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UNIQUENESS AND VALUE-SHARING OF MEROMORPHIC FUNCTIONS WITH MULTIPLICITY

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ABSTRACT

In this paper, we introduce the notion of multiplicity and study with a weighted sharing method the uniqueness problems of meromorphic functions sharing one value and obtain some results which extend and generalises the theorems given by Xiao -Yu Zhang and Weichuan Lin [9].

1. INTRODUCTION:

In this paper, we assume all the functions are non-constant meromorphic functions in the complex plane ***C***. We shall use the following standard notations of value distribution theory: T(r, f), m(r, f), N(r, f), $\overline{N}$(r, f), S(r, f)...

We denote by S(r, f) any function satisfying S(r, f) = o{T(r, f)}, as r -> +∞, possibly outside of a set with finite Lebesgue measure in R.

Let a be a finite complex number, and k be a positive integer. We denote by $N_{k)}\left(r, \frac{1}{(f-a)}\right)$ the counting function for the zeros of f(z) - a with multiplicity $\leq k$ and by $\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)$ the corresponding one for which multiplicity is not counted. Let $N_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the counting function for the zeros of f(z)-a with multiplicity $\geq$ k, and $\overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$ be the corresponding one for which multiplicity is not counted. Moreover,we set
$N_k\left(r, \frac{1}{(f-a)}\right) = \overline{N}\left(r, \frac{1}{(f-a)}\right) + \overline{N}_{(2}\left(r, \frac{1}{(f-a)}\right) + \cdots + \overline{N}_{(k}\left(r, \frac{1}{(f-a)}\right)$.

In the same way, we can define N_k(r, f).

If for some a ∈ C U { ∞} the zeros of f - a and g - a coincide in locations and multiplicity we say that f and g share the value a CM.

For the sake of simplicity, we also use the notations $C^j{}_k=\binom{k}{j}$ and m*:=X_μm, where $X_\mu=\begin{cases}0, & \mu=0,\\ 1, & \mu\neq 0.\end{cases}$

Nevanlinna's five-value theorem states that any five distinct values are enough to identify arbitrary two nonconstant meromorphic functions which share all those values. Further in general, this number 'five' cannot be replaced by any smaller number, if multiplicities are not taken into account at all. Recently, corresponding to one famous question of Hayman [5], Fang and Hua[3], Yang and Hua [11] obtained the following unicity theorem.

Theorem: A *Let f(z) and g(z) be two nonconstant entire functions, n ≥ 6 be a positive integer. If $f^n(z)\, f'(z)$ and $g^n(z)$ $g'(z)$ share 1 CM, then either f(z) = c_1e^{cz}, g(z) =c_2e^{-cz}, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2$= -1, or f(z) = t g(z) for a constant t such that t^{n+1}= 1.*

Recently, notice that $f^n(z)\, f'(z) =\frac{1}{n+1}(f^{n+1})'$, Fang [2] considered k^{th} derivative instead of 1^{st} derivative and proved the following theorems.

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Theorem: B *Let f(z) and g(z) be two nonconstant entire functions and let n, k be two positive integers with n > 2k + 4. If* $[f^n(z)]^{(k)}$ *and* $[g^n(z)]^{(k)}$ *share 1 CM, then either f(z) =* c_1e^{cz}*, g(z) =*c_2e^{-cz}*, where* c_1, c_2 *and c are three constants satisfying* $(-1)^k\ (c_1c_2)^n(nc)^{2k}$*= 1, or f(z) = t g(z) for a constant t such that* t^n*= 1.*

Theorem: C *Let f(z) and g(z) be two nonconstant entire functions and let n, k be two positive integers with n ≥ 2k+8. If* $[f^n(z)(f(z)-1)]^{(k)}$ *and* $[g^n(z)(g(z)-1)]^{(k)}$ *share 1 CM, then f (z) ≡ g (z).*

In 2008, X. Y. Zhang and W. C. Lin [9] extended Theorems B and C for some general differential polynomials such as $[f^n(f^m-1)]^{(k)}$ or $[f^n(f-1)^m]^{(k)}$ and proved the following theorem.

Theorem: D *Let f(z) and g(z) be two nonconstant entire functions and let n, m and k be three positive integers with n > 2k + m* + 4, and λ, μ be constants such that* $|\lambda|+|\mu|\neq 0$*. If* $[f^n(z)(\mu f^m(z)+\lambda)]^{(k)}$ *and* $[g^n(z)(\mu g^m(z)+\lambda)](k)$ *share 1 CM, then*

(i) when $\lambda\mu\neq 0$*, f(z)≡ g(z);*

(ii)when λμ = 0, either f(z) = tg(z), where t is a constant satisfying $t^{n+m^*}=1$*, or f(z) =* c_1e^{cz}*, g(z) =*c_2e^{-cz}*, where* c_1, c_2 *and c are three constants satisfying*

$(-1)^k\ \lambda^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k}$*= 1 or* $(-1)^k\ \mu^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k}$*= 1.*

Naturally, one may ask the following question: Is it really possible to relax in any way the nature of sharing 1 in the above result? The purpose of this paper is to extend above results to meromorphic functions and to discuss this problem. To do this, we introduce the noition of multiplicity and use the idea of weighted sharing introduced by I. Lahari [8], we will study the problem that $[f^n(z)(\mu f^m(z)+\lambda)]^{(k)}$ and $[g^n(z)(\mu g^m(z)+\lambda)]^{(k)}$ sharing one value with the weighted sharing method and obtain the following theorems, which improve and extend the above theorem.

