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# TRUNCATED BILATERAL HYPERGEOMETRIC SERIES ASSOCIATED WITH NEGATIVE UNIT ARGUMENT

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**ABSTRACT**

In this paper we obtain some summation theorems for truncated bilateral generalized hypergeometric series involving negative unit argument given by

$${}_BH_B[(g_B);1+(h_B);-1]_{2P-\delta}^{2R-\theta},\ {}_{B+1}H_{B+1}[(g_B),1-\varepsilon;1+(h_B),-\varepsilon;-1]_{2P-\delta}^{2R-\theta},$$

$${}_{B+2}H_{B+2}[(g_B),1-\varpi,1-\rho;1+(h_B),-\varpi,-\rho;-1]_{2P-\delta}^{2R-\theta}$$

$${}_{B+3}H_{B+3}[(g_B),1-\upsilon,1-\varphi,1-\eta;1+(h_B),-\varphi,-\eta,-\upsilon;-1]_{2P-\delta}^{2R-\theta},$$

and $\quad {}_{B+E}H_{B+E}[(g_B),1-(\Xi_E);1+(h_B),-(\Xi_E);-1]_{2P-\delta}^{2R-\theta}$

using series iteration techniques; where $\varepsilon,\varpi,\rho,\varphi,\upsilon,\eta$ and $\Xi_E$ are the functions of parameters $g_1,g_2,\ldots,g_B,h_1,h_2,\ldots,h_B$. Applying Rainville's limit formula for certain infinite products, some non terminating bilateral hypergeometric summation theorems with negative unit argument are also deduced, in terms of Gamma functions subject to certain conditions. The results presented here are presumably new.

***Keywords and Phrases*:** Pochhammer Symbol; Gamma function; Rainville's limit formula; Unilateral and bilateral series; Truncated and non terminating series.

***2010 AMS Subject Classifications*:** 33-Special Functions, Primary 33C99; Secondary 33C20.

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## 1. INTRODUCTION:

### Rainville's Limit Formula

If $$U_n=\frac{(n+a_1)(n+a_2)\cdots(n+a_k)}{(n+b_1)(n+b_2)\cdots(n+b_\ell)} \tag{1.1}$$

then product $\prod U_n$ can only converge if $k=\ell$ and $\sum a_i=\sum b_i$. When these conditions are satisfied, we can express the infinite product in terms of Gamma functions [3,p.115(Q.No.11)]. Now limit formula for certain infinite products can be written in the following form, if

$$(1+a_1)+(1+a_2)+(1+a_3)+\cdots+(1+a_s)=(1+b_1)+(1+b_2)+(1+b_3)+\cdots+(1+b_s) \tag{1.2}$$

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and no $a_s$ or $b_s$ is a negative integer, then without any loss of absolute convergence, we have the following theorem [3,p.128(Q.1);5,pp.6-7(1.3.8);7,pp.14-15(Th.5)].

$$\prod_{n=1}^{\infty}\left\{\frac{(n+a_1)(n+a_2)\cdots(n+a_s)}{(n+b_1)(n+b_2)\cdots(n+b_s)}\right\}=\lim_{k\to\infty}\left\{\frac{(1+a_1)_k(1+a_2)_k\cdots(1+a_s)_k}{(1+b_1)_k(1+b_2)_k\cdots(1+b_s)_k}\right\} \tag{1.3}$$

$$=\frac{\Gamma(1+b_1)\Gamma(1+b_2)\cdots\Gamma(1+b_s)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_s)} \tag{1.4}$$

If condition (1.2) is not true, then product in (1.3) diverges.

**Non Terminating Bilateral Generalized Hypergeometric Series:**

(Dirichlet series or Laurent series) [9,pp.180-182(6.1.1.2,6.1.1.3,6.1.2.3); see also 2;4]
The values of parameters $a_1,a_2,\ldots,a_A,b_1,b_2,\ldots,b_A$ are adjusted in such a manner that each term in the expansion of (1.5) is well defined then

$${}_AH_A\left[\begin{matrix}(a_A)\,; \\ (b_A)\,;\end{matrix}\quad z\right]={}_AH_A\left[\begin{matrix}a_1,\,a_2,\ldots,\,a_A\;; \\ b_1,\,b_2,\ldots,\,b_A\;;\end{matrix}\quad z\right]=\sum_{r=-\infty}^{\infty}\frac{(a_1)_r(a_2)_r\cdots(a_A)_r\,z^r}{(b_1)_r(b_2)_r\cdots(b_A)_r}=\sum_{r=-\infty}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_r\,z^r}{\prod_{j=1}^{A}(b_j)_r} \tag{1.5}$$

where Pochhammer's symbol $(c)_k$ is given by $(c)_k=\prod_{i=0}^{k-1}(c+i)$.

