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NONNIL-NOETHERIAN COMMUTATIVE RINGS AND NONNIL IDEALS

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ABSTRACT

In this paper, it should be noted that a commutative ring R is said to be Nonnil-Noetherian if every Nonnil Ideal of R is finitely generated. A ring R is said to be Nonnil-Noetherian if R is both right and left Nonnil-Noetherian ring. In the first section of this paper we show that many of the properties Noetherian rings are also true for Nonnil-Noetherian rings. In the second section we show that the direct sum of Nonnil-Noetherian ring needs not to be Noetherian.

Keywords: *Nonnil-Noetherian ring; Nonnil Ideal*

0. INTRODUCTION:

We assume throughout that all rings are commutative with $1 \neq 0$. Let *R* be a ring. We denote by *Nil(R)* and $M_n(R)$ the set of all nilpotent element of *R* and the matrix ring over *R* respectively. Recall from [2] and [7] that R is a Nonnil-Noetherian ring if each Nonnil Ideal of *R* is finitely generated. An ideal *I* of a ring *R* is said to be Nonnil Ideal if $I \not\subset Nil(R)$. In [7], the idealization construction to give examples of Nonnil-Noetherian rings that are not Noetherian rings. In this paper, we continue to investigate the properties of Nonnil-Noetherian rings. Clearly Noetherian ring is Nonnil-Noetherian ring. In the first section of this paper, we show that many of the properties Noetherian ring are also true for Nonnil-Noetherian ring. In the second section we are going to use some examples of [7], to show that the direct sum of Nonnil-Noetherian ring needs not to be Noetherian.

1. THE PROPERTIES OF NONNIL-NOETHERIAN RINGS:

Proposition: 1.1 Let $\{M_i\}_{i\in I}$, that is a family of submodules of a non-zero *R*-module *M*. Then $\oplus_{i\in I} M_i$ is Noetherian(Artinian) if and only if

(1) I is finite.
(2) For some *i*, M_i be Noetherian (Artinian).

Proof: is obvious.

Theorem: 1.2 Let $M = \sum_{i=1}^{n} M_i$, then *M* is Artinian(Noetherian) if and only if every M_i be Artinian(Noetherian).

Proof: "$\Rightarrow$" If *M* is Artinian then we know every submodule of *M* is also Artinian, so every M_i is Artinian.
"$\Leftarrow$" Let every M_i be Artinian. Now we difine

$$\phi : \oplus_{i=1}^{n} M_i \longrightarrow \sum_{i=1}^{n} M_i$$
$$\phi\big((m_1,\ldots,m_n)\big) = \sum_{i=1}^{n} m_i$$

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Clearly ϕ is a R-module epimorphism. Then by the isomorphism first theorem of modules $\oplus_{i=1}^{n} M_i / ker\,\phi \cong \sum_{i=1}^{n} M_i$ since $\oplus_{i=1}^{n} M_i$ is Artinian, then $\oplus_{i=1}^{n} M_i / ker\,\phi$ is Artinian and so $\sum_{i=1}^{n} M_i$ becomes Artinian. The other case is proved similarly.

Definition: 1.3 A ring $M_n(R)$ is left Nonnil-Noetherian as an *R*-module if every Nonnil Ideal as $M_n(I)$ be finitely generated, such that *I* be ideal of *R*.

Theorem: 1.4 Let *R* has 1 then *R* is left Nonnil-Noetherian ring, if and only if for some $n \geq 1$, $M_n(R)$ be a left Nonnil-Noetherian ring.

