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A FINITE CAPACITY ERLANG QUEUE

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ABSTRACT

***A** Simple, efficient, numerically stable algorithm to the analysis of a finite Capacity Erlang queue is given in terms of generalised inverses. Using this algorithm the stationary distribution and mean first passage time distributions are obtained. Some performance measures are calculated. Some particular models are also given.*

Keywords: *Generalised inverses–Erlang queue–Stationary distribution–Mean first passage time.*

AMS Subject classification: *60K25, 60K20, 90B22, 15A51, 65F20.*

1. INTRODUCTION:

In this paper we consider a single server finite queue of Erlang distributed inter arrival and service time with mean $\frac{1}{\lambda}$ and $\frac{1}{\mu}$ respectively. An arriving customer passes a series of k independent arrival stages and r independent service stages. Each stage is exponentially distributed. The size of the waiting line is N-1. In this paper this model has been analysed using the method of group generalised inverses. Hunter (1969) established that a square matrix G possesses group inverse if rank of G is equal to rank of G^2. Some notable works in this area are Adi-Ben.Israel and Greville(1974), Boullion and Odell(1971) and Campbell and Meyer(1979). Meyer (1975) gave a formula for a group inverses of an infinitesimal generator of an m state ergodic processes. Using the group inverse he obtained the fixed probability vector. Kemney and snell (1960) and Hunter (1969) have obtained the mean first passage time matrix for m state ergodic processes. Based on these results the queuing model defined in this section has been analysed by proposing an algorithm. In section 2, the mathematical model has been given and the steps of the algorithm have been given. In section 3, the model is introduced and analysed by defining the underlying Markov chain and the infinitesimal generator. The formula for the probabilities in steady state and the mean first passage time are given in matrix form. The operating characteristics (i) mean number of customers in the queue (ii) mean number of customers in the service (iii) mean number of customers in the system (iv) effective rate of arrival (v) system effectiveness (vi) Mean waiting time in the system (vii) the mean waiting time in the queue and (viii) mean waiting time in the service facility have been obtained in section 4. In section 5, some particular models are presented by assuming particular values to λ and μ. In the last section, a conclusion has been given.

2. THE METHOD:

Let $\{(X_t, Y_t), t \geq 0\}$ be a Markov process on the state space $\{(n, j): 0 \leq n \leq N, 1 \leq j \leq a_n\}$ with a block tri –diagonal infinitesimal generator

$$Q = \begin{pmatrix} B_0 & A_0 & 0 & 0 & \dots & \dots & 0 & 0 \\ C_1 & B_1 & B_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & C_2 & B_2 & A_2 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & .,. & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \dots & C_{N-1} & B_{N-1} & A_{N-1} \\ 0 & 0 & 0 & \dots & \dots. & 0 & C_N & B_N \end{pmatrix}$$

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where $B_0, B_1, \ldots B_N$ are square matrices of order $a_0, a_1, \ldots, a_N$ respectively. Their diagonal elements are strictly negative; all other elements are non -negative. The matrices $A_n : 0 \le n \le N-1$ and $C_n : 1 \le n \le N$ are rectangular with appropriate dimensions and their entries are non-negative. The row sums of Q are equal to 0. Therefore we have,

$$B_0 e + A_0 e = 0$$

$$C_i e + B_i e + A_i e = 0: 1 \le i \le N-1$$

$$C_N e + B_N e = 0$$

Where e denotes the column vector with unit elements. The variable Y_t is to be interpreted as the environment state and X_t the state of the process at time t.

For the determination of the stationary probability distribution and moments of first passage time, the following realization of Markov process is useful. Observe the process Q during the interval of time spent at level n, before the original process enters to level n+1 for the first time. Denote P_n, the realization of the original process. The state of P_n is $S_n = \{\{n, j): 1 \le j \le a_n\}$ Clearly all P_n, $0 \le n \le N-1$ are transient Markov processes. The process P_N is the realization of process Q in the states $\{ (N, j): 1 \le j \le a_N\}$, it is an ergodic Markov process. Denote Q_n as the infinitesimal generator of the process P_n, $0 \le n \le N$. The purpose of this paper is to apply the technique of generalised inverses in the above defined process.

Using the construction the following algorithm has been proposed for solving the queuing model defined in this paper.

Step: 1 Write $Q = \begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$ where U is (m-1) × (m-1) matrix.

Step: 2 Test for rank (Q) = rank (Q^2) so that $Q^\#$, the group inverse of Q exists.

Step: 3 Calculate $h' = d' U^{-1}$

$$\beta = 1 - h' j \text{ where } \beta \text{ is non – zero.}$$

Step: 4 $W' = \frac{1}{\beta}[-h', 1]$

Step: 5 $QQ^\# = I - W$ where $W = \begin{pmatrix} W' \\ W' \\ \ldots \\ \ldots \\ W' \end{pmatrix}'$

Step: 6 Calculate $Q^\#$ as $Q^\# = QQ^\# \begin{pmatrix} U^{-1} & 0 \\ 0 & 0 \end{pmatrix} QQ^\#$

Step: 7 Calculate M (Mean First passage time matrix)

$$M = [I - Q^\# + JQ^\#_{dg}]\, D \text{ where } D = \left(\frac{1}{W_{ii}} \right)$$

3. THE MATHEMATICAL MODEL AND ANALYSIS:

Consider the queuing model defined in section 1. Suppose k = r = 2 and N = 2. The Markov process related to the model is $\{X(t)=(N(t), J(t), S(t)): t \ge 0\}$where N(t) denotes the number of customers in the system at time t, J(t), the arrival stage of customers and S(t), the service stage of the customer who is in service. In steady state, the underlying Markov chain has the state space (0, 1, 0),...,(2, 2, 2). Within stages the arrival process is Poisson with mean 2λ and inter service time of the stage is negative exponential with mean $\frac{1}{\mu}$. For the stability, we require that, the fraction of the service facilities in waiting is $\rho = \frac{\lambda}{\mu} < 1$. The infinitesimal generator for the system is

