



GENERALIZED BI-RECURRENT WEYL SPACES

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ABSTRACT

In this paper, we have studied the generalized bi-recurrent Weyl spaces. The bi-recurrent properties of Weyl Conformal curvature, Weyl projective curvature and Weyl concircular curvature tensor have been studied and some relation between them has been derived.

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1. Introduction:

An-n-dimensional differentiable manifold W_n is said to be Weyl space if it has a symmetric connection ∇^* and a symmetric conformal metric tensor g_{ij} preserved by ∇^* . Accordingly, in local coordinates there exists a covariant vector field T_k (complementary vector field) satisfying the condition [1], [2], [3],

$$\nabla^*_k g_{ij} - 2T_k g_{ij} = 0 \quad (1.1)$$

The above equation can be extended to

$$\partial_k g_{ij} - g_{hj} \Gamma_{jk}^h - 2T_k g_{ij} = 0 \quad (1.2)$$

where Γ_{jk}^h are the connection coefficients of the symmetric connection ∇ and are defined as

$$\Gamma_{jk}^h = \left\{ \begin{matrix} h \\ jk \end{matrix} \right\} - g^{hm} (g_{mj} T_k + g_{mk} T_j - g_{jk} T_m) \quad (1.3)$$

Moreover, under the renormalization condition

$$\tilde{g}_{ij} = \lambda^2 g_{ij} \quad (1.4)$$

of the metric tensor g_{ij} , the covariant vector field T_k is transformed by the law

$$\tilde{T}_k = T_k + \partial_k \ln \lambda \quad (1.5)$$

where λ is a scalar function defined on W_n . We denote such a Weyl space by $W_n(\Gamma_{jk}^h, g_{ij}, T_k)$ or $W_n(g, T)$.

An n-dimensional differential manifold having an anti-symmetric connection ∇ and anti-symmetric metric tensor g_{ij} preserved by ∇ is called generalized Weyl space [4]. It is denoted by $GW_n(g, T)$.

For such a space, in local co-ordinate system, the compatibility condition are

$$\nabla_k g_{ij} - 2T_k g_{ij} = 0 \quad (1.6)$$

Where T_k are the components of a covariant vector field, called the complementary vector field of the $GW_n(g, T)$ space. Using the concept of covariant differentiation ([5], [6]), the compatibility condition of (1.6) can be written as

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$$\partial_k g_{ik} - g_{hj} L_{ik}^h - g_{ih} L_{jk}^h - 2T_k g_{ij} = 0 \quad (1.7)$$

where L_{ik}^h are the connection coefficient of the anti-symmetric connection ∇ and are obtain from the compatibility condition as

$$L_{ij}^h = \Gamma_{ij}^h + \frac{1}{2}[\chi_{im}^h g_{jh} + \chi_{mj}^h g_{ih} + \chi_{ij}^h g_{hm}] g^{mi} \quad (1.8)$$

Now putting

$$\chi_{ij}^h = \frac{1}{2}[\chi_{im}^h g_{jh} + \chi_{mj}^h g_{ih} + \chi_{ij}^h g_{hm}] g^{mi} \quad (1.9)$$

We obtain

$$L_{ij}^h = \Gamma_{ij}^h + \chi_{ij}^h \quad (1.10)$$

Where Γ_{ij}^h and χ_{ij}^h are respectively the coefficient of a Weyl connection and the torsion tenor of $GW_n(g, T)$ space and are expressed as

$$\Gamma_{ij}^h = \frac{1}{2}[L_{ij}^h + L_{ji}^h] = L_{ij}^h \quad (1.11)$$

and

$$\chi_{kl}^h = \frac{1}{2}[L_{kl}^h - L_{lk}^h] = L_{[kl]}^h \quad (1.12)$$

square bracket stands for anti-symmetry.

The components of mixed curvature tensor and Ricci tensor of $GW_n(g, T)$ are respectively

$$L_{jki}^h = \partial_k L_{ji}^h - \partial_i L_{jk}^h + L_{lk}^h L_{ji}^l - L_{li}^h L_{jk}^l \quad (1.13)$$

$$L_{ij} = L_{ija}^a \quad (1.14)$$

on the other hand scalar curvature of $GW_n(g, T)$ is defined by

$$L = g^{ij} L_{ij} \quad (1.15)$$

It is easy to see that curvature tensor L_{jki}^h of $GW_n(g, T)$ can be written as

$$L_{jki}^h = B_{jki}^h + \chi_{jkl}^h \quad (1.16)$$

Where the tensors B_{jki}^h and χ_{jkl}^h are defined respectively

$$B_{jki}^h = \partial_k \Gamma_{ji}^h - \partial_i \Gamma_{jk}^h + \Gamma_{lk}^h \Gamma_{ji}^l - \Gamma_{lk}^h \Gamma_{jk}^l \quad (1.17)$$

$$\chi_{jki}^h = \nabla_k \chi_{ji}^h - \nabla_i \chi_{jk}^h + \chi_{li}^h \chi_{jk}^l - \chi_{lk}^h \chi_{ji}^l - 2\chi_{jl}^h \chi_{ki}^l \quad (1.18)$$

