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APPROXIMATE SOLUTIONS TO FRACTIONAL SEMILINEAR EVOLUTION EQUATION WITH NONLOCAL CONDITION

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ABSTRACT

In this paper, we study the approximate solutions, uniqueness and other properties of solutions of fractional semilinear evolution equations with nonlocal condition in Banach spaces. The results are obtained by using fractional calculus, Banach fixed theorem and the integral inequality established by E. Hernandez.

Key words: *Approximate solutions, evolution equation, fractional calculus, nonlocal condition, continuous dependance, Hernandez's inequality.*

2000 Mathematics Subject Classification: *34G10, 34G20*

1. INTRODUCTION:

Fractional differential equation can describe many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, polymer, biotechnology, game theory, programming languages etc., we refer the reader to the monographs [11, 15, 17, 19, 20, 22, 24, 26, 27] and the papers [1, 2, 6, 9, 12, 13, 16, 18]. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $x(0), x^{'}(0)$, and so forth the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types, see [25, 29].

Motivated by the work of [3, 21, 30], in this paper we consider the following linear evolution equation of the form:

$$D^q x(t) = A(t)x(t) + f(t, x(t)), \quad t \in J = [0, b], \tag{1.1}$$

$$x(0) + g(x) = x_0, \tag{1.2}$$

where $0 < q < 1$, the unknown $x(\cdot)$ takes values in the Banach space X; $f \in C(J \times X, X)$, $g : C(J, X) \to X$ and $A(t)$ is a bounded linear operator on a Banach space X and x_0 is a given element of X. The operator D^q denotes the Caputo fractional derivative of order q.

The nonlocal Cauchy problem was first considered by Byszewski. As pointed out by Byszewski [4] the study of Cauchy problems with nonlocal conditions are of significance since they have applications in physics and other areas of applied mathematics. Subsequently, several works are devoted to the study of nonlocal problems. For example, see [5, 10, 21, 28] and the references cited therein.

In this paper, our main objective is to study the qualitative properties of its solutions. The method of approximations to the solutions is a very powerful tool which provides valuable information, without the

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need to know in advance the solutions explicitly of various dynamic equations. We apply the method of approximations to the solutions of initial value problem (1.1)--(1.2) and investigate new estimates on the difference between the two approximate solutions of equation (1.1) and other properties of solutions of the problems. Our result generalizes the some result of JinRong Wang, Linli Lv, and Yong Zhou [30]. The main tool employed in the analysis is based on the application of a variant of a certain integral inequality with explicit estimate due to E. Hernandez.

The paper is organized as follows. In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the our main results. Section 4 concerns the properties of solutions. In section 5, we give an example to illustrate the applications of some of our results.

2. PRELIMINARIES AND HYPOTHESES:

We shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

Let X be the Banach space with norm $\|\cdot\|$. Let $B = C(J, X)$ be the Banach space of all continuous functions from J into X endowed with supremum norm

$$\|x\|_B = \sup\{\|x(t)\| : t \in J\}.$$

Definition: 2.1 Let $x_i \in B (i = 1,2)$ be functions such that $D^q x_i(t)$ exist for $t \in J$ and satisfy the inequalities

$$\left\| D^q x_i(t) - A(t)x_i(t) - f(t, x_i(t)) \right\| \le \varepsilon_i, \tag{2.1}$$

for given constants $\varepsilon_i \ge 0$, where it is assumed that the conditions

$$x_i(0) + g(x_i) = x_i^*, \tag{2.2}$$

are fulfilled. Then we call $x_i(t)$ the ε_i-approximate solutions with respect to the fractional differential problem (1.1)--(1.2).

Definition: 2.2 *A real function $f(t), t > 0$, is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(t) = t^p g(t)$, where $g(t) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu, n \in \mathbb{N}$.*

Definition: 2.3 *A function $f \in C_\mu, \mu \ge -1$ is said to be Riemann-Liouville fractional integrable of order $\alpha \in \mathbb{R}^+$ if*

$$I^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)ds < \infty,$$

where Γ is the Euler gamma function and if $\alpha = 0$, then $I^0 f(t) = f(t)$.

Definition: 2.4 *The fractional derivative in the Caputo sense is defined as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = I^{n-\alpha}\left(\frac{d^n f(t)}{dt^n}\right) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)ds$$

for $n-1 < \alpha \le n, n \in \mathbb{N}, t > 0$ and $f \in C_{-1}^n$.

If $0 < \alpha \le 1$, then

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds,$$

where $f^{'}(s)=\dfrac{df(s)}{ds}$ and f is an abstract function with values in X.