$Q = [Q_1, Q_2]$ where

$$Q_1 = \begin{pmatrix} -2\lambda & 2\lambda & 0 & 0 & 0 \\ 0 & -2\lambda & 0 & 2\lambda & 0 \\ 2\mu & 0 & -2(\lambda+\mu) & 0 & 2\lambda \\ 0 & 0 & 2\mu & -2(\lambda+\mu) & 0 \\ 0 & 2\mu & 0 & 0 & -2(\lambda+\mu) \\ 0 & 0 & 0 & 0 & 2\mu \\ 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & 0 & 0 \\ 0 & 2\lambda & 0 & 0 & 0 \\ -2(\lambda+\mu) & 0 & 2\lambda & 0 & 0 \\ 0 & -2(\lambda+\mu) & 0 & 2\lambda & 0 \\ 0 & 2\mu & -2(\lambda+\mu) & 0 & 2\lambda \\ 2\mu & 2\lambda & 0 & -2(\lambda+\mu) & 0 \\ 0 & 0 & 2\lambda & 2\mu & -2(\lambda+\mu) \end{pmatrix}$$

Let $R = -Q$ and $R = [R_1, R_2]$ where

$$R_1 = -Q_1 = \begin{pmatrix} 2\lambda & -2\lambda & 0 & 0 & 0 \\ 0 & 2\lambda & 0 & -2\lambda & 0 \\ -2\mu & 0 & 2(\lambda+\mu) & 0 & -2\lambda \\ 0 & 0 & -2\mu & 2(\lambda+\mu) & 0 \\ 0 & -2\mu & 0 & 0 & 2(\lambda+\mu) \\ 0 & 0 & 0 & 0 & -2\mu \\ 0 & 0 & 0 & -2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_2 = -Q_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 & 0 \\ 0 & -2\lambda & 0 & 0 & 0 \\ 2(\lambda+\mu) & 0 & -2\lambda & 0 & 0 \\ 0 & 2(\lambda+\mu) & 0 & -2\lambda & 0 \\ 0 & -2\mu & 2(\lambda+\mu) & 0 & -2\lambda \\ -2\mu & -2\lambda & 0 & 2(\lambda+\mu) & 0 \\ 0 & 0 & -2\lambda & -2\mu & 2(\lambda+\mu) \end{pmatrix}$$

In order to apply the algorithm of section 2, take R = $\begin{pmatrix} U & c \\ d' & \alpha \end{pmatrix}$

The inverse matrix of U is obtained in block form using block triangular decomposition method

Let U = $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A and D are invertible.

$$\text{Let } A = \begin{pmatrix} 2\lambda & -2\lambda & 0 & 0 \\ 0 & 2\lambda & 0 & -2\lambda \\ -2\mu & 0 & 2(\lambda+\mu) & 0 \\ 0 & 0 & -2\mu & 2(\lambda+\mu) \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 & 0 \\ 0 & -2\lambda & 0 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2(\lambda+\mu) & 0 & -2\lambda & 0 & 0 \\ -2\mu & 2(\lambda+\mu) & 0 & -2\lambda & 0 \\ 0 & 0 & 2(\lambda+\mu) & 0 & -2\lambda \\ 0 & 0 & -2\mu & 2(\lambda+\mu) & 0 \\ 0 & -2\mu & -2\lambda & 0 & 2(\lambda+\mu) \end{pmatrix}$$

$$C = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2\lambda \\ 0 \end{pmatrix}$$

$$d' = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\lambda & -2\mu \end{pmatrix}$$

$$\alpha = \big(2(\lambda+\mu)\big)$$

$$\text{let} \quad U^{-1} = \begin{pmatrix} U_D^{-1} & -A^{-1}BU_A^{-1} \\ -D^{-1}CU_D^{-1} & U_A^{-1} \end{pmatrix}$$

Now $U^{-1} = (u_{rc})_{9\times 9}$ is given by

Let

$$K_1 = 512\mu^{12}\lambda^2 + 5376\mu^{11}\lambda^3 + 25472\mu^{10}\lambda^4 + 71232\mu^9\lambda^5 + 129952\mu^8\lambda^6 + 162544\mu^7\lambda^7 + 143080\mu^6\lambda^8 + 89548\mu^5\lambda^9 + 39676\mu^4\lambda^{10} + 12176\mu^3\lambda^{11} + 2464\mu^2\lambda^{12} + 296\mu\lambda^{13} + 16\lambda^{14}$$

$$K_2 = 576\mu^{14}\lambda + 5568\mu^{13}\lambda^2 + 24688\mu^{12}\lambda^3 + 65376\mu^{11}\lambda^4 + 114288\mu^{10}\lambda^5 + 140188\mu^9\lambda^6 + 123289\mu^8\lambda^7 + 79076\mu^7\lambda^8 + 37072\mu^6\lambda^9 + 12574\mu^5\lambda^{10} + 3005\mu^4\lambda^{11} + 480\mu^3\lambda^{12} + 46\mu^2\lambda^{13} + 2\mu\lambda^{14}$$

$$K_3 = 1024\mu^{13}\lambda^2 + 11264\mu^{12}\lambda^3 + 56320\mu^{11}\lambda^4 + 167936\mu^{10}\lambda^5 + 331136\mu^9\lambda^6 + 455040\mu^8\lambda^7 + 448704\mu^7\lambda^8 + 322176\mu^6\lambda^9 + 168900\mu^5\lambda^{10} + 64028\mu^4\lambda^{11} + 17104\mu^3\lambda^{12} + 3056\mu^2\lambda^{13} + 328\mu\lambda^{14} + 16\lambda^{15}$$