The curvature tensor of $GW_n(g, T)$ satisfies the relation [6].

$$L_{jkl}^h + L_{jlk}^h = 0 \quad (1.19)$$

$$L_{hkl}^j + L_{hkl}^j + L_{klh}^j = 2[\nabla_k \chi_{lh}^j + \nabla_h \chi_{kl}^j + 2\chi_{lm}^j \chi_{hk}^m + 2\chi_{hm}^j \chi_{kl}^m + 2\chi_{km}^j \chi_{lh}^m] \quad (1.20)$$

$$\nabla_m L_{jkl}^i + \nabla_k L_{jlm}^i + \nabla_l L_{jmk}^i = 2[L_{jpl}^i \chi_{mk}^p + L_{jpk}^i \chi_{lm}^p + L_{jpm}^i \chi_{kl}^p] \quad (1.21)$$

The Weyl Conformal curvature C_{ijk}^h , Weyl projective curvature W_{ijk}^h and Weyl concircular curvature tensor Z_{ijk}^h in $GW_n(g, T)$ are respectively given by [7][8].

$$C_{ijk}^h = R_{ijk}^h + \frac{2}{n(n-2)} [\delta_k^h R_{[ij]} - \delta_j^h R_{[ik]} - g_{ik} g^{hm} R_{[mj]} + g_{ij} g^{hm} R_{[mk]} - (n-2) \delta_i^h R_{kj}] \\ - \frac{1}{(n-2)} [\delta_k^h R_{ij} - \delta_j^h R_{ik} - g_{ik} g^{hm} R_{mj} + g_{ij} g^{hm} R_{mj}] + \frac{R}{(n-1)(n-2)} (g_{ij} \delta_k^h - g_{ik} \delta_j^h) \quad (1.22)$$

$$W_{ijk}^h = R_{ijk}^h + \frac{1}{(n-1)}(R_{ik}\delta_j^h - R_{ij}\delta_k^h) \quad (1.23)$$

$$Z_{ijk}^h = R_{ijk}^h + \frac{R}{n(n-1)}(g_{ik}\delta_j^h - g_{ij}\delta_k^h) \quad (1.24)$$

From (1.22) and (1.24) we have

$$C_{ijk}^h = Z_{ijk}^h + \frac{2}{n(n-2)}[\delta_k^h Z_{[ij]} - \delta_j^h Z_{[ik]} - g_{ik}g^{hm}Z_{[mj]} + g_{ij}g^{hm}Z_{[mk]} - (n-2)\delta_i^h Z_{kj}] \\ - \frac{1}{(n-2)}[\delta_k^h Z_{ij} - \delta_j^h Z_{ik} - g_{ik}g^{hm}Z_{mj} + g_{ij}g^{hm}Z_{mk}] \quad (1.25)$$

$$\text{where } Z_{jk} = R_{jk} - \frac{R}{n}g_{jk} \quad (1.26)$$

2. Bi-Recurrent Weyl Spaces:

Definition: 2.1 If the curvature tensor L_{ijk}^h of GW_n satisfies the condition

$$L_{ijk,ab}^h = \lambda_{ab}L_{ijk}^h \quad (2.1)$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called bi-recurrent. Such a space is denoted by GW_n^* .

Definition: 2.2 If the curvature tensor C_{ijk}^h of GW_n satisfies the condition

$$C_{ijk,ab}^h = \lambda_{ab}C_{ijk}^h \quad (2.2)$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Conformal curvature tensor. We denote such a space by $C^* - GW_n$.

Definition: 2.3 If the curvature tensor Z_{ijk}^h of GW_n satisfies the condition

$$Z_{ijk,ab}^h = \lambda_{ab}Z_{ijk}^h \quad (2.3)$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Concircular curvature tensor. We denote such a space by $Z^* - GW_n$.

Definition: 2.4 If the curvature tensor W_{ijk}^h of GW_n satisfies the condition

$$W_{ijk,ab}^h = \lambda_{ab}W_{ijk}^h \quad (2.4)$$

where λ_{ab} is a non-zero covariant tensor field, then GW_n is called generalized Weyl space with bi-recurrent Weyl Projective curvature tensor. We denote such a space by $W^* - GW_n$.

Theorem: 2.1 The necessary and sufficient condition for a Generalized Weyl space GW_n to be $W^* - GW_n$ is that it should be $Z^* - GW_n$.