The problem (1.1)--(1.2) is equivalent to the integral equation

$$x(t)=x_0-g(x)+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}A(s)x(s)ds+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,x(s))ds \quad t\in J. \tag{2.3}$$

Definition: 2.5 *By a solution of the abstract Cauchy problem (1.1)--(1.2), we mean an abstract function* x *such that the following conditions are satisfied:*

(i) $x\in S$ and $x\in D(A(t))$ for all $t\in J$;

(ii) $D^q x$ exists and continuous on J, where $0<q<1$;

(iii) x satisfies the equation (1.1) with the nonlocal condition $x(0)+g(x)=x_0$.

Remark: 2.6 *A function* $y\in C^{'}(J;X)$ *is an* $\varepsilon-$ *approximate solution if and only if there exits a function* $h\in C^{'}(J;X)$ *(which depend on* y *) such that*

(i) $|h(t)|\le\varepsilon, \quad t\in J$;

(ii) $D^q y(t)=A(t)y(t)+f(t,y(t))+h(t), \quad t\in J$.

The following integral inequality established by E. Hernandez in ([14], p198) is crucial in the proof of our main results.

Lemma: 2.7 *Let* $v(\cdot),w(\cdot):[0,b]\to[0,\infty)$ *be continuous functions. If* $w(\cdot)$ *is nondecreasing and there are constants* $p>0, \quad 0<\alpha<1$ *such that*

$$v(t)\le w(t)+p\int_0^t\frac{v(s)}{(t-s)^{1-\alpha}}ds, \quad t\in[0,b],$$

then
$$v(t)\le exp\left\{\frac{[p\Gamma(\alpha)b]^n}{\Gamma(n\alpha)}\right\}\sum_{j=0}^{n-1}\left(\frac{pb^\alpha}{\alpha}\right)^j w(t),$$

for every $t\in[0,b]$ and every $n\in\mathbb{N}$ such that $n\alpha>1$, $\Gamma(\cdot)$ is the gamma function.

We list the following hypotheses for our convenience.

(H_1) $A(t)$ is a bounded linear operator on X for each $t\in J$ and the function $t\to A(t)$ is continuous in the uniform operator topology (see [23]). There exists a constant $M>0$ such that

$$\left\|A(t)\right\|\le M, \quad t\in J.$$

(H_2) There exists a constant $K\ge 0$ such that

$$\left\|f(t,x)-f(t,\bar{x})\right\|\le K\left\|x-\bar{x}\right\|,$$

for every $t\in J$ and $x,\bar{x}\in X$.

(H_3) There exists a constant G such that

$$\left\|g(x)-g(\bar{x})\right\|\le G\left\|x-\bar{x}\right\|,$$

for every $x,\bar{x}\in B$.

3. MAIN RESULTS:

Our main result given in the following theorem estimates the difference between the two approximate solutions of equation (1.1).

Theorem: 3.1 *Suppose that the hypotheses* $(H_1)-(H_3)$ *hold. Let* $x_i(t)(i=1,2)$ *be respectively* ε_i*-approximate solutions of equation (1.1) with (2.2) on* J *such that*

$$\left\|x_1^* - x_2^*\right\| \le \delta, \tag{3.1}$$

where δ is nonnegative constant. Then

$$\left\|x_1(t)-x_2(t)\right\| \le \exp\left\{\frac{[(M+K)b]^n}{(1-G)\Gamma(nq)}\right\}\sum_{j=0}^{n-1}\left(\frac{(M+K)b^q}{(1-G)\Gamma(q+1)}\right)^j [\delta+\frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q]\frac{1}{(1-G)}, \tag{3.2}$$

for every $t\in J$ and every $n\in\mathbb{N}$ such that $nq>1$ and $G<1$.

Proof: We have respectively ε_i-approximate solutions $x_i(t)(i=1,2)$ for $t\in J$ of equation (1.1) with (2.2). Then following Remark 2.6, there exist two functions $h_1, h_2 \in C^{'}(J;X)$ such that

$$|h_1(t)|\le\varepsilon_1, \quad |h_2(t)|\le\varepsilon_2 \quad t\in J, \tag{3.3}$$

and

$$x_1(t) = x_1^* - g(x_1) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x_1(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_1(s))ds$$
$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h_1(s)ds, \quad t\in J; \tag{3.4}$$

and

$$x_2(t) = x_2^* - g(x_2) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}A(s)x_2(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_2(s))ds$$
$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}h_2(s)ds, \quad t\in J. \tag{3.5}$$

