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SOME FIXED POINT THEOREM SATISFING GENERAL CONTRACTIVE CONDITION OF INTEGRAL TYPE

[1]Animesh Gupta*, [2]Dilip Jaiswal and [3]S. S. Rajput

[1]Department of Mathematics, Govt. Benazir Science & Commerce College, Bhopal, (M.P.), India
[2]Department of Mathematics, Moti Lal Vigyan Mahavidhyalaya (MVM), Bhopal, (M.P.), India
[3]Head of Department,(Mathematics), Govt. P.G. Collage, Gadarwara, (M.P.), India

E-mail: animeshgupta10@gmail.com

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ABSTRACT

In this paper, we establish a fixed point theorem for a pair of self maps satisfying a general contractive condition of integral type. This theorem extends and generalizes some early results of Boikanyo [4]. And Jaggi and Doss [12]

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1. INTRODUCTION:

The first well known result on fixed points for contractive map was Banach fixed point theorem, published in 1922, [2]. In general setting of complete metric space, Smart [18], presented the following result.

Theorem: 1.1 Let (X, d) be a complete metric space, $\alpha \in [0,1)$ and let $T: X \to X$ be a map such that for each $x, y \in X$,

$$d(Tx, Ty) \leq \alpha\ d(x, y)$$

Then, T has a unique fixed point $z \in X$ such that for each $x \in X, \lim_{n\to\infty} T^n x = z$.

After the classical result, many theorems dealing with the maps satisfying various types of contractive inequalities have been established see in [4], [6], [20].

In 2002, Branciari [3], obtained the following theorem.

Theorem: 1.2 Let (X, d) be a complete metric space, $\alpha \in (0,1)$ and let $T: X \to X$, be a mapping such that for each $x, y \in X$,

$$\int_0^{d(Tx,Ty)} \xi(t)\ dt \leq \int_0^{d(x,y)} \xi(t)\ dt$$

Where $\xi : [0, +\infty] \to [0, +\infty]$ is a legesgue integrable mapping which is summable on each compact subset of $[0, +\infty]$, non negative, and such that, $\forall\ \varepsilon > 0,\ \int_0^{\varepsilon} \xi(t)\ dt > 0$ Then, T

has unique fixed point $z \in X$ such that for each $x \in X$, $T^n x \to z$ as $n \to \infty$.

It is mentioned in [7], that theorem 1.2 could be extended to more general contractive conditions, for example, in [15], Rhoades established that Theorem1.2, hold s

If we replace $d(x, y)$ by $\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ Other work in this direction include [1, 8, 19, 20], In [17], Suzuki proved that Theorem 1.2 of Branciari is a particular case of the famous Meir-Keeler fixed point Theorem [14], More precisely, he proved that under hypotheses of Theorem 1.2, T is a MKC, that is for every $\delta > 0$ such that,

***Corresponding author:** [1]Animesh Gupta*,*E-mail:* animeshgupta10@gmail.com*

$$d(x,y) < \varepsilon + \delta \implies d(Tx,Ty) < \varepsilon$$

And then T has a unique fixed point.

In this paper, we obtain an extension of Theorem 1.2, through rational expression.

Our obtained result extends and improves the result of Jaggi and Dass [12]. Other results on fixed point theorems through rational expression can be found in [7, 9, 11].

2. MAIN RESULT:

Theorem: 2.1 Let (X, d) be a complete metric space and let $T: X \to X$ be a given mapping. We denote

$$m(x,y) = \frac{d(x,Tx)d(y,Ty)}{d(x,Tx) + d(y,Ty) + d(x,y)} \qquad \forall\ x,y \in X \tag{1}$$

We assume that for each $x,\ y \in X$,

$$\int_0^{d(Tx,Ty)} \xi(t)\ dt \le \alpha \int_0^{m(x,y)} \xi(t)\ dt + \beta \int_0^{d(x,y)} \xi(t)\ dt \tag{2}$$

Where $\alpha > 0,\ \beta > 0,\ 0 < \alpha + \beta < 1$ and $\xi : (0,1) \to (0,1)$ is a lebesgue- integrable mapping which is summable on each compact subset of $(0, \infty)$, nonnegative and such that

$$\int_0^{\varepsilon} \xi(t)\ dt > 0, \forall\ \varepsilon > 0 \tag{3}$$

Then T has unique fixed point $z \in X$ such that for each $x \in X$, $T^n x \to z$ as $n \to +\infty$.