$$K_4 = 2\mu^{23}\lambda^5 + 62\mu^{22}\lambda^6 + 904\mu^{21}\lambda^7 + 8256\mu^{20}\lambda^8 + 53062\mu^{19}\lambda^9 + 255646\mu^{18}\lambda^{10} + 960512\mu^{17}\lambda^{11} + 2888832\mu^{16}\lambda^{12} + 7091706\mu^{15}\lambda^{13} + 14406434\mu^{14}\lambda^{14} + 24468152\mu^{13}\lambda^{15} + 35002784\mu^{12}\lambda^{16} + 42384274\mu^{11}\lambda^{17} + 43556042\mu^{10}\lambda^{18} + 37997872\mu^9\lambda^{19} + 28075328\mu^8\lambda^{20} + 17472492\mu^7\lambda^{21} + 9071928\mu^6\lambda^{22} + 3870640\mu^5\lambda^{23} + 1325664\mu^4\lambda^{24} + 351296\mu^3\lambda^{25} + 67712\mu^2\lambda^{26} + 8448\mu\lambda^{27} + 512\lambda^{28}$$

$$K_5 = 2304\mu^{15}\lambda + 23424\mu^{14}\lambda^2 + 109888\mu^{13}\lambda^3 + 310880\mu^{12}\lambda^4 + 587904\mu^{11}\lambda^5 + 789328\mu^{10}\lambda^6 + 773532\mu^9\lambda^7 + 562882\mu^8\lambda^8 + 306440\mu^7\lambda^9 + 124440\mu^6\lambda^{10} + 37168\mu^5\lambda^{11} + 7930\mu^4\lambda^{12} + 1144\mu^3\lambda^{13} + 100\mu^2\lambda^{14} + 4\mu\lambda^{15}$$

$$U_{11} = \frac{1}{K_4}\{\mu^{26}\lambda + 32\mu^{25}\lambda^2 + 480\mu^{24}\lambda^3 + 4497\mu^{23}\lambda^4 + 29595\mu^{22}\lambda^5 + 145956\mu^{21}\lambda^6 + 562288\mu^{20}\lambda^7 + 1742707\mu^{19}\lambda^8 + 4444501\mu^{18}\lambda^9 + 9500176\mu^{17}\lambda^{10} + 17287152\mu^{16}\lambda^{11} + 27143085\mu^{15}\lambda^{12} + 37199213\mu^{14}\lambda^{13} + 44909052\mu^{13}\lambda^{14} + 48060160\mu^{12}\lambda^{15} + 45719481\mu^{11}\lambda^{16} + 38623654\mu^{10}\lambda^{17} + 28840728\mu^9\lambda^{18} + 18881424\mu^8\lambda^{19} + 10714614\mu^7\lambda^{20} + 5191420\mu^6\lambda^{21} + 2105304\mu^5\lambda^{22} + 695344\mu^4\lambda^{23} + 179744\mu^3\lambda^{24} + 34112\mu^2\lambda^{25} + 4224\mu\lambda^{26} + 256\lambda^{27}\}$$

$$U_{12} = \frac{1}{K_4}\{\mu^{26}\lambda + 34\mu^{25}\lambda^2 + 542\mu^{24}\lambda^3 + 5398\mu^{23}\lambda^4 + 37770\mu^{22}\lambda^5 + 198004\mu^{21}\lambda^6 + 810099\mu^{20}\lambda^7 + 2661072\mu^{19}\lambda^8 + 7167032\mu^{18}\lambda^9 + 16085168\mu^{17}\lambda^{10} + 30473036\mu^{16}\lambda^{11} + 49241736\mu^{15}\lambda^{12} + 68443394\mu^{14}\lambda^{13} + 82376120\mu^{13}\lambda^{14} + 86280743\mu^{12}\lambda^{15} + 78905688\mu^{11}\lambda^{16} + 63099359\mu^{10}\lambda^{17} + 44093678\mu^9\lambda^{18} + 26837724\mu^8\lambda^{19} + 14134938\mu^7\lambda^{20} + 6372844\mu^6\lambda^{21} + 2419220\mu^5\lambda^{22} + 754288\mu^4\lambda^{23} + 185984\mu^3\lambda^{24} + 34112\mu^2\lambda^{25} + 4160\mu\lambda^{26} + 256\lambda^{27}\}$$

$$U_{13} = \frac{1}{K_4}\{\mu^{25}\lambda^2 + 32\mu^{24}\lambda^3 + 480\mu^{23}\lambda^4 + 4496\mu^{22}\lambda^5 + 29564\mu^{21}\lambda^6 + 145504\mu^{20}\lambda^7 + 558160\mu^{19}\lambda^8 + 1716176\mu^{18}\lambda^9 + 4316678\mu^{17}\lambda^{10} + 9020000\mu^{16}\lambda^{11} + 15842736\mu^{15}\lambda^{12} + 23597232\mu^{14}\lambda^{13} + 29995996\mu^{13}\lambda^{14} + 32674976\mu^{12}\lambda^{15} + 30558768\mu^{11}\lambda^{16} + 24527344\mu^{10}\lambda^{17} + 16845633\mu^9\lambda^{18} + 9841792\mu^8\lambda^{19} + 4843760\mu^7\lambda^{20} + 978368\mu^6\lambda^{21} + 655456\mu^5\lambda^{22} + 169984\mu^4\lambda^{23} + 32512\mu^3\lambda^{24} + 4096\mu^2\lambda^{25} + 256\mu\lambda^{26}\}$$

