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COMMON FIXED POINT THEOREM IN MENGER SPACE UNDER STRICT CONTRACTIVE CONDITIONS

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ABSTRACT

In this paper we prove some new common fixed point theorem in Menger space under strict contractive conditions for mappings satisfying the new property.

AMS Mathematical subject classification: *47H10, 54H25.*

Keyword: *Common fixed point, Compatible maps, Weakly compatible maps.*

1. INTRODUCTION:

The concept of probabilistic metric space was first introduced and studied by Menger [8], which is a generalization of the metric space and also the study of this space was expanded rapidly with the pioneering works of Schweizer and Sklar [12, 13]. The theory of probabilistic space is of fundamental importance in probabilistic functional analysis.

It is well known that in the setting of metric space, the strict contractive condition do not ensure the existence of common fixed point unless the space is assumed to be compact or the strict conditions are replaced by some stronger conditions as in [4, 7, 9]. In 1986, Jungck [3] introduced the notion of compatible mappings. This concept was frequently used to prove existence theorems in common fixed point theory. However, the study of common fixed points of non-compatible mappings is also very interesting. Research along this direction has recently been initiated by Pant [9, 10]. Aamri and Moutawakil [1].

The aim of this paper is to define a new property which generalizes the concept of non-compatible mappings and gives some common fixed point theorems in Menger space under strict contractive conditions.

2. PRELIMINARIES:

Definition: 2.1 A probabilistic metric space is a pair (X, F), where X is a non-empty set and F is a mapping from $X \times X$ to L (L is set of all distribution function). For $(u,v) \in X \times X$, the distribution function $F(u,v)$ is denoted by $F_{u,v}$. The function $F(u,v)$ are assumed to satisfy the following conditions:

$(P-1)$: $F_{u,v}(x) = 1$ for every $x > 0$ if and only if $u = v$,

$(P-2)$: $F_{u,v}(0) = 0$ for every $u, v \in X$,

$(P-3)$: $F_{u,v}(x) = F_{v,u}(x)$ for every $u, v \in X$,

$(P-4)$: If $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1$ then $F_{u,w}(x+y) = 1$ for every $u, v, w \in X$.

Definition: 2.2 A Menger space is a triple (X, F, t), where (X, F) is a PM-space and t is $T-$norm with the following condition:

$F_{u,w}(x+y) \geq t(F_{u,v}(x), F_{v,w}(y))$ for every $u, v, w \in X$ and $x, y \in R^{+}$.

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Definition: 2.3 [7] Let (X,F,t) be a Menger space with the T-norm t. A sequence $\{p_n\}$ in X is said to be convergent to a point $p\in X$ if for every $\varepsilon>0$ and $\lambda>0$, there exists an integer $N=N(\varepsilon,\lambda l)$ such that $p_n=\cup p(\varepsilon,\lambda)$ for all $n\geq N$, or equivalently, $F_{p,p_n}(\varepsilon)>1-\lambda$, for all $n\geq N$. We write $p_n\to p$ as $n\to\infty$ or $\lim_{n\to\infty} p_n=p$.

Definition: 2.4 [2] Let (X,F,t) be a Menger space such that the T-norm t continuous and S,T be mapping from X into itself. Then S and T are said to be compatible if

$$\lim_{n\to\infty} F(STx_n,TSx_n)=1$$

For all $x>0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=z$$

For some $z\in X$.

Definition: 2.5 Two self mappings S and T are said to be weakly compatible if they commute at their coincidence points; i.e., if $Tu=Su$ for some $u\in X$, then $TSu=STu$.

Definition: 2.6 Let S and T be two self mappings of a Menger space (X,F,t). We say that S and T satisfy the property (E.A.) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=z$$

for some $z\in X$.

Example: [2.7] [1]: Let $X=[0,+\infty)$. Define $S,T:X\to X$ by

$$Tx=\frac{x}{4} \quad\text{and}\quad Sx=\frac{3x}{4},\ \forall x\in X.$$

Consider the sequence $x_n=\dfrac{1}{n}$. Clearly $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=0$. Then S and T satisfy (E.A.).

Example: [2.8] [1]: Let $X=[2,+\infty)$. Define $S,T:X\to X$ by

$$Tx=x+1 \quad\text{and}\quad Sx=2x+1,\ \forall x\in X.$$

Suppose that the property (E.A.) holds. Then, there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=z$$

for some $z\in X$.

Therefore

$$\lim_{n\to\infty} x_n=z-1 \text{ and } \lim_{n\to\infty} x_n=\frac{z-1}{2}.$$

Thus, $z=1$, which is a contradiction since $1\notin X$. Hence S and T do not satisfy (E.A.).