Theorem :1.1 *Let f(z) and g(z) be two nonconstant meromorphic functions, and let n(≥ 1), m(≥ 1), k(≥ 1) and l (≥ 0) be four integers and λ, μ be two constants such that*$|\lambda|+|\mu|\neq 0$ *. Suppose* $[f^n(z)(\mu f^m(z)+\lambda)]^{(k)}$ *and* $[g^n(z)(\mu g^m(z)+\lambda)]^{(k)}$ *share (1, l). If l ≥ 2 and s(n + m*) > 3k + 4m* + 8 or if l = 1 and s(n + m*) >5k + 5m* + 11 or l = 0 and s(n + m*) > 9k + 7m* + 14, then*

(i) when $\lambda\mu\neq 0$*, if m ≥ 2 and* $\delta(\infty,f)>\frac{3}{n+m*}$ *then f(z) ≡g(z); If m = 1 and* $\theta(\infty,f)>\frac{3}{n+1}$*, then f(z) ≡ g(z).*

(ii) when λμ = 0, if f(z) ≠ ∞ and if g(z) ≠ ∞ , then either f(z) ≡ tg(z), where t is a constant satisfying $t^{n+m^*}=1$*, or f(z) =* c_1e^{cz}*, g(z) =*c_2e^{-cz}*, where* c_1, c_2 *and c are three constants satisfying* $(-1)^k\ \lambda^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k}$*= 1 or* $(-1)^k\ \mu^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k}$*= 1.*

Theorem: 1.2 *Let f(z) and g(z) be two nonconstant meromorphic functions, and let n(≥ 1), m(≥ 0), k(≥ 1) and l (≥ 0) be four integers. Suppose that* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share (1, l),*

(i) when m = 0, if f(z) ≠ ∞ and if g(z) ≠ ∞. If l ≥ 2 and sn > 3k + 8 or if l = 1 and sn > 5k + 11 or l = 0 and sn > 9k + 14, then either f(z) = c_1e^{cz}*, g(z) =*c_2e^{-cz}*, where* c_1, c_2 *and c are three constants satisfying* $(-1)^k\ (c_1c_2)^n(nc)^{2k}$*= 1, or f(z) = t g(z) for a constant t such that* t^n*= 1;*

(ii) when m = 1, if l ≥ 2 and s(n+1) > 3k+12 or if l = 1 and s(n+1) > 5k+16 or l = 0 and s(n + 1) > 9k + 21 and $\theta(\infty,f)>\frac{2}{n}$*, then either* $[f^n(f-1)^m]^{(k)}\ [g^n(g-1)^m]^{(k)}$ *≡1 or f≡ g;*

(iii) when (m ≥ 2), if l ≥ 2 and s(n + m) > 3k + 2m* + 2m + 8 or if l = 1 and s(n+m*) > 5k+2m*+3m+11 or l = 0 and s(n+m*) > 9k+2m*+5m+14 then either* $[f^n(f-1)^m]^{(k)}\ [g^n(g-1)^m]^{(k)}$ *≡1 or f≡ g or f and g satisfy the algebraic equation R(f, g) = 0, where R(x, y) =*$x^n(x-1)^m-y^n(y-1)^m$*.*
The possibility $[f^n(f-1)^m]^{(k)}\ [g^n(g-1)^m]^{(k)}$ *≡1 does not arise for k = 1.*

Remark: 1.1 In Theorem 1.1 giving specific values for s for *l* ≥ 2, we get the following interesting cases:

(i) If s = 1, then n > 3k + 3m* + 8.
(ii)If s = 2, then n > $\frac{3k+4m^*+8}{2}$ -m^*.
(iii)If s = 3, then n >$\frac{3k+4m^*+8}{3}-m^*$.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n. This holds for all cases in Theorem 1.1 and Theorem 1.2.

We use the following definitions to prove our main results.

Definition: 1[8] Let k be a non-negative integer or infinity. For a ∈ C U $\{\infty\}$ we denote by E_k (a, f) the set of all a-points of f where an a-point of multiplicity m is counted m times if m ≤ k and k + 1 times if m > k.

Definition: 2[8] Let k be a non-negative integer or infinity. If for a ∈ C U $\{\infty\}$ such that E_k (a, f) = E_k (a, g), then we say that f and g share the value a with weight k.

We write f, g sharing (a, k) to mean that f, g share the value a with weight k. Clearly, if f; g share (a, k) then f; g share (a, p) for all integers p, 0 ≤p ≤ k. Also we note that f, g share a value a CM if and only if f, g share (a, ∞).

2. MAIN PROPOSITION AND SOME LEMMAS:

For the proof of our results, we discuss the following main propositions.