$$U_{14} = \frac{1}{K_4}\{\mu^{25}\lambda^2 + 33\mu^{24}\lambda^3 + 512\mu^{23}\lambda^4 + 4976\mu^{22}\lambda^5 + 34060\mu^{21}\lambda^6 + 175068\mu^{20}\lambda^7 + 703664\mu^{19}\lambda^8$$
$$+2274336\mu^{18}\lambda^9 + 6032854\mu^{17}\lambda^{10} + 13336678\mu^{16}\lambda^{11} + 24862736\mu^{15}\lambda^{12} + 39439968\mu^{14}\lambda^{13}$$
$$+53593228\mu^{13}\lambda^{14} + 62670972\mu^{12}\lambda^{15} + 63233744\mu^{11}\lambda^{16} + 55086112\mu^{10}\lambda^{17} + 41372977\mu^{9}\lambda^{18}$$
$$+26687425\mu^{8}\lambda^{19} + 14685552\mu^{7}\lambda^{20} + 6822128\mu^{6}\lambda^{21} + 2633824\mu^{5}\lambda^{22} + 825440\mu^{4}\lambda^{23}$$
$$+202496\mu^{3}\lambda^{24} + 36608\mu^{2}\lambda^{25} + 4352\mu\lambda^{26} + 256\lambda^{27}\}$$

$$U_{15} = \frac{1}{K_3}\{1024\mu^{14} + 11264\mu^{13}\lambda + 54784\mu^{12}\lambda^2 + 157184\mu^{11}\lambda^3 + 298880\mu^{10}\lambda^4 + 399744\mu^{9}\lambda^5$$
$$+388224\mu^{8}\lambda^6 + 277824\mu^{7}\lambda^7 + 146724\mu^{6}\lambda^8 + 56540\mu^{5}\lambda^9 + 15472\mu^{4}\lambda^{10} + 2846\mu^{3}\lambda^{11} + 316\mu^{2}\lambda^{12}$$
$$+16\mu\lambda^{13}\}$$

$$U_{16} = \frac{1}{K_3}\{512\mu^{14} + 6144\mu^{13}\lambda + 33792\mu^{12}\lambda^2 + 112128\mu^{11}\lambda^3 + 249536\mu^{10}\lambda^4 + 393088\mu^{9}\lambda^5$$
$$+451872\mu^{8}\lambda^6 + 385440\mu^{7}\lambda^7 + 245538\mu^{6}\lambda^8 + 116464\mu^{5}\lambda^9 + 40566\mu^{4}\lambda^{10} + 10080\mu^{3}\lambda^{11}$$
$$+1692\mu^{2}\lambda^{12} + 172\mu\lambda^{13} + 8\lambda^{14}\}$$

$$U_{17} = \frac{1}{K_5}\{3456\mu^{15} + 31680\mu^{14}\lambda + 133152\mu^{13}\lambda^2 + 340080\mu^{12}\lambda^3 + 589512\mu^{11}\lambda^4 + 734100\mu^{10}\lambda^5$$
$$+677190\mu^{9}\lambda^6 + 470013\mu^{8}\lambda^7 + 246679\mu^{7}\lambda^8 + 97313\mu^{6}\lambda^9 + 28297\mu^{5}\lambda^{10} + 5790\mu^{4}\lambda^{11} + 755\mu^{3}\lambda^{12}$$
$$+68\mu^{2}\lambda^{13} + 3\mu\lambda^{14}\}$$

$$U_{18} = 1$$

$$U_{19} = \frac{1}{K_5}\{3456\mu^{14}\lambda + 28224\mu^{13}\lambda^2 + 104928\mu^{12}\lambda^3 + 235152\mu^{11}\lambda^4 + 354504\mu^{10}\lambda^5 + 379740\mu^{9}\lambda^6$$
$$+297450\mu^{8}\lambda^7 + 172563\mu^{7}\lambda^8 + 74172\mu^{6}\lambda^9 + 23337\mu^{5}\lambda^{10} + 5226\mu^{4}\lambda^{11} + 789\mu^{3}\lambda^{12} + 72\mu^{2}\lambda^{13} + 3\mu\lambda^{14}\}$$

$$U_{21} = \frac{1}{K_4}\{\mu^{26}\lambda + 32\mu^{25}\lambda^2 + 480\mu^{24}\lambda^3 + 4496\mu^{23}\lambda^4 + 29564\mu^{22}\lambda^5 + 145504\mu^{21}\lambda^6 + 558160\mu^{20}\lambda^7$$
$$+1716176\mu^{19}\lambda^8 + 4316678\mu^{18}\lambda^9 + 9020000\mu^{17}\lambda^{10} + 15842736\mu^{16}\lambda^{11} + 23597232\mu^{15}\lambda^{12}$$
$$+29995996\mu^{14}\lambda^{13} + 32674976\mu^{13}\lambda^{14} + 30558768\mu^{12}\lambda^{15} + 24527344\mu^{11}\lambda^{16} + 16845633\mu^{10}\lambda^{17}$$
$$+9841792\mu^{9}\lambda^{18} + 4843760\mu^{8}\lambda^{19} + 1978368\mu^{7}\lambda^{20} + 655456\mu^{6}\lambda^{21} + 169984\mu^{5}\lambda^{22} + 32512\mu^{4}\lambda^{23}$$
$$+4096\mu^{3}\lambda^{24} + 256\mu^{2}\lambda^{25}\}$$