**Truncated Bilateral Generalized Hypergeometric Series:**

In 1967, Verma [10,pp.233-234(3.4)] gave the following definition of truncated bilateral generalized hypergeometric series

$${}_AH_A\left[\begin{matrix}(a_A)\,; \\ (b_A)\,;\end{matrix}\quad z\right]_N^M={}_AH_A\left[\begin{matrix}a_1,\,a_2,\ldots,\,a_A\;; \\ b_1,\,b_2,\ldots,\,b_A\;;\end{matrix}\quad z\right]_N^M=\sum_{r=-M-1}^{N}\frac{\prod_{j=1}^{A}(a_j)_r\,z^r}{\prod_{j=1}^{A}(b_j)_r}$$

where $A,N\in\{1,2,3,\ldots\}$, $M\in\{0,1,2,3,\ldots\}$.

In 1996, R. P. Agarwal[1,p.19(13)] gave the following definition, with slightly modification in the above definition

$${}_AH_A\left[\begin{matrix}(a_A)\,; \\ (b_A)\,;\end{matrix}\quad z\right]_N^M={}_AH_A\left[\begin{matrix}a_1,\,a_2,\ldots,\,a_A\;; \\ b_1,\,b_2,\ldots,\,b_A\;;\end{matrix}\quad z\right]_N^M=\sum_{r=-M}^{N}\frac{\prod_{j=1}^{A}(a_j)_r\,z^r}{\prod_{j=1}^{A}(b_j)_r} \tag{1.6}$$

where $A,M,N\in\{1,2,3,\ldots\}$.

When $M,N\to\infty$ in (1.6), we get non terminating bilateral generalized hypergeometric series (1.5).

In our analysis, the symbol $S_r(g_1,g_2,\ldots,g_B)$ represents the sum of all possible combinations of the products of parameters taken "$r$" at a time from the set of "$B$" parameters $\{g_1,g_2,\ldots,g_B\}$ and we shall apply the modified definition (1.6) of R. P. Agarwal.

In next sections we shall discuss the applications of summation theorems of Slater, Verma, Qureshi and Quraishi with positive unit argument, for truncated bilateral hypergeometric series involving negative unit argument.

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers therefore numerator and denominator parameters are adjusted in such a way that each term of following results is completely well defined and meaningful then without any loss of convergence, we have the following theorems.

## 2. COMPANION OF FIRST THEOREM OF VERMA:

$$ {}_BH_B\left[\begin{matrix}(g_B) & ; \\ 1+(h_B) & ;\end{matrix}\; -1\right]_{2P-\delta}^{2R-\theta} = W_1 - \frac{\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)}W_2 - \frac{\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{B}(-1+g_j)}W_3 + \frac{\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{B}(1-g_j)_2}W_4 \tag{2.1} $$

where

$$ W_1 = \frac{(2)_{P-\delta}\prod_{j=1}^{B}\left\{\left(\frac{2+g_j}{2}\right)_{P-\delta}\left(\frac{3+g_j}{2}\right)_{P-\delta}\right\}}{(P-\delta)!\prod_{j=1}^{B}\left\{\left(\frac{1+h_j}{2}\right)_{P-\delta}\left(\frac{2+h_j}{2}\right)_{P-\delta}\right\}} \tag{2.2} $$

$$ W_2 = \frac{(2)_{P-1}\prod_{j=1}^{B}\left\{\left(\frac{3+g_j}{2}\right)_{P-1}\left(\frac{4+g_j}{2}\right)_{P-1}\right\}}{(P-1)!\prod_{j=1}^{B}\left\{\left(\frac{2+h_j}{2}\right)_{P-1}\left(\frac{3+h_j}{2}\right)_{P-1}\right\}} \tag{2.3} $$