Proof: "$\Rightarrow$" $M_n(R)$ is finitely generated and generated by $c_{11}, c_{12}, \ldots, c_{nn} \in M_n(R)$. By definition 1.3 $M_n(R)$ is left Nonnil-Noetherian ring. Now we show that $M_n(R)$-module is left Nonnil-Noetherian ring. Since

$I_1 \subseteq I_2 \subseteq \ldots$

Is a ascending chain of submoduls of $M_n(R)$, $M_n(R)$ is left Nonnil-Noetherian *R*-module. then every I_j is Nonnil-Noetherian R-module, because for some $r \in I$, $rI \in M_n(R)$. Since $M_n(R)$ is left Nonnil-Noetherian as an *R*-module so there exist $k \in \mathbb{N}$ such tha $I_k = I_{k+1}$. Then $M_n(R)$ is left Nonnil-Noetherian as a ring.

"$\Leftarrow$" Let $M_n(R)$ be left Nonnil-Noetherian ring and

$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$

Is a ascending chain of left Nonnil Ideal of *R*. Then

$M_n(I_1) \subsetneq M_n(I_2) \subsetneq M_n(I_3) \subsetneq \ldots$

Is a strictly ascending chain of left Nonnil Ideal of $M_n(R)$, a contradiction. Thus *R* is left Nonnil-Noetherian ring. ■

The following theorem shows that many of the properties of Noetherian rings are also true for Nonnil-Noetherian rings.

Theorem: 1.5 The following statements are equivalent:

(1) *R* is a right Nonnil-Noetherian ring.
(2) Every non-empty set of nonnil right ideal of *R* has a maximal element at least.
(3) Every nonnil right ideal of *R* is finitely generated.

Proof: $(1) \Leftrightarrow (3)$ is obvious.

$(1) \Rightarrow (2)$ Let *S* be a non-empty set of nonnil right ideal of *R*. Assume that S has no maximal element. Since $S \neq \emptyset$, there exist $I_1 \in S$ and since *S* has no maximal element, there exist $I_2 \in S$ such that $I_1 \subseteq I_2$. Again I_2 is not a maximal element of *S*, so there is $I_3 \in S$ such that $I_1 \subseteq I_2 \subseteq I_3$. Proceed in this way we find an infinit ascending chain of nonnil right ideal of *R* which never stops.

$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$

This is a contradiction. So every non-empty set of nonnil right ideal of *R* has maximal element at least.

$(2) \Rightarrow (1)$ Let $I_1 \subseteq I_2 \subseteq \ldots$ is ascending chain of Nonnil Ideals of *R*. set $S = \{I_i | i = 1,2,3,\ldots\}$ is non-empty set of Nonnil Ideals of *R*, then *S* has a maximal element I_n. It is clear that for some $i \geq n$, $I_i = I_n$. So *R* is right Nonnil-Noetherian ring.

We then poit out that various ring-theoretic operations on commutative Noetherian rings produce again commutative Nonnil-Noetherian rings.

Lemma: 1.6 Let *R* and R' be commutative rings, and let $f : R \longrightarrow R'$ be a surjective ring homomorphism. If *R* is Nonnil-Noetherian ring, then so too is R'. In particular, if *I* is a Nonnil Ideal of *R* and *R* is Nonnil-Noetherian, then R/I is also a commutative Nonnil-Noetherian ring.

Proof: By the isomorphism first theorem for commutative ring [6,2.13], we have $R/ker\, f \cong R'$. Since it is clear that if two commutative rings are isomorphic, then one is Nonnil-Noetherian, it is enough to prove the second claim that R/I is Nonnil-Noetherian whenewer *I* is a Nonnil Ideal of *R* and *R* is Nonnil-Noetherian. This we do by [6, 2.37] and [6, 2.39], an ascending chain of ideals of R/I will have the form

$$I_1/I \subseteq I_2/I \subseteq \ldots \subseteq I_i/I \subseteq I_{i+1}/I \subseteq \ldots$$

Where

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_i \subseteq I_{i+1} \subseteq \ldots$$

Is an ascending chain of Nonnil Ideals of *R* all of wich contain *I*. Since *R* is Nonnil-Noetherian ring, there exist $k \in \mathbb{N}$ such that $I_k = I_{k+i}$ for all $i \in \mathbb{N}$, and so $I_k/I = I_{k+i}/I$ for all $i \in \mathbb{N}$, therefor R/I is Nonnil-Noetherian ring.