Remark: 2.9 It is clear from the definition of Sharma and Deshpande [14], Sharma and Choubey [15] and Jungck [3] that two self mappings S and T of a Menger space (X,F,t) will be non compatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=z$$

for some $z\in X$.

But $\lim_{n\to\infty} F(STx_n, TSx_n)$ is either not equal to 1 or non-existent. Therefore two non-compatible self mappings of a Menger space (X, F, t) satisfy the property (E.A.).

I. Kubiaczyk and Sushil Sharma [7] have proved the following result.

Theorem: 1.2 Let (X, F, t) be a Menger space with $t(x, y) = \min(x, y)$ for all $x, y \in [0,1]$ and S and T be weakly compatible mappings of X into itself such that

(i) S and T satisfy the property (E.A.)

(ii) there exists a number $k \in (0,1)$ such that

$$F_{Tu,Tv}(kx) \geq \min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}$$

For all $u, v \in X$.

(iii) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ be a closed subset of X, then S and T have a unique common fixed point.

MAIN RESULT:

Theorem: Let (X, F, t) be a Menger space with $t(x, y) = \min(x, y)$ for all $x, y \in [0,1]$ and S and T be weakly compatible mappings of X into itself such that

(i) S and T satisfy the property (E.A.)

(ii) there exists a number $k \in (0,1)$ such that

$$F^2_{Tu,Tv}(kx) \geq [\min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}]^2$$

For all $u, v \in X$.

If $S(X)$ be a closed subset of X, then S and T have a unique common fixed point.

Proof: Since S and T satisfy the property (E.A.) there exists a sequence $\{x_n\}$ in X satisfying $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$. Suppose that $S(X)$ is closed.

Then $\lim_{n\to\infty} Sx_n = Sa$ for some $a \in X$, also $\lim_{n\to\infty} Tx_n = Sa.$

By condition (ii)

$$F^2_{Tu,Tv}(kx) \geq [\min\{F_{Su,Sv}(x), F_{Su,Tu}(x), F_{Sv,Tv}(x), F_{Sv,Tu}(x), F_{Su,Tv}(x)\}]^2$$

$$F^2_{Tx_n,Ta}(kx) \geq [\min\{F_{Sx_n,Sa}(x), F_{Sx_n,Tx_n}(x), F_{Sa,Ta}(x), F_{Sa,Tx_n}(x), F_{Sx_n,Ta}(x)\}]^2$$

Taking $n \to \infty$, we get

$$F^2_{Sa,Ta}(kx) \geq F^2_{Sa,Ta}(x) \geq F^2_{Sa,Ta}(kx)$$

Therefore

$$F^2_{Sa,Ta}(t) = F^2_{Sa,Ta}(x) \quad \forall \quad t \in [kx, x]$$

For any $t \in [k^2 x, kx]$,

$F^2_{Sa,Ta}(t) = F^2_{Sa,Ta}(x)$

……. ……..

We can prove that

$F^2_{Sa,Ta}(t) = F^2_{Sa,Ta}(x) \;\; \forall \;\; x \in [x, \frac{x}{k}]$

$F^2_{Sa,Ta}(t) = F^2_{Sa,Ta}(x) \;\; \forall \;\; x \in [\frac{x}{k}, \frac{x}{k^2}]$

……. ……..

Continuing this way we can prove that

$F^2_{Sa,Ta}(x) = 1 \;\; \forall \;\; x > 0.$

Therefore $Ta = Sa.$

Since S, T are weakly compatible $STa = TSa$

$\therefore TTa = TSa = STa = SSa$

We prove that Ta is a common fixed point of T and $S.$

Consider

$$F^2_{Ta,TTa}(kx) \geq [\min\{F_{Sa,STa}(x), F_{Sa,Ta}(x), F_{STa,TTa}(x), F_{STa,Ta}(t), F_{Sa,TTa}(x)\}]^2$$
$$= F^2_{Ta,TTa}(x)$$

Hence $F^2_{Ta,TTa}(kx) \geq F^2_{Ta,TTa}(x)$

$\Rightarrow F^2_{Ta,TTa}(t) = F^2_{Ta,TTa}(x) \;\; \forall \;\; t \in [kx, x]$

$\Rightarrow F^2_{Ta,TTa}(t) = 1 \;\; \forall \;\; t \in [0, \infty)$

$\Rightarrow Ta = TTa$

$\therefore TTa = Ta$ and $STa = TTa = Ta$

i.e. Ta is a common fixed point for S and $T.$

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