$$ W_3 = \frac{(2)_{R-1}\prod_{j=1}^{B}\left\{\left(\frac{3-h_j}{2}\right)_{R-1}\left(\frac{4-h_j}{2}\right)_{R-1}\right\}}{(R-1)!\prod_{j=1}^{B}\left\{\left(\frac{2-g_j}{2}\right)_{R-1}\left(\frac{3-g_j}{2}\right)_{R-1}\right\}} \tag{2.4} $$

$$ W_4 = \frac{(2)_{R-1-\theta}\prod_{j=1}^{B}\left\{\left(\frac{4-h_j}{2}\right)_{R-1-\theta}\left(\frac{5-h_j}{2}\right)_{R-1-\theta}\right\}}{(R-1-\theta)!\prod_{j=1}^{B}\left\{\left(\frac{3-g_j}{2}\right)_{R-1-\theta}\left(\frac{4-g_j}{2}\right)_{R-1-\theta}\right\}} \tag{2.5} $$

subject to the following conditions given by

$$ S_r(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) = S_r(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) \tag{2.6} $$

$$ S_{2B+1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) \neq 0 \tag{2.7} $$

when

$$ r = 1,2,3,\ldots,(2B);\ \theta,\delta \in \{0,1\}, B \in \{1,2,3,\ldots\},\ P \in \{2,3,4,\ldots\} \text{ and } R \in \{3,4,5,\ldots\}. \tag{2.8} $$

### Proof of (2.1)

Consider the finite series identities (2.9) and (2.11):

$$\sum_{i=0}^{2R-1-\theta}\Phi(i)=\sum_{i=0}^{R-1}\Phi(2i)+\sum_{i=0}^{R-1-\theta}\Phi(2i+1) \tag{2.9}$$

which is the unification of

$$\sum_{i=0}^{2R-1}\Phi(i)=\sum_{i=0}^{R-1}\Phi(2i)+\sum_{i=0}^{R-1}\Phi(2i+1) \text{ and } \sum_{i=0}^{2R-2}\Phi(i)=\sum_{i=0}^{R-1}\Phi(2i)+\sum_{i=0}^{R-2}\Phi(2i+1) \tag{2.10}$$

where $\theta\in\{0,1\}$ and $R\in\{3,4,5,\ldots\}$.

$$\sum_{i=0}^{2P-\delta}\Phi(i)=\sum_{i=0}^{P-\delta}\Phi(2i)+\sum_{i=0}^{P-1}\Phi(2i+1) \tag{2.11}$$

which is the unification of

$$\sum_{i=0}^{2P}\Phi(i)=\sum_{i=0}^{P}\Phi(2i)+\sum_{i=0}^{P-1}\Phi(2i+1) \text{ and } \sum_{i=0}^{2P-1}\Phi(i)=\sum_{i=0}^{P-1}\Phi(2i)+\sum_{i=0}^{P-1}\Phi(2i+1) \tag{2.12}$$

where $\delta\in\{0,1\}$ and $P\in\{2,3,4,\ldots\}$.

Consider the left hand side of (2.1)

$$\Xi=\sum_{i=-2R+\theta}^{2P-\delta}\frac{(g_1)_i(g_2)_i\cdots(g_B)_i(-1)^i}{(1+h_1)_i(1+h_2)_i\cdots(1+h_B)_i}$$

$$=\sum_{i=-2R+\theta}^{-1}\frac{(g_1)_i(g_2)_i\cdots(g_B)_i(-1)^i}{(1+h_1)_i(1+h_2)_i\cdots(1+h_B)_i}+\sum_{i=0}^{2P-\delta}\frac{(g_1)_i(g_2)_i\cdots(g_B)_i(-1)^i}{(1+h_1)_i(1+h_2)_i\cdots(1+h_B)_i} \tag{2.13}$$

$$=-\frac{\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{B}(-1+g_j)}\sum_{i=0}^{2R-1-\theta}\frac{(1-h_1)_i(1-h_2)_i\cdots(1-h_B)_i(-1)^i}{(2-g_1)_i(2-g_2)_i\cdots(2-g_B)_i}+\sum_{i=0}^{2P-\delta}\frac{(g_1)_i(g_2)_i\cdots(g_B)_i(-1)^i}{(1+h_1)_i(1+h_2)_i\cdots(1+h_B)_i} \tag{2.14}$$