Proof: Let $x \in X$ and we define the sequence $\{x_n\}$ in X, defined as, $x_{n+1} = Tx_n$ for each integer $n \ge 1$. from (2), we claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

To prove (4), we required to show that,

$$\int_0^{d(x_{n+1},x_{n+2})} \xi(t)\ dt \le r^{n+1} \int_0^{d(x_0,x_1)} \xi(t)\ dt$$

Where, $r = (\alpha + \beta)$

By Using (2),

$$\int_0^{d(Tx_n,Tx_{n+1})} \xi(t)\ dt \le \alpha \int_0^{m(x_n,x_{n+1})} \xi(t)\ dt + \beta \int_0^{d(x_n,x_{n+1})} \xi(t)\ dt$$

By Using (1)

$$m(x_n, x_{n+1}) = \frac{d(x_n,x_{n+1})d(x_{n+2},x_{n+1})}{d(x_n,x_{n+1}) + d(x_{n+2},x_{n+1}) + d(x_n,x_{n+1})}$$

$$m(x_n, x_{n+1}) = d(x_n, x_{n+1})$$

Which implies,

$$\int_0^{d(x_{n+1},x_{n+2})} \xi(t)\ dt \le \alpha \int_0^{d(x_n,x_{n+1})} \xi(t)\ dt + \beta \int_0^{d(x_n,x_{n+1})} \xi(t)\ dt$$

$$\int_0^{d(x_{n+1},x_{n+2})} \xi(t)\ dt \le (\alpha + \beta) \int_0^{d(x_n,x_{n+1})} \xi(t)\ dt$$

In general we can write,

$$\int_0^{d(x_{n+1},x_{n+2})} \xi(t)\ dt \le (\alpha + \beta)^{n+1} \int_0^{d(x_0,x_1)} \xi(t)\ dt$$

Since $(\alpha + \beta) < 1$ and, it follows that,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 \tag{4}$$

Now we show that, $\{x_n\}$ is a Cauchy sequence in X. Suppose that it is not then there exists an $\varepsilon > 0$ such that for each $p \in N$ there are $m(p)$ and $n(p)$ in N, with $m(p) > n(p) > p$, such that,

$$d(Tx_{m(p)}, Tx_{n(p)}) \geq \varepsilon, \; d(Tx_{m(p)-1}, Tx_{n(p)}) < \varepsilon$$

Hence

$$\varepsilon \leq d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)-1}) + d(Tx_{m(p)-1}, Tx_{n(p)})$$

$$d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)-1}) + \varepsilon$$

Using (4) and taking $p \to +\infty$, we get

$$d(Tx_{m(p)}, Tx_{n(p)}) \to \varepsilon^+ \tag{5}$$

This implies that there exists $k \in N$ such that $p > k \implies d(Tx_{m(p)+1}, Tx_{n(p)+1}) < \varepsilon$

In fact if there exists a subsequence $(p_k) \in N$, $p_k > k$, $d(Tx_{m(p)+1}, Tx_{n(p)+1}) \geq \varepsilon$

We obtain from (2)

$$\int_0^{\varepsilon} \xi(t)\, dt \leq \alpha \int_0^{m(x_{m(p_k)}, x_{n(p_k)})} \xi(t)\, dt + \beta \int_0^{d(x_{m(p_k)}, x_{n(p_k)})} \xi(t)\, dt \tag{6}$$