Then from equations (3.4), (3.5) and using the hypotheses, we have

$$\begin{aligned}
\left\|x_1(t)-x_2(t)\right\| &\le \left\|x_1^*-x_2^*\right\| + \left\|g(x_1)-g(x_2)\right\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|A(s)\right\|\left\|x_1(s)-x_2(s)\right\|ds \\
&\quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\left\|f(s,x_1(s))-f(s,x_2(s))\right\|ds \\
&\quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|h_1(s)-h_2(s)|ds \\
&\le \delta + G\left\|x_1(t)-x_2(t)\right\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}M\left\|x_1(s)-x_2(s)\right\|ds \\
&\quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}K\left\|x_1(s)-x_2(s)\right\|ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}(\varepsilon_1+\varepsilon_2)ds \\
&\le \delta + G\left\|x_1(t)-x_2(t)\right\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}M\left\|x_1(s)-x_2(s)\right\|ds \\
&\quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}K\left\|x_1(s)-x_2(s)\right\|ds + \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)}t^q.
\end{aligned} \tag{3.6}$$

Let $u(t) = \|x_1(t) - x_2(t)\|, \quad t \in J$, then the equation (3.6) becomes

$$u(t) \le \delta + Gu(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Mu(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Ku(s)ds + \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)} t^q. \tag{3.7}$$

Therefore,

$$u(t) \le [\delta + \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)} t^q]\frac{1}{(1-G)} + \frac{[M+K]}{(1-G)\Gamma(q)}\int_0^t (t-s)^{q-1} u(s)ds. \tag{3.8}$$

Now a suitable application of Lemma 2.7 (with $p = \frac{1}{(1-G)\Gamma(q)}[M+K]$ and

$w(t) = [\delta + \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)} t^q]\frac{1}{(1-G)}$) to above inequality yields

$$u(t) \le \exp\left\{\frac{[(M+K)b]^n}{(1-G)\Gamma(nq)}\right\}\sum_{j=0}^{n-1}\left(\frac{(M+K)b^q}{(1-G)\Gamma(q+1)}\right)^j [\delta + \frac{(\varepsilon_1+\varepsilon_2)}{\Gamma(q+1)} t^q]\frac{1}{(1-G)}. \tag{3.9}$$

This completes the proof of Theorem 3.1.

Remark: 3.2 *We note that the estimate obtained in (3.9) yields the bound on the difference between the two approximate solutions of equation (1.1). If* $x_1(t)$ *is a solution of equation (1.1) with* $x_0 = x_1^*$, *then we have* $\varepsilon_1 = 0$ *and from (3.9), we see that* $x_2(t) \to x_1(t)$ *as* $\varepsilon_2 \to 0$ *and* $\delta \to 0$. *Moreover, if we put* (i) $\varepsilon_1 = \varepsilon_2 = 0$ *and* $x_1^* = x_2^*$ *in (3.9), then the uniqueness of solutions of equation (1.1) is established and* (ii) $\varepsilon_1 = \varepsilon_2 = 0$, *then we get the bound which shows the dependency of solutions of equation (1.1) on given initial values. This also shows that the equation (1.1) is Ulam-Hyers stable, for more details see [30].*

So the Ulam-Hyers stabilities of the fractional differential equations are some special types of data dependence of the solutions of fractional differential equations.

Theorem: 3.3 *Let* $0 < q < 1$, *if* $y \in C'(J;X)$ *is an* $\varepsilon-$ *approximate solution, then* y *is a solution of the following integral inequality*

$$| y(t) - y(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} A(s)y(s)ds - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds | \le \frac{t^q}{\Gamma(q+1)}\varepsilon, \quad t \in J. \tag{3.10}$$

Proof: By Remark 2.6 (ii) we have that

$$D^q y(t) = A(t)y(t) + f(t, y(t)) + h(t), \quad t \in J.$$

Then by observing (2.3), we have

$$\begin{aligned} y(t) = y(0) &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} A(s)y(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds \\ &+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)ds, \quad t \in J. \end{aligned} \tag{3.11}$$

From (3.11) and the Remark 2.6 (i), we obtain

$$\begin{aligned} &| y(t) - y(0) - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} A(s)y(s)ds - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, y(s))ds | \\ &= | \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} h(s)ds | \end{aligned}$$

$$\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \mid h(s) \mid ds$$
$$\leq \frac{\varepsilon}{\Gamma(q)}\int_0^t (t-s)^{q-1} ds$$
$$\leq \frac{\varepsilon}{\Gamma(q+1)} t^q$$
$$\leq \frac{t^q}{\Gamma(q+1)}\varepsilon. \quad t \in J \tag{3.12}$$

This completes the proof of Theorem 3.3.