Proposition: 1 Let f(z) be a transcendental meromorphic function, and let n, k, and m be three positive integers with n ≥ k+3, and *and* λ, μ *are complex numbers such that* $|\lambda| + |\mu| \neq 0$. *Then* $[f^n(z)(\mu f^m(z) + \lambda)]^{(k)}=1$ has infinitely many solutions.

Proposition: 2 Let f(z) be a transcendental meromorphic function, and let n, k, and m be three positive integers with n ≥ k + 3 ≥ 4. Then $[f^n(z)(f(z)-1)^m]^{(k)} = 1$ has infinitely many solutions.

In order to prove the above proposition, we require the following results.

Lemma: 2.1 ([10]) *Let f(z) be a nonconstant meromorphic function and let* $a_n(z)(\neq 0), a_{n-1}(z), \ldots, a_0(z)$ *be meromorphic functions such that*

T(r, a_i*) = S(r, f), (i = 0, 1, ,..., n). Then T(,* $a_nf^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0$*) = nT(r, f) + S(r, f).*

Lemma: 2.2 ([12]) *Let f(z) be a transcendental meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^k-c}\right) + N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f)$$

$$\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^k-c}\right) - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f),$$

where $N_0\left(r,\frac{1}{f^{k+1}}\right)$ *is the counting function which only counts those points such that*

$f^{(k+1)} = 0$ *but* $f(f^{(k)} - c) \neq 0$.

Proof of proposition: 1 By Lemma 2.1 and Lemma 2.2, we have

$$(n + m^*)T(r, f) = T(r, f^n(\mu f^m + \lambda)) + S(r, f),$$

$$\leq \overline{N}(r, f^n(\mu f^m + \lambda)) + N_{k+1}\left(r,\frac{1}{f^n(\mu f^m + \lambda)}\right) + N\left(r,\frac{1}{[f^n(\mu f^m + \lambda)]^{(k)}-1}\right) + S(r,f)$$

$$\leq (k+2+m^*)T(r, f) + N\left(r,\frac{1}{[f^n(\mu f^m + \lambda)]^{(k)}-1}\right) + S(r,f),$$

thus, we get

$$(n - k - 2)T(r, f) \leq N\left(r,\frac{1}{[f^n(\mu f^m + \lambda)]^{(k)}-1}\right) + S(r,f). \tag{2.1}$$

Hence, we deduce by (2.1) and n ≥ k+3 that $f^n(z)(\mu f^m(z) + \lambda)$=1 has infinitely many solutions.

This completes the proof of proposition 1.

Proof of proposition: 2 proceeding as in the proof of proposition 1, we can easily obtain the conclusion of proposition 2.

Next, for the proof of our theorems, we still need the following Lemmas.

Lemma: 2.3 ([12]) *Let f(z) be a transcendental meromorphic function, and let* $a_1(z), a_2(z)$ *be two meromorphic functions functions such that*
$T(r, a_i) = S(r, f)$ $\quad i = 1, 2$ *and* $a_1 \neq a_2$. *Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r, f).$$

Lemma: 2.4 ([13]) Let f(z) be a nonconstant meromorphic function, and let k, p be two positive integers then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \text{ Clearly } \overline{N}\left(r, \frac{1}{f^{(k)}}\right) = N_1\left(r, \frac{1}{f^{(k)}}\right).$$

Lemma: 2.5 *([4]) Let f(z) be a non-constant entire function, and let* $k(\geq 2)$ *be a positive integer. If* $f(z) f^{(k)}(z) \neq 0$, *then* $f(z) = e^{az+b}$, *where* $a \neq 0$, *b are constants.*

Lemma: 2.6 *([7]) Let f(z) and g(z) be two non constant meromorphic functions,* $k\ (\geq 1)$ *and* $l\ (\geq 0)$ *be integers. Suppose that* $f^{(k)}(z)$ *and* $g^{(k)}(z)$ *share (1, l). If one of the following conditions holds, then either* $f^{(k)}(z)\, g^{(k)}(z) \equiv 1$ *or* $f(z) \equiv g(z)$.

(1) $l \geq 2$ *and* $(k+2)\,\theta(\infty, f) + 2\,\theta(\infty, g) + \theta(0, f) + \theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7$;

(2) $l = 1$ *and* $(2k+3)\,\theta(\infty, f) + 2\,\theta(\infty, g) + \theta(0, f) + \theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9$;

(3) $l = 0$ *and* $(2k+3)\,\theta(\infty, f) + (2k+4)\,\theta(\infty, g) + \theta(0, f) + \theta(0, g) + 2\,\delta_{k+1}(0, f) + 3\,\delta_{k+1}(0, g) > 4k + 13$.

Lemma: 2.7 *Let f and g be two non-constant meromorphic functions, and let* $n(\geq 1)$, $k(\geq 1)$ *and* $m(\geq 1)$ *be three integers. Then* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)} \neq 1$, *for* $k = 1$ *and* $n \geq m + 3$.

Proof: If possible, let $[f^n(f-1)^m]^{(k)}\,[g^n(g-1)^m]^{(k)} \equiv 1$, for k = 1. That is,

$f^{n-1}(f-1)^{m-1}(af-b)f' g^{n-1}(g-1)^{m-1}(ag-b)g' \equiv 1$,
where a = n + m and b = n.

