$ be two $m-$convex functions with $m \in [0,1]$ and $0 \le a < b$. If $f, g \in L_1[a,b]$. Then the following inequality holds

$$\begin{aligned}
& \frac{f\left(\frac{a+mb}{2}\right)}{mb-a}\int_a^{mb} g(x)dx + \frac{g\left(\frac{a+mb}{2}\right)}{mb-a}\int_a^{mb} f(x)dx \\
\le\ & \frac{1}{2(mb-a)}\int_a^{mb} f(x)g(x)dx \\
& + \frac{M(a,b;m)}{12} + \frac{N(a,b;m)}{6} + f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right),
\end{aligned} \tag{2.3}$$

where $M(a,b;m)$, and $N(a,b;m)$ are same as those in theorem 2.1.

Proof: From the fact that f, g are $(\alpha, m)-$convex functions on $[a,b]$, then for $t \in [0,1]$ we have

$$\begin{aligned}
f\left(\frac{a+mb}{2}\right) &= f\left(\frac{ta+m(1-t)b}{2} + \frac{(1-t)a+mtb}{2}\right) \\
&= \frac{f[ta+m(1-t)b] + f[(1-t)a+mtb]}{2}
\end{aligned}$$

and

$$g\left(\frac{a+mb}{2}\right)=g\left(\frac{ta+m(1-t)b}{2}+\frac{(1-t)a+mtb}{2}\right)$$
$$=\frac{g[ta+m(1-t)b]+g[(1-t)a+mtb]}{2}.$$

Again, using the inequality $ER+FP\le EP+FR$, which holds for $E\le F$ and $P\le R$, we obtain

$$\begin{aligned}
&f\left(\frac{a+mb}{2}\right)\frac{g[ta+m(1-t)b]+g[(1-t)a+mtb]}{2}\\
&+g\left(\frac{a+mb}{2}\right)\frac{f[ta+m(1-t)b]+f[(1-t)a+mtb]}{2}\\
&\le\frac{f[ta+m(1-t)b]+f[(1-t)a+mtb]}{2}\,\frac{g[ta+m(1-t)b]+g[(1-t)a+mtb]}{2}\\
&+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{2}f\left(\frac{a+mb}{2}\right)\{g[ta+m(1-t)b]+g[(1-t)a+mtb]\}\\
&+\frac{1}{2}g\left(\frac{a+mb}{2}\right)\{f[ta+m(1-t)b]+f[(1-t)a+mtb]\}\\
&\le\frac{1}{4}\{f[ta+m(1-t)b]g[ta+m(1-t)b]+f[(1-t)a+mtb]g[(1-t)a+mtb]\}\\
&\frac{1}{4}\{f[ta+m(1-t)b]g[(1-t)a+mtb]+f[(1-t)a+mtb]g[ta+m(1-t)b]\}\\
&+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&\le\frac{1}{4}\{f[ta+m(1-t)b]g[ta+m(1-t)b]+f[(1-t)a+mtb]g[(1-t)a+mtb]\}\\
&+\frac{1}{4}\{[tf(a)+m(1-t)f(b)][(1-t)g(a)+mtg(b)]\}\\
&+\frac{1}{4}\{[(1-t)f(a)+mtf(b)][tg(a)+m(1-t)g(b)]\}\\
&+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right)\\
&=\frac{1}{4}\{f[ta+m(1-t)b]g[ta+m(1-t)b]+f[(1-t)a+mtb]g[(1-t)a+mtb]\}\\
&+\frac{1}{4}2t(1-t)[f(a)g(a)+m^2f(b)g(b)]\\
&+\frac{1}{4}m[t^2+(1-t)^2][f(a)g(b)+f(b)g(a)]\\
&+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right).
\end{aligned}$$

Hence, integrating the above inequality over $[0,1]$, we get

$$\frac{1}{2} f\left(\frac{a+mb}{2}\right)\int_0^1 \{g[ta+m(1-t)b]+g[(1-t)a+mtb]\}dt$$
$$+\frac{1}{2} g\left(\frac{a+mb}{2}\right)\int_0^1 \{f[ta+m(1-t)b]+f[(1-t)a+mtb]\}dt$$
$$\leq \frac{1}{4}\int_0^1 \{f[ta+m(1-t)b]g[ta+m(1-t)b]+f[(1-t)a+mtb]g[(1-t)a+mtb]\}dt$$
$$+\frac{1}{4}[f(a)g(a)+m^2 f(b)g(b)]2\int_0^1 t(1-t)dt$$
$$+\frac{1}{4}m[f(a)g(b)+f(b)g(a)]\int_0^1 [t^2+(1-t)^2]dt$$
$$+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right).$$

Now, we substitute $ta+m(1-t)b=x$ and $(1-t)a+mtb=x$ to the above integrals, and taking into account that

$$\int_0^1 [t^2+(1-t)^2]dt=\frac{2}{3}, \quad \int_0^1 2t(1-t)dt=\frac{1}{3}$$

$$\int_0^1 f[ta+m(1-t)b]g[ta+m(1-t)b]dt=$$
$$\int_0^1 f[(1-t)a+mtb]g[(1-t)a+mtb]dt=\frac{1}{mb-a}\int_a^{mb} f(x)g(x)dx$$
$$\int_0^1 f[ta+m(1-t)b]dt=\int_0^1 f[(1-t)a+mtb]dt=\frac{1}{mb-a}\int_a^{mb} f(x)dx$$
$$\int_0^1 g[ta+m(1-t)b]dt=\int_0^1 g[(1-t)a+mtb]dt=\frac{1}{mb-a}\int_a^{mb} g(x)dx$$

we finally obtain

$$\frac{f\left(\frac{a+mb}{2}\right)}{mb-a}\int_a^{mb} g(x)dx+\frac{g\left(\frac{a+mb}{2}\right)}{mb-a}\int_a^{mb} f(x)dx$$
$$\leq \frac{1}{2(mb-a)}\int_a^{mb} f(x)g(x)dx+\frac{M(a,b;m)}{12}+\frac{N(a,b;m)}{6}+f\left(\frac{a+mb}{2}\right)g\left(\frac{a+mb}{2}\right).$$

With this the proof of the theorem is completed.

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