The following theorem shows the uniqueness of solutions to (1.1)--(1.2) in the whole space B, without the existence part

Theorem: 3.4 *If the hypotheses $(H_1)-(H_3)$ are satisfied, then the initial value problem (1.1)--(1.2) has at most one mild solution on J.*

Proof: Let $x_1(t)$ and $x_2(t)$ be two solutions of the initial value problem (1.1)--(1.2) and $u(t) = \|x_1(t) - x_2(t)\|$. Then by hypotheses, we have

$$u(t) \leq G\|x_1(t) - x_2(t)\| + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \|A(s)\| \|x_1(s) - x_2(s)\| ds$$
$$+ \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} \|f(s, x_1(s)) - f(s, x_2(s))\| ds$$
$$\leq Gu(t) + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Mu(s)ds + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Ku(s)ds,$$

which implies

$$u(t) \leq \frac{1}{(1-G)\Gamma(q)}[M+K]\int_0^t (t-s)^{q-1} u(s)ds.$$

Now a suitable application of Lemma 2.7 (with $p = \frac{1}{(1-G)\Gamma(q)}[M+K]$ and $w(t) = 0$) to above inequality yields

$$u(t) = \|x_1(t) - x_2(t)\| \leq 0,$$

which implies $x_1(t) = x_2(t)$ for $t \in J$. Thus there is at most one solution to the initial value problem (1.1)--(1.2) on J. The proof is complete.

4 CONTINUOUS DEPENDENCE:

In this section we study the continuous dependence of solutions of (1.1) on the given initial data, and on certain parameters.

The following theorem deals with the continuous dependence of solutions of equation (1.1) on given initial values.

Theorem: 4.1 *Suppose that the hypotheses $(H_1)-(H_3)$ hold. Let $x_1(t)$ and $x_2(t)$ be solutions of the equation (1.1) for $0 \leq t \leq b$ with given initial conditions*

$$x_1(0) + g(x_1) = x_0^{*}, \quad and \quad x_2(0) + g(x_2) = x_0^{**},$$

respectively, where x_0^{*}, x_0^{**} are elements of X. Then

$$\|x_1(t)-x_2(t)\| \le \frac{\|x_0^{*}-x_0^{**}\|}{(1-G)} \exp\left\{\frac{[(M+K)b]^n}{(1-G)\Gamma(nq)}\right\} \sum_{j=0}^{n-1}\left(\frac{(M+K)b^q}{(1-G)\Gamma(q+1)}\right)^j,$$

for every $t\in J$ and every $n\in\mathbb{N}$ such that $nq>1$.

Proof: Let $x_1(t)$ and $x_2(t)$ be two solutions of the problem (1.1) and $u(t)=\|x_1(t)-x_2(t)\|$. Then by hypotheses, we have

$$u(t)\le \|x_0^{*}-x_0^{**}\|+G\|x_1-x_2\|+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A(s)\|\|x_1(s)-x_2(s)\|ds$$

$$+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|f(s,x_1(s))-f(s,x_2(s))\|ds$$

$$\le \|x_0^{*}-x_0^{**}\|+Gu(t)\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Mu(s)ds+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}Ku(s)ds,$$

which yields

$$u(t)\le \frac{\|x_0^{*}-x_0^{**}\|}{(1-G)}+\frac{1}{(1-G)\Gamma(q)}[M+K]\int_0^t (t-s)^{q-1}u(s)ds. \tag{4.1}$$

Now a suitable application of Lemma 2.7 (with $p=\frac{1}{(1-G)\Gamma(q)}[M+K]$ and $w(t)=\frac{\|x_0^{*}-x_0^{**}\|}{(1-G)}$) to (4.1) yields

$$\|x_1(t)-x_2(t)\| \le \frac{\|x_0^{*}-x_0^{**}\|}{(1-G)} \exp\left\{\frac{[(M+K)b]^n}{(1-G)\Gamma(nq)}\right\} \sum_{j=0}^{n-1}\left(\frac{(M+K)b^q}{(1-G)\Gamma(q+1)}\right)^j.$$

This completes the proof of the Theorem 4.1.

We consider the differential equations of fractional order:

$$\frac{d^q x}{dt^q}=A(t)x(t)+F(t,x(t),\mu_1), \tag{4.2}$$

$$\frac{d^q x}{dt^q}=A(t)x(t)+F(t,x(t),\mu_2), \tag{4.3}$$

for $t\in J$, where $F\in C(J\times X\times\mathbb{R},X)$, and μ_1,μ_2 are real parameters and with nonlocal condition given by (1.2).