On other hand we have,

$$m(x_{m(p_k)}, x_{n(p_k)}) = \frac{d(x_{m(p_k)}, Tx_{m(p_k)})\, d(x_{n(p_k)}, Tx_{n(p_k)})}{d(x_{m(p_k)}, Tx_{m(p_k)}) + d(x_{n(p_k)}, Tx_{n(p_k)}) + d(x_{m(p_k)}, x_{n(p_k)})}$$

$$m(x_{m(p_k)}, x_{n(p_k)}) \to 0 \text{ as } k \to \infty,$$

Tacking $k \to +\infty$ in (6) we get

$$\int_0^{\varepsilon} \xi(t)\, dt \leq \beta \int_0^{\varepsilon} \xi(t)\, dt$$

Which is contradiction being $\beta \in (0,1)$ and the integral being positive. Let us prove now that there is $\sigma_\varepsilon \in (0, \varepsilon)$, $p_\varepsilon \in N$ such that $p > p_\varepsilon \Rightarrow d(Tx_{m(p)+1}, Tx_{n(p)+1}) < \varepsilon - \sigma_\varepsilon$

If it is not true, then there exists a subsequence$(p_\varepsilon) \in N$ such that

$$d(Tx_{m(p)+1}, Tx_{n(p)+1}) \to \varepsilon^- \text{ as } k \to +\infty.$$

By (2), we obtain

$$\int_0^{d(Tx_{m(p_k)-1}, x_{n(p_k)-1})} \xi(t)\, dt \leq \alpha \int_0^{m(x_{m(p_k)}, x_{n(p_k)})} \xi(t)\, dt + \beta \int_0^{d(x_{m(p_k)}, x_{n(p_k)})} \xi(t)\, dt$$

On taking as $k \to +\infty$, we get

$$\int_0^{\varepsilon} \xi(t)\, dt \leq \beta \int_0^{\varepsilon} \xi(t)\, dt$$

Which contradiction, since $\beta \in (0,1)$. now we can deduce the Cauchy character of$\{Tx_n\}$. In fact for each natural number$p > p_\varepsilon$, we have

$$\varepsilon \leq d(Tx_{m(p)}, Tx_{n(p)}) < d(Tx_{m(p)}, Tx_{m(p)+1}) + d(Tx_{m(p)+1}, Tx_{n(p)+1}) + d(Tx_{n(p)+1}, Tx_{n(p)})$$

$$d(Tx_{m(p)}, Tx_{n(p)}) \to \varepsilon - \sigma_\varepsilon \text{ as } p \to +\infty$$

Thus, $\varepsilon \leq \varepsilon - \sigma_\varepsilon$

Which is contradiction. We conclude that $\{Tx_n\}$ is Cauchy.

By the completeness of X, there is $z \in X$ such that $Tx_n \to z$ as $n \to \infty$. We shall show that $Tz = z$. Suppose by contradiction that $d(z, Tz) > 0$. We have

$$0 \leq d(z, Tz) \leq d(z, Tx_{n+1}) + d(Tz, Tx_{n+1}) \tag{7}$$

First, let us prove that $d(Tx_{n+1}, z)$ is convergent sequence and it is converges to zero, then $d(Tx_{n+1}, Tz)$ is bounded sequence. Assume that there exists a subsequnce

$d(Tx_{n_k+1}, Tz)$ such that $d(Tx_{n_k+1}, Tz) \to l \in (0, +\infty)$ as $k \to +\infty$, we obtain

$$\int_0^{d(Tx_{n_k+1},Tz)} \xi(t)\ dt \leq \alpha \int_0^{m(x_{n_k+1},z)} \xi(t)\ dt + \beta \int_0^{d(x_{n_k+1},z)} \xi(t)\ dt$$

from (1) and as $k \to +\infty$

$$\int_0^{l} \xi(t)\ dt \leq \beta \int_0^{d(z,Tz)} \xi(t)\ dt$$

Which is contradiction the hypothsis,

Since $0 < \alpha < 1$, then $d(Tx_{n+1}, Tz) \to 0$ as $n \to +\infty$

Now letting $n \to +\infty$ in (7) we get

$$0 < d(z, Tz) \leq d(z, Tx_{n+1}) + d(Tz, Tx_{n+1}) \to 0$$

We deduce that z is a fixed point of T.