Theorem: 1.7 Let *R* and R' be commutative rings, and let $f : R \longrightarrow R'$ be a ring homomorphism. Assume that *R* is Nonnil-Noetherian ring and that R', when viewed as an *R*-module by maens of *f*, is finitely generated, then R' is a Nonnil-Noetherian ring.

Proof: By [6,7.22] R' is a Nonnil-Noetherian *R*-module, however, every Nonnil Ideal of R' is automatically an *R*-submodule of R', and so, since R' satisfies the ascending chain for *R*-submodules, it automatically satisfies the ascending chain for Nonnil Ideals, then R' is Nonnil-Noetherian ring.

2. EXAMPLES OF NONNIL-NOETHERIAN RINGS:

In this section, we show that the direct sum of Nonnil-Noetherian Rings is not necessarily Noetherian. Recall that a commutative ring *R* is said to be Nonnil-Noetherian if every Nonnil Ideal of *R* is said to be Nonnil-Noetherian if every Nonnil Ideal of *R* is finitely generated. It was proved in [2] and [7] that many of the properties of Noetherian rings are also true for Nonnil-Noetherian Rings.

In the example [7, 3. 1], let *S* be a Noetherian domain with quotient field *K* such that $dim(S) = 1$ and *S* has many infinitely maximal ideals. Then $T(S,K) = S \oplus K$ is a Nonnil-Noetherian Ring with krull dimension 1 which is not a Noetherian Ring. In particular $Q \oplus \mathbb{Z}$ is a Nonnil-Noetherian Ring with krull dimension 1 which is not a Noetherian Ring. (where $\mathbb{Z}$ is the set of all integer numbers with quotient field *Q*) More precisely, by proposition [2, 2.1], let $D = S \oplus K \in H$. We have $dim(D) = 1$. Since *K* is not finitely generated over *R* by proposition [2, 2. 3], we conclude that *K* is not a finitely generated *R*-module. Hence, $Nil(D)$ is not finitely generated ideal of *D* by lemma [2, 2. 2] thus, *D* is not a Noetherian Ring. By proposition [2,2.1], we have $Nil(D) = \{0\} \oplus K$ is a prime ideal of *D*. To show that $Nil(D)$ is divided: let $(0,k) \in Nil(D)$, and $(a,c) \in D \setminus Nil(D)$. Hence, $a \neq 0$, thus, $(0,k) = (a,c)(0,k/a)$ and so $Nil(D)$ is divided in *D*.

Thus $D \in H$, it is easy to see that $D/Nil(D)$ is ring-isomorphic to *R*. Since *R* is a Noetherian domain, we conclude that $D/Nil(D)$ is a Noetherian domain. Hence, *D* is a Nonnil-Noetherian Ring, by theorem [7,2.5] and that this every finite sum of left Noetherian Rings is left Noetherian. The example [7,3.3] shows that the direct sum of Nonnil-Noetherian Rings need not be Nonnil-Noetherian. Suppose that *S* be a Noetherian domain with quotient field *K* such that $dim(S) = 1$ and *S* has infinitely many maximal ideals. Then $R = T(S,K)$ is a Nonnil-Noetherian Ring, But not Noetherian. Thus $R \oplus R$ is not a Nonnil-Noetherian Ring. In fact, let *I* be a Nonnil Ideal of *R*. Then *I* is finitely generated by condition. Since *R* is not Noetherian, there exists $J \leq R$ with $J \subseteq Nil(R)$ such that *J* is not finitely generated. Hence, $I \oplus J$ is a Nonnil Ideal of $R \oplus R$, but $I \oplus J$ is not finitely generated. It follows that the direct sum of Nonnil-Noetherian Rings need not be Nonnil-Noetherian.

Now, we conclude that $R \oplus R$ is not necessarily Noetherian Ring. So the direct sum of Nonnil-Noetherian Ring needs not to be Noetherian.

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