The next theorem states the continuous dependency of solutions to (4.2)--(1.2) and (4.3)--(1.2) on parameters.

Theorem 4.2 *Assume that the function* F *satisfy the conditions*

$$\|F(t,x,\mu)-F(t,\bar{x},\mu)\|\le L_1\|x-\bar{x}\|, \tag{4.4}$$

$$\|F(t,x,\mu)-F(t,x,\bar{\mu})\|\le L_2\,|\,\mu-\bar{\mu}\,|, \tag{4.5}$$

where $L_1,L_2\ge 0$ and the hypotheses (H_1) and (H_3) hold. Let $x_1(t)$ and $x_2(t)$ be the solutions of (4.2)--(1.2) and (4.3)--(1.2) respectively. Then

$$\|x_1(t)-x_2(t)\| \le \frac{b^q L_2\,|\,\mu_1-\mu_2\,|}{(1-G)\Gamma(q+1)} \exp\left\{\frac{[(M+L_1)b]^n}{(1-G)\Gamma(nq)}\right\} \sum_{j=0}^{n-1}\left(\frac{(M+L_1)b^q}{(1-G)\Gamma(q+1)}\right)^j, \tag{4.6}$$

for every $t\in J$ and every $n\in\mathbb{N}$ such that $nq>1$.

Proof: Define $u(t)$ as in the proof of Theorem 4.1. Using the facts that $x_1(t)$ and $x_2(t)$ are solutions of the initial value problem (4.2)--(1.2) and (4.3)--(1.2) on J and hypotheses, we have

$$u(t) \le Gu(t) + \frac{1}{\Gamma(q)} M \int_0^t (t-s)^{q-1} u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\| G(s, x_1(s), \mu_1) - G(s, x_2(s), \mu_2) \right\| ds$$

$$\le Gu(t) + \frac{1}{\Gamma(q)} M \int_0^t (t-s)^{q-1} u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\| G(s, x_1(s), \mu_1) - G(s, x_2(s), \mu_1) \right\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\| G(s, x_2(s), \mu_1) - G(s, x_2(s), \mu_2) \right\| ds$$

$$\le Gu(t) + \frac{1}{\Gamma(q)} M \int_0^t (t-s)^{q-1} u(s)ds + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_1 \left\| x_1(s) - x_2(s) \right\| ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} L_2 \mid \mu_1 - \mu_2 \mid ds$$

$$\le Gu(t) + \frac{1}{\Gamma(q)} [M + L_1] \int_0^t (t-s)^{q-1} u(s)ds + \frac{b^q L_2 \mid \mu_1 - \mu_2 \mid}{\Gamma(q+1)},$$

implies that

$$u(t) \le \frac{b^q L_2 \mid \mu_1 - \mu_2 \mid}{(1-G)\Gamma(q+1)} + \frac{1}{(1-G)\Gamma(q)} [M + L_1] \int_0^t (t-s)^{q-1} u(s)ds. \tag{4.7}$$

Now an application of Lemma 2.7 to the equation (4.7) yields (4.6).

5 APPLICATION:

Finally, we give an example to illustrate the existence and uniqueness results obtained in this paper. Let us consider the fractional initial value problem of the form:

$$\frac{d^q x}{dt^q} = \frac{t}{20} x + \frac{Lt^2}{(1+t^2)} x(t), \quad t \in J = [0,1], \quad 0 < q < 1, \tag{5.1}$$

$$x(0) + \sum_{i=1}^{m} c_i x(t_i) = 0, \tag{5.2}$$

where $L > 0$.

Set

$$f(t,x) = \frac{Lt^2}{(1+t^2)} x(t), \quad (t,x) \in J \times R^+ = [0, \infty),$$

and
$$A(t) = \frac{t}{20}.$$

Let $x, y \in R^+$ and $t \in J$. Then we have

$$\begin{aligned} \mid f(t,x) - f(t,y) \mid &= \mid \frac{Lt^2}{(1+t^2)} x(t) - \frac{Lt^2}{(1+t^2)} y(t) \mid \\ &= \frac{Lt^2}{(1+t^2)} \mid x - y \mid \\ &\le L \mid x - y \mid. \end{aligned}$$

Hence the hypotheses $(H_1)-(H_2)$ holds with $K = L$. Also we have $M = \frac{1}{20}$. Further we have the function $g(x) = \sum_{i=1}^{m} c_i x(t_i)$ and the constant $G_1 = \sum_{i=1}^{m} c_i$ is selected properly by choosing c_i and consequently (H_3) satisfied. Therefore we ca easily verify the results.

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