Let z_0 be a 1-point of f with multiplicity p (≥ 1), and a pole of g with multiplicity q (≥ 1) such that mp-1 = (n+m) q+1, i.e., mp = (n+m) s+2. i.e., p $\geq \frac{(n+m)s+2}{m}$.

Let z_1 be a zero of af - b with multiplicity p_1 (≥ 1), and a pole of g with Multiplicity q_1 (≥ 1) such that

$2p_1 - 1 = (n + m)\, q_1 + 1$, i.e., $2p_1 = (n + m)\, s + 2$,

i.e., $p_1 \geq \frac{(n+m)s+2}{2}$.

Let z_2 be a zero of f with multiplicity p_2 (≥ 1), and a pole of g with multiplicity q_2 (≥ 1). Then

$$n\, p_2 - 1 = (n + m)\, q_2 + 1. \tag{2.2}$$

From (2.2) we get $mq_2 + 2 = n(p_2 - q_2) \geq n$, i.e., $q_2 \geq \frac{n-2}{m}$. Thus from (2.2) we get

$n\, p_2 = (n + m)\, q_2 + 2 \geq \frac{(n+m)(n-2)}{m} + 2$,

i.e., $p_2 \geq \frac{n+m-2}{m}$.

Since a pole of f is either a zero of g(g - 1)(ag - b) or a zero of g', we have

$$\overline{N}(r, f) \leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g-1}\right) + \overline{N}\left(r, \frac{1}{g-b/a}\right) + \overline{N}_0\left(r, \frac{1}{g'}\right) + S(r, f) + S(r, g),$$

$$\leq \left(\frac{m+2}{(n+m)s+2}+\frac{m}{n+m-2}\right)T(r,g)+\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g),$$

Where $\overline{N}_0\left(r,\frac{1}{g'}\right)$ denotes the reduced counting function of those zeros of g' which are not the zeros of g(g - 1)(ag - b).

Then by the second fundamental theorem of Nevanlinna we get

$$2\,T(r,f) \leq \overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+\overline{N}\left(r,\frac{1}{f-b/a}\right)+\overline{N}(r,f)-\overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f)$$

i.e.,
$$2\,T(r,f) \leq \left(\frac{m+2}{(n+m)s+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0\left(r,\frac{1}{f'}\right)+\overline{N}_0\left(r,\frac{1}{g'}\right)+S(r,f)+S(r,g). \quad (2.3)$$

Similarly,

$$2\,T(r,g) \leq \left(\frac{m+2}{(n+m)s+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0\left(r,\frac{1}{g'}\right)+\overline{N}_0\left(r,\frac{1}{f'}\right)+S(r,f)+S(r,g). \quad (2.4)$$

Adding (2.3) and (2.4) and substituting s = 1 we obtain

$$\left(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which is a contradiction for n ≥ m + 3. This proves the Lemma.

Lemma: 2.8 (**[1]**) *Let f, g be two non-constant entire functions and k(≥ 1) and n > 3k + 8 be two integers. If* $[f^n(z)]^{(k)}$ $[g^n(z)]^{(k)} \equiv b^2$,*where b(≠ 0), be a constant, then f(z) =* c_1e^{cz}, *g(z) =*c_2e^{-cz}, *where* c_1, c_2 *and c are three constants satisfying* $(-1)^k\ (c_1c_2)^n(nc)^{2k}= b^2$.

3. PROOFS OF THE THEOREMS:

In this section we present the proof of the main results.

Proof of Theorem: 1.1

Set F =$f^n(\mu f^m+\lambda)$, G=$g^n(\mu g^m+\lambda)$, consider

$$\overline{N}\left(r,\frac{1}{F}\right)=\overline{N}\left(r,\frac{1}{f^n(\mu f^m+\lambda)}\right) \leq \frac{1}{s(n+m^*)}\,N\left(r,\frac{1}{F}\right) \leq \frac{1}{s(n+m^*)}\,[T(r,F)+O(1)],$$

then we have

$$\theta(0,F)=1-\lim_{r\to\infty} sup\ \frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)} \geq 1-\frac{1+m^*}{s(n+m^*)}.$$

Similarly, we have

$$\theta(0,G) \geq 1-\frac{1+m^*}{s(n+m^*)},\ \theta(\infty,F) \geq 1-\frac{1}{s(n+m^*)},\ \theta(\infty,G) \geq 1-\frac{1}{s(n+m^*)}.$$

Consider

$$N_{k+1}\left(r,\frac{1}{F}\right)=N_{k+1}\left(r,\frac{1}{f^n(\mu f^m+\lambda)}\right)=(k+1)\,\overline{N}\left(r,\frac{1}{f^n(\mu f^m+\lambda)}\right) \leq \frac{k+m^*+1}{s(n+m^*)}\,[T(r,F)+O(1)],$$

Next we have

$$\delta_{k+1}(0,F)=1-\lim_{r\to\infty} sup\ \frac{\overline{N}_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)} \geq 1-\frac{k+m^*+1}{s(n+m^*)},$$

similarly, we have

$$\delta_{k+1}(0,G) \geq 1-\frac{k+m^*+1}{s(n+m^*)}.$$

If $l \geq 2$, we have from Lemma 2.6

$$\Delta_1 = (k+2)\,\theta(\infty,F) + 2\,\theta(\infty,G) + \theta(0,F) + \theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)\,,$$
$$\geq (k+8) - \frac{3k+4m^*+8}{s(n+m^*)}.$$

Note that from Lemma 2.6, we have $s(n+m^*) \leq 3k+4m^*+8$, which contradicts our hypothesis $s(n+m^*) > 3k + 4m^* + 8$.