$$U_{22} = \frac{1}{K_4}\{\mu^{26}\lambda + 34\mu^{25}\lambda^2 + 542\mu^{24}\lambda^3 + 5398\mu^{23}\lambda^4 + 37770\mu^{22}\lambda^5 + 198004\mu^{21}\lambda^6 + 810098\mu^{20}\lambda^7$$
$$+2661056\mu^{19}\lambda^8 + 7166920\mu^{18}\lambda^9 + 16084712\mu^{17}\lambda^{10} + 30471822\mu^{16}\lambda^{11} + 49239476\mu^{15}\lambda^{12}$$
$$+68440282\mu^{14}\lambda^{13} + 82372612\mu^{13}\lambda^{14} + 86276678\mu^{12}\lambda^{15} + 78900000\mu^{11}\lambda^{16} + 63091655\mu^{10}\lambda^{17}$$
$$+44086214\mu^{9}\lambda^{18} + 26834476\mu^{8}\lambda^{19} + 14137910\mu^{7}\lambda^{20} + 6379980\mu^{6}\lambda^{21} + 2426424\mu^{5}\lambda^{22}$$
$$+758896\mu^{4}\lambda^{23} + 187936\mu^{3}\lambda^{24} + 34624\mu^{2}\lambda^{25} + 4224\mu\lambda^{26} + 256\lambda^{27}\}$$

$$U_{23}=\frac{1}{K_4}\{\mu^{25}\lambda^2+32\mu^{24}\lambda^3+480\mu^{23}\lambda^4+4496\mu^{22}\lambda^5+29564\mu^{21}\lambda^6+145504\mu^{20}\lambda^7+558160\mu^{19}\lambda^8$$
$$+1716176\mu^{18}\lambda^9+4316678\mu^{17}\lambda^{10}+9020000\mu^{16}\lambda^{11}+15842736\mu^{15}\lambda^{12}+23597232\mu^{14}\lambda^{13}$$
$$+29995996\mu^{13}\lambda^{14}+32674976\mu^{12}\lambda^{15}+30558768\mu^{11}\lambda^{16}+24527344\mu^{10}\lambda^{17}+16845633\mu^9\lambda^{18}$$
$$+9841792\mu^8\lambda^{19}+4843760\mu^7\lambda^{20}+1978368\mu^6\lambda^{21}+655456\mu^5\lambda^{22}+169984\mu^4\lambda^{23}+32512\mu^3\lambda^{24}$$
$$+4096\mu^2\lambda^{25}+256\mu\lambda^{26}\}$$

$$U_{24}=\frac{1}{K_4}\{\mu^{25}\lambda^2+33\mu^{24}\lambda^3+512\mu^{23}\lambda^4+4976\mu^{22}\lambda^5+34060\mu^{21}\lambda^6+175068\mu^{20}\lambda^7+703664\mu^{19}\lambda^8$$
$$+2083836\mu^{18}\lambda^9+4508854\mu^{17}\lambda^{10}+8764678\mu^{16}\lambda^{11}+18004736\mu^{15}\lambda^{12}+33534468\mu^{14}\lambda^{13}$$
$$+50545228\mu^{13}\lambda^{14}+61908972\mu^{12}\lambda^{15}+63233744\mu^{11}\lambda^{16}+55086112\mu^{10}\lambda^{17}+41372977\mu^9\lambda^{18}$$
$$+26687425\mu^8\lambda^{19}+14685552\mu^7\lambda^{20}+6822128\mu^6\lambda^{21}+2633824\mu^5\lambda^{22}+825440\mu^4\lambda^{23}+202496\mu^3\lambda^{24}$$
$$+36608\mu^2\lambda^{25}+4352\mu\lambda^{26}+256\lambda^{27}\}$$

$$U_{25}=\frac{1}{K_3}\{1024\mu^{14}+11264\mu^{13}\lambda+54784\mu^{12}\lambda^2+157184\mu^{11}\lambda^3+298880\mu^{10}\lambda^4+399744\mu^9\lambda^5$$
$$+388224\mu^8\lambda^6+277824\mu^7\lambda^7+146724\mu^6\lambda^8+56540\mu^5\lambda^9+15472\mu^4\lambda^{10}+2846\mu^3\lambda^{11}+316\mu^2\lambda^{12}$$
$$+16\mu\lambda^{13}\}$$

$$U_{26}=\frac{1}{K_3}\{512\mu^{14}+6144\mu^{13}\lambda+33792\mu^{12}\lambda^2+112128\mu^{11}\lambda^3+249536\mu^{10}\lambda^4+393088\mu^9\lambda^5$$
$$+451872\mu^8\lambda^6+385440\mu^7\lambda^7+245538\mu^6\lambda^8+116464\mu^5\lambda^9+40566\mu^4\lambda^{10}+10080\mu^3\lambda^{11}$$
$$+1692\mu^2\lambda^{12}+172\mu\lambda^{13}+8\lambda^{14}\}$$

$$U_{27}=\frac{1}{K_5}\{3456\mu^{15}+31680\mu^{14}\lambda+133152\mu^{13}\lambda^2+340080\mu^{12}\lambda^3+589512\mu^{11}\lambda^4+734100\mu^{10}\lambda^5$$
$$+677190\mu^9\lambda^6+470013\mu^8\lambda^7+246679\mu^7\lambda^8+97313\mu^6\lambda^9+28297\mu^5\lambda^{10}+5790\mu^4\lambda^{11}$$
$$+755\mu^3\lambda^{12}+68\mu^2\lambda^{13}+3\mu\lambda^{14}\}$$