Now applying the series identities (2.9), (2.11) and properties of Pochhammer's symbol, after simplification we get

$$\Xi=-\frac{\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{B}(-1+g_j)}\sum_{i=0}^{R-1}\frac{\left(\frac{1-h_1}{2}\right)_i\left(\frac{2-h_1}{2}\right)_i\left(\frac{1-h_2}{2}\right)_i\left(\frac{2-h_2}{2}\right)_i\cdots\left(\frac{1-h_B}{2}\right)_i\left(\frac{2-h_B}{2}\right)_i}{\left(\frac{2-g_1}{2}\right)_i\left(\frac{3-g_1}{2}\right)_i\left(\frac{2-g_2}{2}\right)_i\left(\frac{3-g_2}{2}\right)_i\cdots\left(\frac{2-g_B}{2}\right)_i\left(\frac{3-g_B}{2}\right)_i}+$$

$$+\frac{\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{B}(1-g_j)_2}\sum_{i=0}^{R-1-\theta}\frac{\left(\frac{2-h_1}{2}\right)_i\left(\frac{3-h_1}{2}\right)_i\left(\frac{2-h_2}{2}\right)_i\left(\frac{3-h_2}{2}\right)_i\cdots\left(\frac{2-h_B}{2}\right)_i\left(\frac{3-h_B}{2}\right)_i}{\left(\frac{3-g_1}{2}\right)_i\left(\frac{4-g_1}{2}\right)_i\left(\frac{3-g_2}{2}\right)_i\left(\frac{4-g_2}{2}\right)_i\cdots\left(\frac{3-g_B}{2}\right)_i\left(\frac{4-g_B}{2}\right)_i}+$$

$$+\sum_{i=0}^{P-\delta}\frac{\left(\frac{g_1}{2}\right)_i\left(\frac{1+g_1}{2}\right)_i\left(\frac{g_2}{2}\right)_i\left(\frac{1+g_2}{2}\right)_i\cdots\left(\frac{g_B}{2}\right)_i\left(\frac{1+g_B}{2}\right)_i}{\left(\frac{1+h_1}{2}\right)_i\left(\frac{2+h_1}{2}\right)_i\left(\frac{1+h_2}{2}\right)_i\left(\frac{2+h_2}{2}\right)_i\cdots\left(\frac{1+h_B}{2}\right)_i\left(\frac{2+h_B}{2}\right)_i}-$$

$$-\frac{\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)}\sum_{i=0}^{P-1}\frac{\left(\frac{1+g_1}{2}\right)_i\left(\frac{2+g_1}{2}\right)_i\left(\frac{1+g_2}{2}\right)_i\left(\frac{2+g_2}{2}\right)_i\cdots\left(\frac{1+g_B}{2}\right)_i\left(\frac{2+g_B}{2}\right)_i}{\left(\frac{2+h_1}{2}\right)_i\left(\frac{3+h_1}{2}\right)_i\left(\frac{2+h_2}{2}\right)_i\left(\frac{3+h_2}{2}\right)_i\cdots\left(\frac{2+h_B}{2}\right)_i\left(\frac{3+h_B}{2}\right)_i} \tag{2.15}$$

Now write all finite series of (2.15) in truncated unilateral generalized hypergeometric notation, we get