If $l = 1$, then from Lemma 2.6

$$\Delta_2 = (2k+3)\,\theta(\infty,F) + 2\,\theta(\infty,G) + \theta(0,F) + \theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)\delta_{k+2}(0,F),$$
$$\geq (2k+10) - \frac{5k+5m^*+11}{s(n+m^*)}.$$

Note that from Lemma 2.6, we have $s(n+m^*) \leq 5k+5m^*+11$ which contradicts our hypothesis

$s(n+m^*) > 5k+5m^*+11$.

If $l = 0$, then from Lemma 2.6 we have

$$\Delta_3 = (2k+3)\,\theta(\infty,F) + (2k+4)\,\theta(\infty,G) + \theta(0,F) + \theta(0,G) + 2\,\delta_{k+1}(0,F) + 3\,\delta_{k+1}(0,G),$$
$$\geq (4k+14) - \frac{9k+7m^*+14}{s(n+m^*)}.$$

Note that from Lemma 2.6, we have $s(n+m^*) \leq 9k+7m^*+14$, which contradicts our hypothesis $s(n+m^*) > 9k+7m^*+14$.

Here $[F]^{(k)}=[f^n(\mu f^m+\lambda)]^{(k)}$, $[G]^{(k)}=[g^n(\mu g^m+\lambda)]^{(k)}$ share the value (1, l), then by Lemma 2.6, we get either $F^{(k)}G^{(k)} \equiv 1$ *or* $F \equiv G$.

Next we consider two cases:

Case: 1 $F^{(k)}G^{(k)} \equiv 1$ That is,

$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)} \equiv 1. \tag{3.1}$$

Considering the following two subcases.

Case: 1.1. $\lambda\,\mu = 0$.
By $|\lambda| + |\mu| \neq 0$, when $\lambda = 0$, $\mu \neq 0$, (3.1) becomes $[\mu f^{n+m}]^{(k)}\,[\mu g^{n+m}]^{(k)} \equiv 1$. Obviously $f(z) \neq 0$, $g(z) \neq 0$.

In fact, suppose f(z) have a zero z_0, then z_0 is a zero of $[\mu f^{n+m}]^{(k)}$, thus z_0 is a pole of $[\mu g^{n+m}]^{(k)}$, which contradicts that $g \neq \infty$. Hence $f(z) \neq 0$, $g(z) \neq 0$. So we have $[\mu f^{n+m}]^{(k)} \neq 0$ *and* $[\mu g^{n+m}]^{(k)} \neq 0$.

By Lemma 2.3, we have $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and c are three constants satisfying* $(-1)^k \mu^2 (c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$ when $k > 2$.

Next we consider $[\mu f^{n+m}]^{(k)}\,[\mu g^{n+m}]^{(k)} \equiv 1$, for the case $k = 1$. That is

$$(n+m)^2\mu^2 f^{n+m-1}f'g^{n+m-1}g' \equiv 1. \tag{3.2}$$

From above, there exists two entire functions $\alpha(z)$ and $\beta(z)$ such that $f(z) = e^{\alpha(z)}$ and $g(z) = e^{\beta(z)}$. From this and (3.2) we have

$$(n+m)^2\mu^2\,\alpha'\beta'\;\; e^{(n+m)(\alpha+\beta)} \equiv 1. \tag{3.3}$$

Thus α' and β' have no zeros and we may set

$$\alpha' = e^{\delta(z)}, \quad \beta' = e^{\gamma(z)}. \tag{3.4}$$

where δ and γ are entire functions. By (3.3) and (3.4), we get

$(n+m)^2\mu^2 e^{(n+m)(\alpha+\beta)+\delta+\gamma} \equiv 1.$

Differentiating this yields in view of (3.4).

$$(n+m)\left(e^{\delta}+e^{\gamma}\right)+\delta'+\gamma' \equiv 0. \tag{3.5}$$

i.e., $(n+m)\, e^{\delta}+\delta' \equiv -(n+m)\, e^{\gamma}-\gamma'$.

Since δ and γ are entire, we get
$T(r,\delta') = m(r,\delta') = m\left(r, \frac{(e^{\delta})'}{e^{\delta}}\right) = S(r, e^{\delta})$,

$T(r,\gamma') = m(r,\gamma') = m\left(r, \frac{(e^{\gamma})'}{e^{\gamma}}\right) = S(r, e^{\gamma})$.

Thus, from this we have

$T(r, e^{\delta}) = T(r, e^{\gamma}) + S(r, e^{\delta}) + S(r, e^{\gamma})$,

which implies $S(r, e^{\delta}) = S(r, e^{\gamma}) := S(r)$. Let $\omega = -(\delta'+\gamma')$. Then $T(r,\omega) = S(r)$.