$$U_{28}=1$$

$$U_{29}=\frac{1}{K_5}\{3456\mu^{14}\lambda+28224\mu^{13}\lambda^2+104928\mu^{12}\lambda^3+235152\mu^{11}\lambda^4+354504\mu^{10}\lambda^5+379740\mu^9\lambda^6$$
$$+297450\mu^8\lambda^7+172563\mu^7\lambda^8+74172\mu^6\lambda^9+23337\mu^5\lambda^{10}+5226\mu^4\lambda^{11}+789\mu^3\lambda^{12}+72\mu^2\lambda^{13}+3\mu\lambda^{14}\}$$

$$U_{31}=\frac{1}{K_4}\{\mu^{26}\lambda+32\mu^{25}\lambda^2+480\mu^{24}\lambda^3+4497\mu^{23}\lambda^4+29594\mu^{22}\lambda^5+145925\mu^{21}\lambda^6+561840\mu^{20}\lambda^7$$
$$+1738689\mu^{19}\lambda^8+4419376\mu^{18}\lambda^9+9383477\mu^{17}\lambda^{10}+16868204\mu^{16}\lambda^{11}+25948563\mu^{15}\lambda^{12}$$
$$+34440670\mu^{14}\lambda^{13}+39674087\mu^{13}\lambda^{14}+39808344\mu^{12}\lambda^{15}+34832819\mu^{11}\lambda^{16}+26543751\mu^{10}\lambda^{17}$$
$$+17542463\mu^9\lambda^{18}+9980652\mu^8\lambda^{19}+4832872\mu^7\lambda^{20}+1958480\mu^6\lambda^{21}+647936\mu^5\lambda^{22}+168512\mu^4\lambda^{23}$$
$$+32384\mu^3\lambda^{24}+4096\mu^2\lambda^{25}+256\mu\lambda^{26}\}$$

$$U_{32} = \frac{1}{K_4}\{ \mu^{26}\lambda + 34\mu^{25}\lambda^2 + 542\mu^{24}\lambda^3 + 5398\mu^{23}\lambda^4 + 37770\mu^{22}\lambda^5 + 198005\mu^{21}\lambda^6 + 810066\mu^{20}\lambda^7$$
$$+ 2660398\mu^{19}\lambda^8 + 7161128\mu^{18}\lambda^9 + 16051579\mu^{17}\lambda^{10} + 30332926\mu^{16}\lambda^{11} + 48790422\mu^{15}\lambda^{12}$$
$$+ 67287518\mu^{14}\lambda^{13} + 79975918\mu^{13}\lambda^{14} + 82182718\mu^{12}\lambda^{15} + 73094658\mu^{11}\lambda^{16} + 56210667\mu^{10}\lambda^{17}$$
$$+37244480\mu^9\lambda^{18} + 21126268\mu^8\lambda^{19} + 10154744\mu^7\lambda^{20} + 4072848\mu^6\lambda^{21} + 1331680\mu^5\lambda^{22}$$
$$+342336\mu^4\lambda^{23} + 65152\mu^3\lambda^{24} + 8192\mu^2\lambda^{25} + 512\mu\lambda^{26}\}$$

$$U_{33} = \frac{1}{K_4}\{ \mu^{25}\lambda^2 + 32\mu^{24}\lambda^3 + 480\mu^{23}\lambda^4 + 4497\mu^{22}\lambda^5 + 29594\mu^{21}\lambda^6 + 145925\mu^{20}\lambda^7 + 561840\mu^{19}\lambda^8$$
$$+1738689\mu^{18}\lambda^9 + 4419376\mu^{17}\lambda^{10} + 9383477\mu^{16}\lambda^{11} + 16868204\mu^{15}\lambda^{12} + 25948563\mu^{14}\lambda^{13}$$
$$+34440670\mu^{13}\lambda^{14} + 39674087\mu^{12}\lambda^{15} + 39808344\mu^{11}\lambda^{16} + 34832819\mu^{10}\lambda^{17} + 26543751\mu^9\lambda^{18}$$
$$+17542463\mu^8\lambda^{19} + 9980652\mu^7\lambda^{20} + 4832872\mu^6\lambda^{21} + 1958480\mu^5\lambda^{22} + 647936\mu^4\lambda^{23} + 168512\mu^3\lambda^{24}$$
$$+32384\mu^2\lambda^{25} + 4096\mu\lambda^{26} + 256\lambda^{27}\}$$

$$U_{34} = \frac{1}{K_4}\{ \mu^{25}\lambda^2 + 33\mu^{24}\lambda^3 + 512\mu^{23}\lambda^4 + 4976\mu^{22}\lambda^5 + 34060\mu^{21}\lambda^6 + 175067\mu^{20}\lambda^7 + 703637\mu^{19}\lambda^8$$
$$+2273998\mu^{18}\lambda^9 + 6030602\mu^{17}\lambda^{10} + 13325197\mu^{16}\lambda^{11} + 24813025\mu^{15}\lambda^{12} + 39274916\mu^{14}\lambda^{13}$$
$$+53181608\mu^{13}\lambda^{14} + 61866945\mu^{12}\lambda^{15} + 61961485\mu^{11}\lambda^{16} + 53424856\mu^{10}\lambda^{17} + 39565317\mu^9\lambda^{18}$$
$$+25046318\mu^8\lambda^{19} + 13447733\mu^7\lambda^{20} + 6052022\mu^6\lambda^{21} + 2243444\mu^5\lambda^{22} + 667064\mu^4\lambda^{23} + 152592\mu^3\lambda^{24}$$
$$+25120\mu^2\lambda^{25} + 2624\mu\lambda^{26} + 128\lambda^{27}\}$$