Proof: Let GW_n satisfies the relation (2.3) then (2.3) in view of (1.24) reduces to

$$R_{ijk,ab}^h = \lambda_{ab}R_{ijk}^h + \frac{(R_{ab}-\lambda_{ab}R)}{n(n-1)}(g_{ik}\delta_j^h - g_{ij}\delta_k^h) \quad (2.5)$$

contracting the indices h and k we have

$$R_{ij,ab}^h = \lambda_{ab}R_{ij}^h + \frac{(R_{ab}-\lambda_{ab}R)}{n}g_{ij} \quad (2.6)$$

Now differentiating (1.23) covariantly with respect to a and then again differentiating the result thus obtained with respect to b we have

$$W_{ijk,ab}^h = R_{ijk,ab}^h + \frac{1}{(n-1)}(R_{ik,ab}\delta_j^h - R_{ij,ab}\delta_k^h) \quad (2.7)$$

Above equation in view of (2.5) and (2.6) reduces to

$$W_{ijk,ab}^h = \lambda_{ab} R_{ijk}^h + \frac{(R_{ab}-\lambda_{ab}R)}{n(n-1)} (g_{ik}\delta_j^h - g_{ij}\delta_k^h) + \frac{1}{(n-1)} \left[\lambda_{ab} R_{ik}\delta_j^h - \frac{(R_{ab}-\lambda_{ab}R)}{n} g_{ik} \right] - \delta_k^h \left[\lambda_{ab} R_{ij} - \frac{(R_{ab}-\lambda_{ab}R)}{n} g_{ij} \right] \quad (2.8)$$

$$= \lambda_{ab} \left[R_{ijk}^h + \frac{1}{(n-1)} (R_{ik}\delta_j^h - R_{ij}\delta_k^h) \right] \quad (2.9)$$

$$= \lambda_{ab} W_{ijk}^h$$

therefore the space is $W^* - GW_n$.

Conversely let the space be $W^* - GW_n$ then (2.4) in view of (1.23) gives

$$R_{ijk,ab}^h + \frac{1}{(n-1)} (R_{ik,ab}\delta_j^h - R_{ij,ab}\delta_k^h) = \lambda_{ab} \left[R_{ijk}^h + \frac{1}{(n-1)} (R_{ik}\delta_j^h - R_{ij}\delta_k^h) \right] \quad (2.10)$$

transecting (2.10) by g^{ij} we obtain

$$R_{ij,ab} = \lambda_{ab} R_{ij} - \frac{1}{n} (\lambda_{ab} R - R_{,ab}) g_{ij} \quad (2.11)$$

Therefore we have

$$\delta_k^h R_{ij,ab} - \delta_j^h R_{ik,ab} = \lambda_{ab} (\delta_k^h R_{ij} - \delta_j^h R_{ik}) - \frac{R}{n} \lambda_{ab} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) + \frac{R_{,ab}}{n} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \quad (2.12)$$

Substituting (2.12) in (2.10) we have

$$R_{ijk,ab}^h - \frac{R_{,ab}}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) = \lambda_{ab} \left[R_{ijk}^h - \frac{R}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \right]$$

which reduces to

$$Z_{ijk,ab}^h = \lambda_{ab} Z_{ijk}^h$$

Theorem: 2.2 The necessary and sufficient condition for a Generalized Weyl space GW_n to be $C^* - GW_n$ is that it should be $Z^* - GW_n$.

Proof: Let GW_n satisfies the relation (2.3) then (2.3) in view of (1.24) reduces to (2.5) and (2.6).

Now differentiating (1.24), (1.25) and (1.26) covariantly with respect to a and then again differentiating the result thus obtained with respect to b we have respectively

$$Z_{ijk,ab}^h = R_{ijk,ab}^h + \frac{R_{,ab}}{n(n-1)} (g_{ik}\delta_j^h - g_{ij}\delta_k^h), \quad (2.13)$$

$$C_{ijk,ab}^h = Z_{ijk,ab}^h + \frac{2}{n(n-2)} [\delta_k^h Z_{[ij],ab} - \delta_j^h Z_{[ik],ab} - g_{ik} g^{hm} Z_{[mj],ab} + g_{ij} g^{hm} Z_{[mk],ab} - (n-2)\delta_i^h Z_{[kj],ab}] - \frac{1}{(n-2)} [\delta_k^h Z_{ij,ab} - \delta_j^h Z_{ik,ab} - g_{ik} g^{hm} Z_{mj,ab} + g_{ij} g^{hm} Z_{mk,ab}] \quad (2.14)$$