$$\begin{aligned}\Xi = {}_{2B+1}F_{2B}&\left[\begin{matrix}1,\frac{g_1}{2},\frac{1+g_1}{2},\frac{g_2}{2},\frac{1+g_2}{2},\ldots,\frac{g_B}{2},\frac{1+g_B}{2} & ; \\ \frac{1+h_1}{2},\frac{2+h_1}{2},\frac{1+h_2}{2},\frac{2+h_2}{2},\ldots,\frac{1+h_B}{2},\frac{2+h_B}{2} & ;\end{matrix}\ 1\right]_{P-\delta} - \\
&-\frac{\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)}\,{}_{2B+1}F_{2B}\left[\begin{matrix}1,\frac{1+g_1}{2},\frac{2+g_1}{2},\frac{1+g_2}{2},\frac{2+g_2}{2},\ldots,\frac{1+g_B}{2},\frac{2+g_B}{2} & ; \\ \frac{2+h_1}{2},\frac{3+h_1}{2},\frac{2+h_2}{2},\frac{3+h_2}{2},\ldots,\frac{2+h_B}{2},\frac{3+h_B}{2} & ;\end{matrix}\ 1\right]_{P-1} - \\
&-\frac{\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{B}(-1+g_j)}\,{}_{2B+1}F_{2B}\left[\begin{matrix}1,\frac{1-h_1}{2},\frac{2-h_1}{2},\frac{1-h_2}{2},\frac{2-h_2}{2},\ldots,\frac{1-h_B}{2},\frac{2-h_B}{2} & ; \\ \frac{2-g_1}{2},\frac{3-g_1}{2},\frac{2-g_2}{2},\frac{3-g_2}{2},\ldots,\frac{2-g_B}{2},\frac{3-g_B}{2} & ;\end{matrix}\ 1\right]_{R-1} + \\
&+\frac{\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{B}(1-g_j)_2}\,{}_{2B+1}F_{2B}\left[\begin{matrix}1,\frac{2-h_1}{2},\frac{3-h_1}{2},\frac{2-h_2}{2},\frac{3-h_2}{2},\ldots,\frac{2-h_B}{2},\frac{3-h_B}{2} & ; \\ \frac{3-g_1}{2},\frac{4-g_1}{2},\frac{3-g_2}{2},\frac{4-g_2}{2},\ldots,\frac{3-g_B}{2},\frac{4-g_B}{2} & ;\end{matrix}\ 1\right]_{R-1-\theta}\end{aligned} \tag{2.16}$$

By the applications of Slater's theorem[8;9,pp.83-84(2.6.1.1, 2.6.1.7); see also 6,equations (3.5.1)-(3.5.3)] in the right hand side of (2.16), we obtain the right hand side of (2.1).

### Deduction of (2.1)

In $W_1, W_2, W_3, W_4$ of (2.1), (1.2) type conditions associated with (1.3) type products are satisfied, therefore we can take the limit $R, P \to \infty$

$${}_BH_B\left[\begin{matrix}(g_B) & ; \\ 1+(h_B) & ;\end{matrix}\ -1\right] = W_5 - \frac{\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{B}(1+h_j)}W_6 - \frac{\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{B}(-1+g_j)}W_7 + \frac{\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{B}(1-g_j)_2}W_8 \tag{2.17}$$

where

$$W_5 = \frac{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{1+h_j}{2}\right)\Gamma\left(\frac{2+h_j}{2}\right)\right\}}{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{2+g_j}{2}\right)\Gamma\left(\frac{3+g_j}{2}\right)\right\}} \tag{2.18}$$

$$W_6 = \frac{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{2+h_j}{2}\right)\Gamma\left(\frac{3+h_j}{2}\right)\right\}}{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{3+g_j}{2}\right)\Gamma\left(\frac{4+g_j}{2}\right)\right\}} \tag{2.19}$$

$$W_7 = \frac{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{2-g_j}{2}\right)\Gamma\left(\frac{3-g_j}{2}\right)\right\}}{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{3-h_j}{2}\right)\Gamma\left(\frac{4-h_j}{2}\right)\right\}} \tag{2.20}$$

$$W_8 = \frac{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{3-g_j}{2}\right)\Gamma\left(\frac{4-g_j}{2}\right)\right\}}{\prod_{j=1}^{B}\left\{\Gamma\left(\frac{4-h_j}{2}\right)\Gamma\left(\frac{5-h_j}{2}\right)\right\}} \tag{2.21}$$

subject to the conditions (2.6)-(2.8).

## 3. COMPANION OF SECOND THEOREM OF VERMA:

If we proceed on the same parallel lines of preceding section and apply Verma theorem [10,p.233(3.3); 1,p.19(4.12); see also 6,equations (3.2.1)-(3.2.4)], we obtain

$${}_{B+1}H_{B+1}\left[\begin{matrix}(g_B),1-\varepsilon & ; \\ 1+(h_B),-\varepsilon & ;\end{matrix}\; -1\right]_{2P-\delta}^{2R-\theta} = W_1 - \frac{(\varepsilon-1)\prod_{j=1}^{B}(g_j)}{\varepsilon\prod_{j=1}^{B}(1+h_j)}W_2 - \frac{(1+\varepsilon)\prod_{j=1}^{B}(h_j)}{\varepsilon\prod_{j=1}^{B}(-1+g_j)}W_3 + \frac{(2+\varepsilon)\prod_{j=1}^{B}(-h_j)_2}{\varepsilon\prod_{j=1}^{B}(1-g_j)_2}W_4 \tag{3.1}$$