If $\omega \neq 0$, then we rewrite (3.5) as

$\frac{e^{\delta}}{\omega}+\frac{e^{\gamma}}{\omega}=\frac{1}{n+m}.$

From this and the second fundamental theorem, we obtain

$$\begin{aligned} T(r, e^{\delta}) &\le T\left(r, \frac{e^{\gamma}}{\omega}\right) + S(r) \\ &\le \overline{N}\left(r, \frac{e^{\gamma}}{\omega}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\gamma}}{\omega}}\right) + \overline{N}\left(r, \frac{1}{\frac{e^{\gamma}}{\omega}-\frac{1}{n+m}}\right) + S(r) \\ &\le S(r), \end{aligned}$$

which implies e^{δ} is constant. Similarly, e^{γ} is also constant. This shows that $\omega \equiv 0$, which a contradiction is. Therefore ,$\omega = -(\delta'+\gamma') \equiv 0$. It follows from this and (3.5) that $e^{\delta}+e^{\gamma} \equiv 0$, which deduces that $\delta \equiv \gamma + (2\rho+1)\pi i$ for some integer ρ. Thus by (3.5) we have $\delta' \equiv \gamma' \equiv 0$, so that δ and γ are constants. ie., α' and β' are constants. From this we can also obtain the above results.

In the same manner as above, when $\lambda \neq 0$, *$\mu=0$,* we can also get the results Which is *$f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k\, \lambda^2(c_1c_2)^n[nc]^{2k} = 1$* . Therefore, the case (ii) of Theorem 1.1 holds.

Case: 1.2. *when* $\lambda\mu \neq 0$.

We can rewrite (3.1) as

$$[f^n(f-a_1)\ldots(f-a_m)]^{(k)}[g^n(g-a_1)\ldots(g-a_m)]^{(k)} \equiv 1 \tag{3.6}$$
where $a_1, a_2, \ldots, a_m$ are roots of $\mu\omega^m+\lambda = 0$.

Let z_0 be zero of f of order p. From (3.6) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q. Again by (3.6), we obtain np - k≡ (n + m)q + k

ie., n(p - q) = mq + 2k, which implies that p ≥ q + 1 and mq + 2k ≥ n. Hence $p \ge \frac{n-2k}{m}+1$.

Let z_{i1} be a zero of f - a_i i = 1, 2,...,m of order p_{i1}, then z_{i1} is a zero of $f^n(\mu f^m+\lambda)$ of order $p_{i1}-$k. Therefore, from (3.6), we obtain z_{i1} is a pole of g of order q_{i1} and $p_{i1}-$ k = (n + m) q_{i1} + k. ie., $p_{i1}-k$ = (n + m) q_{i1} + 2k,

ie., $p_{i1} \ge$(n + m)s + 2k.

Let z_2 be a zero of f' of order p_2 that not a zero of f$(f-a_1)$ ... $(f-a_{m)}$, as above, we obtain from (3.6).

ie., p_2 -(k -1) = (n+m) q_2 +k. ie., $p_2 \geq$ (n+m)s+2k -1.

Moreover, in the same manner as above, we have similar results for the zero of $g^n(\mu g^m + \lambda)$.

On the other hand, suppose z_3 is a pole of f. From (3.6), we get z_3 is a pole of $g^n(\mu g^m + \lambda)$. Thus

$$\bar{N}(r,f) \leq \bar{N}\left(r,\frac{1}{g}\right) + \sum_{i=1}^{m} \bar{N}\left(r,\frac{1}{f-a_i}\right) + \bar{N}\left(r,\frac{1}{g'}\right)$$

$$\leq \frac{m}{n-2k+m} N\left(r,\frac{1}{g}\right) + \frac{1}{(n+m)s+2k}\sum_{i=1}^{m} \bar{N}\left(r,\frac{1}{g-a_i}\right) + \frac{1}{(n+m)s+2k-1} N\left(r,\frac{1}{g\prime}\right).$$

We get

$$\bar{N}(r,f) \leq \left(\frac{m}{n-2k+m} + \frac{m}{(n+m)s+2k} + \frac{1}{(n+m)s+2k-1}\right) T(r,g) + S(r,g).$$

From this and the second fundamental theorem we obtain

$$\text{mT(r, f)} \leq \bar{N}(r,f) + \sum_{i=1}^{m} \bar{N}\left(r,\frac{1}{f-a_i}\right) + \bar{N}\left(r,\frac{1}{f}\right) + S(r,f),$$

$$\leq \left(\frac{m}{n-2k+m} + \frac{m}{(n+m)s+2k} + \frac{1}{(n+m)s+2k-1}\right) T(r,g) \left(\frac{m}{n-2k+m} + \frac{m}{(n+m)s+2k}\right) T(r,f) + S(r,f) + S(r,g).$$

Similarly, we have

$$\text{mT(r, f)} \leq \left(\frac{m}{n-2k+m} + \frac{m}{(n+m)s+2k} + \frac{1}{(n+m)s+2k-1}\right) T(r,f) \left(\frac{m}{n-2k+m} + \frac{m}{(n+m)s+2k}\right) T(r,g) + S(r,f) + S(r,g).$$

We can deduce from above

$$\text{m}[T(r,f) + T(r,g)] \leq \left(\frac{2m}{n-2k+m} + \frac{2m}{(n+m)s+2k} + \frac{2}{(n+m)s+2k-1}\right) [T(r,f) + T(r,g)] + S(r,f) + S(r,g).$$

Since (n+m)s> $3k + 2m^* + 8$, we obtain at a contradiction.