and

$$Z_{jk,ab} = R_{jk,ab} - \frac{R_{,ab}}{n} g_{jk} \quad (2.15)$$

Equation (1.14) in view of (2.13) and (2.15) reduces to

$$C_{ijk,ab}^h = R_{ijk,ab}^h + \frac{R_{,ab}}{n(n-1)} (g_{ik}\delta_j^h - g_{ij}\delta_k^h) + \frac{2}{n(n-2)} [\delta_k^h (R_{ij,ab} - \frac{R_{,ab}}{n} g_{ij}) - \delta_j^h (R_{ik,ab} - \frac{R_{,ab}}{n} g_{ik}) - g_{ik} g^{hm} (R_{mj,ab} - \frac{R_{,ab}}{n} g_{mj}) + g_{ij} g^{hm} (R_{mk,ab} - \frac{R_{,ab}}{n} g_{mk}) - (n-2)\delta_i^h (R_{kj,ab} - \frac{R_{,ab}}{n} g_{kj}) - \frac{1}{(n-2)} [\delta_k^h (R_{ij,ab} - \frac{R_{,ab}}{n} g_{ij}) - \delta_j^h (R_{ik,ab} - \frac{R_{,ab}}{n} g_{ik}) - g_{ik} g^{hm} (R_{mj,ab} - \frac{R_{,ab}}{n} g_{mj}) + g_{ij} g^{hm} (R_{mk,ab} - \frac{R_{,ab}}{n} g_{mk})]. \quad (2.16)$$

Equation (2.16) in view of (2.5), (1.24) and (2.6) reduces to

$$C_{ijk,ab}^h = \lambda_{ab} \left\{ Z_{ijk}^h + \frac{2}{n(n-2)} [\delta_k^h Z_{[ij]} - \delta_j^h Z_{[ik]} - g_{ik} g^{hm} Z_{[mj]} + g_{ij} g^{hm} Z_{[mk]} - (n-2) \delta_i^h Z_{kj}] \right. \\ \left. - \frac{1}{(n-2)} [\delta_k^h Z_{ij} - \delta_j^h Z_{ik} - g_{ik} g^{hm} Z_{mj} + g_{ij} g^{hm} Z_{mk}] \right\}$$

which reduces to

$$C_{ijk,ab}^h = \lambda_{ab} C_{ijk}^h$$

Conversely, let the space be $C^* - GW_n$ then (2.2) in view of (1.25) reduces to

$$Z_{ijk,ab}^h + \frac{2}{n(n-2)} [\delta_k^h Z_{[ij],ab} - \delta_j^h Z_{[ik],ab} - g_{ik} g^{hm} Z_{[mj],ab} + g_{ij} g^{hm} Z_{[mk],ab} - (n-2) \delta_i^h Z_{kj,ab}] \\ - \frac{1}{(n-2)} [\delta_k^h Z_{ij,ab} - \delta_j^h Z_{ik,ab} - g_{ik} g^{hm} Z_{mj,ab} + g_{ij} g^{hm} Z_{mk,ab}] = \lambda_{ab} \left\{ Z_{ijk}^h + \frac{2}{n(n-2)} [\delta_k^h Z_{[ij]} - \delta_j^h Z_{[ik]} \right. \\ \left. - g_{ik} g^{hm} Z_{[mj]} + g_{ij} g^{hm} Z_{[mk]} - (n-2) \delta_i^h Z_{kj}] - \frac{1}{(n-2)} [\delta_k^h Z_{ij} - \delta_j^h Z_{ik} - g_{ik} g^{hm} Z_{mj} + g_{ij} g^{hm} Z_{mk}] \right\}$$

which reduces to

$$Z_{ijk,ab}^h - \lambda_{ab} Z_{ijk}^h = \frac{2}{n(n-2)} [\delta_k^h (Z_{[ij],ab} - \lambda_{ab} Z_{[ij]}) - \delta_j^h (Z_{[ik],ab} - \lambda_{ab} Z_{[ik]}) \\ - g_{ik} g^{hm} (Z_{[mj],ab} - \lambda_{ab} Z_{[mj]}) + g_{ij} g^{hm} (Z_{[mk],ab} - \lambda_{ab} Z_{[mk]}) \\ - (n-2) \delta_i^h (Z_{kj,ab} - \lambda_{ab} Z_{kj})] - \frac{1}{(n-2)} [\delta_k^h (Z_{ij,ab} - \lambda_{ab} Z_{ij}) - \delta_j^h (Z_{ik,ab} - \lambda_{ab} Z_{ik}) \\ - g_{ik} g^{hm} (Z_{mj} - \lambda_{ab} Z_{mj}) + g_{ij} g^{hm} (Z_{mk,ab} - \lambda_{ab} Z_{mk})] \quad (2.17)$$

Equation (2.17) in view of (1.26)(2.6) and (2.15) reduces to

$$Z_{ijk,ab}^h - \lambda_{ab} Z_{ijk}^h = 0$$

therefore the space is $Z^* - GW_n$.

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