subject to the following conditions given by

$$S_r(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B) = S_r(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B) \tag{3.2}$$

$$S_{2B}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B) \neq S_{2B}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B) \tag{3.3}$$

$$S_{2B+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B) \neq 0 \tag{3.4}$$

$$\varepsilon = \frac{-S_{2B+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)}{\{S_{2B}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B) - S_{2B}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}} \tag{3.5}$$

when

$r = 1,2,3,\ldots,(2B-1);\ \theta,\delta \in \{0,1\}, B \in \{1,2,3,\ldots\},\ P \in \{2,3,4,\ \ldots\}$ and $R \in \{3,4,5,\ldots\}$. (3.6)

and $W_1, W_2, W_3, W_4$ are given by the equations (2.2), (2.3), (2.4), (2.5) respectively.

**Deduction of (3.1)**

Similarly in $W_1, W_2, W_3, W_4$ of (3.1), (1.2) type conditions associated with (1.3) type products are satisfied, therefore we can take the limit $R, P \to \infty$

$$ {}_{B+1}H_{B+1}\left[\begin{matrix}(g_B),\ 1-\varepsilon\ ; \\ 1+(h_B),\ -\varepsilon\ ;\end{matrix}\ -1\right] = W_5 - \frac{(\varepsilon-1)\prod_{j=1}^{B}(g_j)}{\varepsilon\prod_{j=1}^{B}(1+h_j)}W_6 - \frac{(1+\varepsilon)\prod_{j=1}^{B}(h_j)}{\varepsilon\prod_{j=1}^{B}(-1+g_j)}W_7 + \frac{(2+\varepsilon)\prod_{j=1}^{B}(-h_j)_2}{\varepsilon\prod_{j=1}^{B}(1-g_j)_2}W_8 \tag{3.7} $$

subject to the conditions (3.2)-(3.6) and $W_5, W_6, W_7, W_8$ are given by the equations (2.18), (2.19), (2.20), (2.21) respectively.

**4. COMPANION OF FIRST THEOREM OF QURESHI AND QURAISHI:**

If we proceed on the same parallel lines of preceding sections and apply first theorem of authors[6,equations (3.3.1)-(3.3.6)], we obtain

$$ {}_{B+2}H_{B+2}\left[\begin{matrix}(g_B),\ 1-\varpi, 1-\rho\ ; \\ 1+(h_B),\ -\varpi, -\rho\ ;\end{matrix}\ -1\right]_{2P-\delta}^{2R-\theta} $$

$$ = W_1 - \frac{(\varpi-1)(\rho-1)\prod_{j=1}^{B}(g_j)}{\varpi\rho\prod_{j=1}^{B}(1+h_j)}W_2 - \frac{(1+\varpi)(1+\rho)\prod_{j=1}^{B}(h_j)}{\varpi\rho\prod_{j=1}^{B}(-1+g_j)}W_3 + \frac{(2+\varpi)(2+\rho)\prod_{j=1}^{B}(-h_j)_2}{\varpi\rho\prod_{j=1}^{B}(1-g_j)_2}W_4 \tag{4.1} $$

subject to the following conditions given by

$$ S_r(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) = S_r(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) \tag{4.2} $$

$$ S_{2B-1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) \neq S_{2B-1}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) \tag{4.3} $$

$$ S_{2B+1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) \neq 0 \tag{4.4} $$

$$ \varpi = \frac{-S_{2B}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) + S_{2B}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) + \sqrt{D}}{2\{S_{2B-1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) - S_{2B-1}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B)\}} \tag{4.5} $$

$$ \rho = \frac{-S_{2B}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) + S_{2B}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) - \sqrt{D}}{2\{S_{2B-1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) - S_{2B-1}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B)\}} \tag{4.6} $$

$$ \begin{aligned} D = \{&S_{2B}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) - S_{2B}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B)\}^2 - \\ &-4\{S_{2B+1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B)\}\times \\ &\times\{S_{2B-1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) - S_{2B-1}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B)\} \end{aligned} \tag{4.7} $$

 

when $r = 1,2,3,\ldots,(2B-2);\ \theta,\delta \in \{0,1\}, B, P \in \{2,3,4,\ \ldots\}$ and $R \in \{3,4,5,\ldots\}$. (4.8)

and $W_1, W_2, W_3, W_4$ are given by the equations (2.2), (2.3), (2.4), (2.5) respectively.