Case: 2 F$\equiv G$.

That is

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{3.7}$$

If $\lambda\mu = 0$, then from $|\lambda| + |\mu| \neq 0$, we can get f(z) = tg(z) where t is a constant satisfying $t^{n+m^*} \equiv 1$.

If $\lambda\mu \neq 0$, then we suppose that h=$\frac{f}{g}$. If $h \neq 1$ then substituting f = hg into (3.7) we have

$$g^m = -\frac{\lambda}{\mu} * \frac{1-h^n}{1-h^{n+m}} = -\frac{\lambda}{\mu} * \frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^{n+m-1}}. \tag{3.8}$$

If $m \geq 2$, then from above we get that every pole of $f^m = -\frac{\lambda}{\mu} * \frac{(1+h+\cdots+h^{n-1})h^m}{1+h+\cdots+h^{n+m}}$.

It follows that T(r, f) = $\frac{n+m}{m}$ T(r, h) + S(r, f). On the other hand, every poles of f of order p must be a zero of h^{n+m} - 1 of order mp. Hence N(r, f) = $\frac{1}{m}\sum_{i=1}^{n+m} N\left(r,\frac{1}{h-a_i}\right)$ where a_i ($\neq$ 1)(i = 1, 2, ..., (n+m-1)) are distinct root of the algebraic equation h^{n+m} = 1. Therefore, we deduce

$$N(r, f) = \frac{1}{m}\sum_{i=1}^{n+m-1} N\left(r, \frac{1}{h-a_i}\right) \geq \frac{1}{m}\sum_{i=1}^{n+m-1} \overline{N}\left(r, \frac{1}{h-a_i}\right) \geq \frac{n+m-3}{m}T(r,h) + S(r,f),$$

We have

$$\delta(\infty, f) = 1 - \lim_{r\to\infty}\sup\frac{\frac{n+m-3}{m}T(r,h)+S(r,f)}{\frac{n+m}{m}T(r,h)+S(r,f)} \leq \frac{3}{n+m},$$

which contradicts the assumption $\delta(\infty, f) > \frac{3}{n+m}$.

If m = 1, (3.8) is $g = -\frac{\lambda}{\mu} * \frac{1+h+\cdots+h^{n-1}}{1+h+\cdots+h^n}$, from f = hg, we have

$f == -\frac{\lambda}{\mu} * \frac{(1+h+\cdots+h^{n-1})h}{1+h+\cdots+h^n}$, where h is a nonconstant meromorphic function.

It follows that T(r, f) = T(r, gh) = (n + 1)T(r, h) + S(r, f).

On the other hand, by the second fundamental theorem, we deduce

$$\overline{N}(r,f) = \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{h-a_j}\right) \leq (n-2)T(r,h) + S(r,f),$$

where a_j ($\neq 1$)(j = 1, 2, ..., n) are distinct roots of the algebraic equation $h^{n+1} = 0$.
We have

$$\theta(\infty, f) = 1 - \lim_{r\to\infty}\sup\frac{\overline{N}(r,f)}{T(r,f)}$$

$$\leq \lim_{r\to\infty}\sup\frac{(n-2)T(r,h)+S(r,f)}{(n+1)T(r,h)+S(r,f)} \leq \frac{3}{n+1},$$

which contradicts the assumption $\theta(\infty, f) > \frac{3}{n+1}$.

Thus h ≡ 1, that is f(z) ≡ g(z).

This completes the proof of Theorem 1.1.

Proof of Theorem: 1.2

Set F = $f^n(f-1)^m$ and G = $g^n(g-1)^m$..

Consider

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{f^n(f-1)^m}\right) \leq \frac{1}{s(n+m)} N\left(r, \frac{1}{F}\right) \leq \frac{1}{s(n+m)}[T(r,F)+O(1)],$$

then we have

$$\theta(0, F) = 1 - \lim_{r\to\infty} sup\frac{\overline{N}\left(r, \frac{1}{F}\right)}{T(r,F)} \geq 1 - \frac{1+m^*}{s(n+m)},$$

where $m^* = \begin{pmatrix} 0 & \text{if } m=0 \\ 1 & if \quad m\geq 1 \end{pmatrix}$.