$$U_{35} = \frac{1}{K_3}\{1024\mu^{14} + 11264\mu^{13}\lambda + 54784\mu^{12}\lambda^2 + 157696\mu^{11}\lambda^3 + 303488\mu^{10}\lambda^4 + 417664\mu^9\lambda^5$$
$$+428160\mu^8\lambda^6 + 334848\mu^7\lambda^7 + 201828\mu^6\lambda^8 + 93500\mu^5\lambda^9 + 32746\mu^4\lambda^{10} + 8384\mu^3\lambda^{11}$$
$$+1478\mu^2\lambda^{12} + 160\mu\lambda^{13} + 8\lambda^{14}\}$$

$$U_{36} = \frac{1}{K_3}\{512\mu^{14} + 6144\mu^{13}\lambda + 33792\mu^{12}\lambda^2 + 112128\mu^{11}\lambda^3 + 249536\mu^{10}\lambda^4 + 393088\mu^9\lambda^5$$
$$+451872\mu^8\lambda^6 + 385440\mu^7\lambda^7 + 245538\mu^6\lambda^8 + 116464\mu^5\lambda^9 + 40566\mu^4\lambda^{10} + 10080\mu^3\lambda^{11}$$
$$+1692\mu^2\lambda^{12} + 172\mu\lambda^{13} + 8\lambda^{14}\}$$

$$U_{37} = \frac{1}{K_5}\{3456\mu^{15} + 31680\mu^{14}\lambda + 133152\mu^{13}\lambda^2 + 341232\mu^{12}\lambda^3 + 597768\mu^{11}\lambda^4 + 760964\mu^{10}\lambda^5$$
$$+ 728854\mu^9\lambda^6 + 536517\mu^8\lambda^7 + 306755\mu^7\lambda^8 + 136387\mu^6\lambda^9 + 46800\mu^5\lambda^{10} + 12151\mu^4\lambda^{11} + 2349\mu^3\lambda^{12}$$
$$+315\mu^2\lambda^{13} + 26\mu\lambda^{14} + \lambda^{15}\}$$

$$U_{38} = 1$$

$$U_{39} = \frac{1}{K_5}\{3456\mu^{14}\lambda + 28224\mu^{13}\lambda^2 + 104928\mu^{12}\lambda^3 + 236304\mu^{11}\lambda^4 + 361608\mu^{10}\lambda^5 + 399356\mu^9\lambda^6$$
$$+329498\mu^8\lambda^7 + 207019\mu^7\lambda^8 + 99792\mu^6\lambda^9 + 36791\mu^5\lambda^{10} + 10219\mu^4\lambda^{11} + 2073\mu^3\lambda^{12} + 290\mu^2\lambda^{13}$$
$$+25\mu\lambda^{14} + \lambda^{15}\}$$

$$U_{41}=\frac{1}{K_4}\{\mu^{26}\lambda+32\mu^{25}\lambda^2+480\mu^{24}\lambda^3+4496\mu^{23}\lambda^4+29564\mu^{22}\lambda^5+145504\mu^{21}\lambda^6+558160\mu^{20}\lambda^7$$
$$+1716176\mu^{19}\lambda^8+4316678\mu^{18}\lambda^9+9020000\mu^{17}\lambda^{10}+15842736\mu^{16}\lambda^{11}+23597232\mu^{15}\lambda^{12}$$
$$+29995996\mu^{14}\lambda^{13}+32674976\mu^{13}\lambda^{14}+30558768\mu^{12}\lambda^{15}+24527344\mu^{11}\lambda^{16}+16845633\mu^{10}\lambda^{17}$$
$$+9841792\mu^9\lambda^{18}+4843760\mu^8\lambda^{19}+1978368\mu^7\lambda^{20}+655456\mu^6\lambda^{21}+169984\mu^5\lambda^{22}+32512\mu^4\lambda^{23}$$
$$+4096\mu^3\lambda^{24}+256\mu^2\lambda^{25}\}$$

$$U_{42}=\frac{1}{K_4}\{\mu^{26}\lambda+34\mu^{25}\lambda^2+542\mu^{24}\lambda^3+5397\mu^{23}\lambda^4+37739\mu^{22}\lambda^5+197552\mu^{21}\lambda^6+805970\mu^{20}\lambda^7$$
$$+2634525\mu^{19}\lambda^8+7039097\mu^{18}\lambda^9+15604536\mu^{17}\lambda^{10}+29027406\mu^{16}\lambda^{11}+45693615\mu^{15}\lambda^{12}$$
$$+61236937\mu^{14}\lambda^{13}+70137640\mu^{13}\lambda^{14}+68771558\mu^{12}\lambda^{15}+57697511\mu^{11}\lambda^{16}+41293666\mu^{10}\lambda^{17}$$
$$+25060014\mu^9\lambda^{18}+12769820\mu^8\lambda^{19}+5381944\mu^7\lambda^{20}+1833456\mu^6\lambda^{21}+487072\mu^5\lambda^{22}+95040\mu^4\lambda^{23}$$
$$+12160\mu^3\lambda^{24}+768\mu^2\lambda^{25}\}$$

$$U_{43}=\frac{1}{K_4}\{\mu^{25}\lambda^2+33\mu^{24}\lambda^3+480\mu^{23}\lambda^4+4496\mu^{22}\lambda^5+29564\mu^{21}\lambda^6+145504\mu^{20}\lambda^7+558160\mu^{19}\lambda^8$$
$$+1716176\mu^{18}\lambda^9+4316678\mu^{17}\lambda^{10}+9020000\mu^{16}\lambda^{11}+15842736\mu^{15}\lambda^{12}+23597232\mu^{14}\lambda^{13}$$
$$+29995996\mu^{13}\lambda^{14}+32674976\mu^{12}\lambda^{15}+30558768\mu^{11}\lambda^{16}+24527344\mu^{10}\lambda^{17}+16845633\mu^9\lambda^{18}$$
$$+9841792\mu^8\lambda^{19}+4843760\mu^7\lambda^{20}+1978368\mu^6\lambda^{21}+655456\mu^5\lambda^{22}+169984\mu^4\lambda^{23}+32512\mu^3\lambda^{24}$$
$$+4096\mu^2\lambda^{25}+256\mu\lambda^{26}\}$$