### Deduction of (4.1)

When $R, P \to \infty$ in (4.1), then

$$ {}_{B+2}H_{B+2}\left[\begin{matrix}(g_B),\ 1-\varpi, 1-\rho\ ; \\ 1+(h_B),\ -\varpi, -\rho\ ;\end{matrix}\ -1\right] $$

$$ = W_5 - \frac{(\varpi-1)(\rho-1)\prod_{j=1}^{B}(g_j)}{\varpi\rho\prod_{j=1}^{B}(1+h_j)}W_6 - \frac{(1+\varpi)(1+\rho)\prod_{j=1}^{B}(h_j)}{\varpi\rho\prod_{j=1}^{B}(-1+g_j)}W_7 + \frac{(2+\varpi)(2+\rho)\prod_{j=1}^{B}(-h_j)_2}{\varpi\rho\prod_{j=1}^{B}(1-g_j)_2}W_8 \tag{4.9} $$

subject to the conditions (4.2)-(4.8) and $W_5, W_6, W_7, W_8$ are given by the equations (2.18), (2.19), (2.20), (2.21) respectively.

### 5. COMPANION OF SECOND THEOREM OF QURESHI AND QURAISHI:

If we proceed on the same parallel lines of preceding sections and apply second theorem of authors[6,equations (3.4.1)-(3.4.8)], we can obtain

$$ {}_{B+3}H_{B+3}\left[\begin{matrix}(g_B),\ 1-\varphi, 1-\eta, 1-\upsilon\ ; \\ 1+(h_B),\ -\varphi, -\eta, -\upsilon\ ;\end{matrix}\ -1\right]_{2P-\delta}^{2R-\theta} = W_1 - \frac{(\varphi-1)(\eta-1)(\upsilon-1)\prod_{j=1}^{B}(g_j)}{\varphi\eta\upsilon\prod_{j=1}^{B}(1+h_j)}W_2 - $$

$$ - \frac{(1+\varphi)(1+\eta)(1+\upsilon)\prod_{j=1}^{B}(h_j)}{\varphi\eta\upsilon\prod_{j=1}^{B}(-1+g_j)}W_3 + \frac{(2+\varphi)(2+\eta)(2+\upsilon)\prod_{j=1}^{B}(-h_j)_2}{\varphi\eta\upsilon\prod_{j=1}^{B}(1-g_j)_2}W_4 \tag{5.1} $$

subject to the following conditions given by

$$ S_r(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) = S_r(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) \tag{5.2} $$

$$ S_{2B-2}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) \neq S_{2B-2}(-1+h_1, h_1, -1+h_2, h_2, \ldots, -1+h_B, h_B) \tag{5.3} $$

$$ S_{2B+1}(2, g_1, 1+g_1, g_2, 1+g_2, \ldots, g_B, 1+g_B) \neq 0 \tag{5.4} $$

where $\frac{\varphi}{2}, \frac{\eta}{2}, \frac{\upsilon}{2}$ are the roots of the following cubic equation

$$[\{S_{2B-2}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B-2}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)^3+$$
$$+\{S_{2B-1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B-1}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)^2+$$
$$+\{S_{2B}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)+$$
$$+\{S_{2B+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)\}]=0 \quad (5.5)$$

when $r=1,2,3,\ldots,(2B-3);\ \theta,\delta\in\{0,1\}, B,P\in\{2,3,4,\ldots\}$ and $R\in\{3,4,5,\ldots\}$. (5.6)

and $W_1,W_2,W_3,W_4$ are given by the equations (2.2), (2.3), (2.4), (2.5) respectively.