Uniqueness:

Suppose that there is another fixed point of T say w, different from z in X, then from(2) we have

$$\int_0^{d(Tw,Tz)} \xi(t)\ dt \leq \alpha \int_0^{m(w,z)} \xi(t)\ dt + \beta \int_0^{d(w,z)} \xi(t)\ dt$$

$$\int_0^{d(Tw,Tz)} \xi(t)\ dt \leq \beta \int_0^{d(w,z)} \xi(t)\ dt$$

which contradiction of hypothis. so T has unique fixed point in X

REMARK:

By taking $\xi(t) = 1$ in the above theorem then we get the result of Jaggi and Das [12].

REFERENCES:

[1] A. Aliouche. "A common fixed point theorem for weakly compatible mappings in symmetric Spaces satisfying a contractive condition of integral type. Journal of Mathematical Analysis and Applications, vol. 322, no. 2 (2006), 796-802.

[2] S. Banach, 1922. "Sur les operations dans les ensembles abstraits et leur application aux equations integrals", Fund. Math. 3: 133 – 181 (in French)

[3] A. Branciari. "A fixed point theorem for mappings satisfying a general contractive condition of integral type". Int. J. Math. Math. Sci. 29 (2002), no. 9, 531-536.

[4] O. A. Boikanyo, 2007." Some fixed point theorems for mappings satisfying a general contractive condition of integral types", Far East J. Math. Sci., (FJMS) 26 (1)219 – 230.

[5] R. Caccioppoli, 1930. "Un teorema generale sull' esistenza di elementi uniti in una transformazione funzionale, "Rend. Acad. Dei Lincei 11: 794 – 799 (in Italian).

[6] S.K. Chatterjea, 1972. "Fixed point theorems", C.R. Acad. Balgare Sci. 25: 727 – 730.

[7] H. Chatterjee. “On generalization of Banach Contraction Principle.”, Indian J. Pure appl. Math., 10 (1979), 400-403.

[8] A. Djoudi and A. Aliouche. “Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type.” Journal of Mathematical Analysis and Applications, vol. 329, no. 1 (2007), 31-45.

[9] P. Das. “Some results on fixed points in complete metric spaces.” Indian J. Pure appl. Math., 13(3) (1982), 303-308.

[10] G. E. Hardy and T. D. Rogers, 1973. “A generalization of a fixed point theorem of Reich,” Bull. Cand. Math. Soc. 16:201 –206.

[11] G. S. Jeong and B. E. Rhoades. “Some remarks for improving fixed point theorems for more than two maps.” Indian Journal of Pure and Applied Mathematics, 28(9) (1997), 1177-1196.

[12] D. S. Jaggi and B.K. Dass, “Bull. Calcutta Math. Soc. 72(1980), 261-262.

[13] R. Kannan, “Some results on fixed points”, Bull. Calcutta Math. Soc. 60 (1968), 71 – 76.

[14] A. Meir and E. Keeler. “A theorem on contraction mappings. Journal of Mathematical Analysis and Applications”, vol. 28, no. 2 (1969), 326-329.

[15] B. E. Rhoades.” Two fixed point theorems for mappings satisfying a general contractive condition of integral type.” IJMMS. 63 (2003), 4007-4013.

[16] B. E. Rhoades, “A comparison of various definitions of contractive mappings”, Trans. Amer. Math. Soc. 226 (1977), 257 – 290.

[17] T. Suzuki. “ Meir-Keeler Contractions of Integral Type Are Still Meir-Keeler Contractions “ International Journal of Mathematics and Mathematical Sciences, Volume 2007, Article ID 39281, 6 pages.

[18] D. R. Smart, Fixed point theorems, Cambridge University Press, London, 1974.

[19] D. Turkoglu and I. Altun.” A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation”. Boletn de la Sociedad Matematica Mexicana, 13 (1) (2007), 195-205.

[20] P. Vijayaraju, B. E. Rhoades and R. Mohanraj. “A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type”. International Journal of Mathematics and Mathematical Sciences, vol. 2005, no. 15, 2359-2364.

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