Similarly, we have

$$\theta(0, G) \geq 1 - \frac{1+m^*}{s(n+m)},\ \theta(\infty, F) \geq 1 - \frac{1}{s(n+m)},\ \theta(\infty, G) \geq 1 - \frac{1}{s(n+m)}.$$

Consider

$N_{k+1}\left(r,\frac{1}{F}\right)$=$N_{k+1}\left(r,\frac{1}{f^n(f-1)^m}\right)=(k+1)\,\overline{N}\left(r,\frac{1}{f^n(f-1)^m}\right)\leq\frac{k+m+1}{s(n+m)}$ [T(r,F)+O(1)],

next we have

$$\delta_{k+1}(0,F)=1-\lim_{r\to\infty} sup\ \frac{\overline{N}_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)}\geq 1-\frac{k+m+1}{s(n+m)},$$

similarly, we have

$$\delta_{k+1}(0,G)\geq 1-\frac{k+m+1}{s(n+m)}.$$

If l ≥ 2, we have from Lemma 2.6

$$\Delta_1\geq(k+8)-\frac{3k+2m^*+2m+8}{s(n+m)}.$$

Note that from Lemma 2.6, we have $s(n+m)\ \leq\ 3k+2m^*+2m+8$, which contradicts our hypothesis $s(n+m)>$ $3\mathrm{k}+2m^*+2m+8$.

If l = 1, then from Lemma 2.6

$$\Delta_2\geq(2k+10)-\frac{5k+2m^*+3m+11}{s(n+m)}.$$

Note that from Lemma 2.6, we have $s(n+m)\ \leq 5k+2m^*+3m+11$ which contradicts our hypothesis $s(n+m)>5k+2m^*+3m+11$.

If l = 0, then from Lemma 2.6 we have

$$\Delta_3\ \geq(4k+14)-\frac{9k+2m^*+5m+14}{s(n+m)}.$$

Note that from Lemma 2.6, we have $s(n+m)\ \leq 9k+2m^*+5m+14$, which contradicts our hypothesis $s(n+m)>9k+2m^*+5m+14$.

Here $[F]^{(k)}$=$[f^n(f-1)^m]^{(k)}$, $[G]^{(k)}$= $[g^n(g-1)^m]^{(k)}$ share the value (1, l), then by Lemma 2.6, we get either

$F^{(k)}G^{(k)}\equiv 1$ *or* $F\equiv G.$

Let m=0. Since f(z) ≠ ∞ and g(z) ≠ ∞ , by $F^{(k)}G^{(k)}\equiv 1$ and Lemma 2.8 we obtain

f(z) = c_1e^{cz}, *g(z) =*c_2e^{-cz}, *where* c_1, c_2 *and c are three constants satisfying* $(-1)^k\ (c_1c_2)^n(nc)^{2k}$*= 1. Also by Lemma 2.7 the case* $F^{(k)}G^{(k)}$ *≡1 does not arise for k=1 and m*≥ 1.

Let F ≡ G. i.e.

$$f^n(f-1)^m\equiv g^n(g-1)^m. \tag{3.9}$$

Now we consider the following three cases.

Case i. Let m= 0. Then from (3.9) we get f= t g for a constant t such that $t^n=1$.

Case ii. Let m = 1. Then from (3.9), we have

$$f^n(f-1)\equiv g^n(g-1). \tag{3.10}$$

Suppose f ≠ g. Let h = $\frac{f}{g}$ be a constant. Then from (3.10) it follows that

h≠ $1, h^n\neq 1,\ h^{n+1}\neq 1$ and g= $\frac{1-h^n}{1-h^{n+1}}$ = constant, a contradiction. So we suppose that h is not a constant. Since f ≠ g,

We have h≠ 1. From (3.10), we obtain g = $\frac{1-h^n}{1-h^{n+1}}$ and f= ($\frac{1-h^n}{1-h^{n+1}}$) h.

Hence it follows that T(r, f) = nT(r, h) + S(r, f).

Again, by second fundamental theorem of Nevanlinna, we have

$$\bar{N}(r,f) = \sum_{j=1}^{n} \bar{N}\left(r,\frac{1}{h-\alpha_j}\right) \leq (n-2)T(r,h) + S(r,f),$$

where α_j ($\neq 1$)(j = 1, 2, ..., n) are distinct roots of the algebraic equation $h^{n+1} = 1$. So we obtain

$\theta(\infty, f) = 1 - \lim_{r\to\infty} \sup \frac{\bar{N}(r,f)}{T(r,f)} \leq \frac{2}{n}$, which contradicts the assumption $\theta(\infty, f) > \frac{2}{n}$.

Thus f ≡ g.

Case: iii Let m ≥ 2. Then from (3.9) we obtain

$$f^n(f^m + \cdots + (-1)^i mc_{m-i} f^{m-i} + \cdots + (-1)^m) = g^n(g^m + \cdots + (-1)^i mc_{m-i} g^{m-i} + \cdots + (-1)^m). \quad (3.11)$$

Let h = $\frac{f}{g}$. If h is a constant, then substituting f = hg into (3.11), we obtain

$$g^{n+m}(h^{n+m} - 1) + \cdots + (-1)^i \, mc_{m-i} g^{m+n-i}(h^{n+m-i} - 1) + \cdots + g^n(h^n - 1) = 0$$

which implies h = 1. Thus f(z) ≡ g(z). If h is not a constant, then we know by (3.11) that f and g satisfy the algebraic equation R (f, g) = 0, where R(x, y) = $x^n(x-1)^m - y^n(y-1)^m$.

This completes the proof.

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