### Deduction of (5.1)

When $R,P\to\infty$ in (5.1), then

$$ {}_{B+3}H_{B+3}\left[\begin{matrix}(g_B),\ 1-\varphi,1-\eta,1-\upsilon\ ; \\ 1+(h_B),\ -\varphi,-\eta,-\upsilon\ ;\end{matrix}\ -1\right]=W_5-\frac{(\varphi-1)(\eta-1)(\upsilon-1)\prod_{j=1}^{B}(g_j)}{\varphi\eta\upsilon\prod_{j=1}^{B}(1+h_j)}W_6-$$
$$-\frac{(1+\varphi)(1+\eta)(1+\upsilon)\prod_{j=1}^{B}(h_j)}{\varphi\eta\upsilon\prod_{j=1}^{B}(-1+g_j)}W_7+\frac{(2+\varphi)(2+\eta)(2+\upsilon)\prod_{j=1}^{B}(-h_j)_2}{\varphi\eta\upsilon\prod_{j=1}^{B}(1-g_j)_2}W_8 \quad (5.7)$$

subject to the conditions (5.2)-(5.6) and $W_5,W_6,W_7,W_8$ are given by the equations (2.18), (2.19), (2.20), (2.21) respectively.

## 6. COMPANION OF THIRD THEOREM OF QURESHI AND QURAISHI:

If we apply third theorem of authors[6,equations (3.6.1)-(3.6.4)] and proceed on the same parallel lines of preceding sections, we can obtain

$$ {}_{B+E}H_{B+E}\left[\begin{matrix}(g_B),\ 1-\Xi_1,1-\Xi_2,\ldots,1-\Xi_E\ ; \\ 1+(h_B),\ -\Xi_1,-\Xi_2,\ldots,-\Xi_E\ ;\end{matrix}\ -1\right]_{2P-\delta}^{2R-\theta}$$
$$=W_1-\frac{\prod_{j=1}^{E}(\Xi_j-1)\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(1+h_j)}W_2-\frac{\prod_{j=1}^{E}(1+\Xi_j)\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(-1+g_j)}W_3+\frac{\prod_{j=1}^{E}(2+\Xi_j)\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(1-g_j)_2}W_4 \quad (6.1)$$

subject to the following conditions given by

$$S_r(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)=S_r(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B) \quad (6.2)$$

$$S_{2B-E+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)\neq S_{2B-E+1}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B) \quad (6.3)$$

$$S_{2B+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)\neq 0 \quad (6.4)$$

where $\frac{\Xi_1}{2},\frac{\Xi_2}{2},\cdots,\frac{\Xi_E}{2}$ are the roots of the following equation

 

$$[\{S_{2B-E+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B-E+1}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)^E+$$
$$\{S_{2B-E+2}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B-E+2}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)^{E-1}+$$
$$+\cdots+\{S_{2B-1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B-1}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)^2+$$
$$+\{S_{2B}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)-S_{2B}(-1+h_1,h_1,-1+h_2,h_2,\ldots,-1+h_B,h_B)\}(2m)+$$
$$+\{S_{2B+1}(2,g_1,1+g_1,g_2,1+g_2,\ldots,g_B,1+g_B)\}]=0 \tag{6.5}$$

when $r=1,2,3,\ldots,(2B-E)$; $\theta,\delta\in\{0,1\}, B\in\{1,2,3,\ldots\}, P\in\{2,3,4,\ldots\}$, $R\in\{3,4,5,\ldots\}, E<2B$ (6.6)
and $W_1,W_2,W_3,W_4$ are given by the equations (2.2), (2.3), (2.4), (2.5) respectively.

## Deduction of (6.1)

When $R,P\to\infty$ in (6.1), then

$$_{B+E}H_{B+E}\left[\begin{matrix}(g_B),1-\Xi_1,1-\Xi_2,\ldots,1-\Xi_E & ; \\ 1+(h_B),-\Xi_1,-\Xi_2,\ldots,-\Xi_E & ;\end{matrix}\ -1\right]$$

$$=W_5-\frac{\prod_{j=1}^{E}(\Xi_j-1)\prod_{j=1}^{B}(g_j)}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(1+h_j)}W_6-\frac{\prod_{j=1}^{E}(1+\Xi_j)\prod_{j=1}^{B}(h_j)}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(-1+g_j)}W_7+\frac{\prod_{j=1}^{E}(2+\Xi_j)\prod_{j=1}^{B}(-h_j)_2}{\prod_{j=1}^{E}(\Xi_j)\prod_{j=1}^{B}(1-g_j)_2}W_8 \tag{6.7}$$

subject to the conditions (6.2)-(6.6) and $W_5,W_6,W_7,W_8$ are given by the equations (2.18), (2.19), (2.20), (2.21) respectively.

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