$$U_{44}=\frac{1}{K_4}\{\mu^{25}\lambda^2+33\mu^{24}\lambda^3+512\mu^{23}\lambda^4+4976\mu^{22}\lambda^5+34060\mu^{21}\lambda^6+175068\mu^{20}\lambda^7+703664\mu^{19}\lambda^8$$
$$+2274336\mu^{18}\lambda^9+6032854\mu^{17}\lambda^{10}+13336678\mu^{16}\lambda^{11}+24862736\mu^{15}\lambda^{12}+39439968\mu^{14}\lambda^{13}$$
$$+53593228\mu^{13}\lambda^{14}+62670972\mu^{12}\lambda^{15}+63233744\mu^{11}\lambda^{16}+55086112\mu^{10}\lambda^{17}+41372977\mu^9\lambda^{18}$$
$$+26687425\mu^8\lambda^{19}+14685552\mu^7\lambda^{20}+6822128\mu^6\lambda^{21}+2633824\mu^5\lambda^{22}+825440\mu^4\lambda^{23}+202496\mu^3\lambda^{24}$$
$$+36608\mu^2\lambda^{25}+4352\mu\lambda^{26}+256\lambda^{27}\}$$

$$U_{45}=\frac{1}{K_3}\{1024\mu^{14}+11264\mu^{13}\lambda+54784\mu^{12}\lambda^2+157184\mu^{11}\lambda^3+298880\mu^{10}\lambda^4+399744\mu^9\lambda^5$$
$$+388224\mu^8\lambda^6+277824\mu^7\lambda^7+146724\mu^6\lambda^8+56540\mu^5\lambda^9+15472\mu^4\lambda^{10}+2846\mu^3\lambda^{11}+316\mu^2\lambda^{12}$$
$$+16\mu\lambda^{13}\}$$

$$U_{46}=\frac{1}{K_3}\{512\mu^{14}+6144\mu^{13}\lambda+33792\mu^{12}\lambda^2+112128\mu^{11}\lambda^3+249536\mu^{10}\lambda^4+393088\mu^9\lambda^5$$
$$+451872\mu^8\lambda^6+385440\mu^7\lambda^7+245538\mu^6\lambda^8+116464\mu^5\lambda^9+40566\mu^4\lambda^{10}+10080\mu^3\lambda^{11}$$
$$+1692\mu^2\lambda^{12}+172\mu\lambda^{13}+8\lambda^{14}\}$$

$$U_{47}=\frac{1}{K_5}\{3456\mu^{15}+31680\mu^{14}\lambda+133152\mu^{13}\lambda^2+340080\mu^{12}\lambda^3+589512\mu^{11}\lambda^4+734100\mu^{10}\lambda^5$$
$$+677190\mu^9\lambda^6+470013\mu^8\lambda^7+246679\mu^7\lambda^8+97313\mu^6\lambda^9+28297\mu^5\lambda^{10}+5790\mu^4\lambda^{11}+755\mu^3\lambda^{12}$$
$$+68\mu^2\lambda^{13}+3\mu\lambda^{14}\}$$

$$U_{48}=1$$

$$U_{49} = \frac{1}{K_5}\{3456\mu^{14}\lambda + 28224\mu^{13}\lambda^2 + 104928\mu^{12}\lambda^3 + 235152\mu^{11}\lambda^4 + 354504\mu^{10}\lambda^5 + 379740\mu^9\lambda^6 + 297450\mu^8\lambda^7 + 172563\mu^7\lambda^8 + 74172\mu^6\lambda^9 + 23337\mu^5\lambda^{10} + 5226\mu^4\lambda^{11} + 789\mu^3\lambda^{12} + 72\mu^2\lambda^{13} + 3\mu\lambda^{14}\}$$

$$U_{51} = \frac{1}{K_4}\{\mu^{26}\lambda + 32\mu^{25}\lambda^2 + 480\mu^{24}\lambda^3 + 4496\mu^{23}\lambda^4 + 29563\mu^{22}\lambda^5 + 145477\mu^{21}\lambda^6 + 557822\mu^{20}\lambda^7 + 1713564\mu^{19}\lambda^8 + 4302677\mu^{18}\lambda^9 + 8964529\mu^{17}\lambda^{10} + 15673674\mu^{16}\lambda^{11} + 23189892\mu^{15}\lambda^{12} + 29204809\mu^{14}\lambda^{13} + 31418543\mu^{13}\lambda^{14} + 28911330\mu^{12}\lambda^{15} + 22732948\mu^{11}\lambda^{16} + 15218222\mu^{10}\lambda^{17} + 8614699\mu^9\lambda^{18} + 4079190\mu^8\lambda^{19} + 1589372\mu^7\lambda^{20} + 497080\mu^6\lambda^{21} + 120080\mu^5\lambda^{22} + 21024\mu^4\lambda^{23} + 2368\mu^3\lambda^{24} + 128\mu^2\lambda^{25}\}$$

$$U_{52} = \frac{1}{K_4}\{\mu^{26}\lambda + 34\mu^{25}\lambda^2 + 542\mu^{24}\lambda^3 + 5398\mu^{23}\lambda^4 + 37768\mu^{22}\lambda^5 + 197946\mu^{21}\lambda^6 + 809312\mu^{20}\lambda^7 + 2654426\mu^{19}\lambda^8 + 7127794\mu^{18}\lambda^9 + 15912542\mu^{17}\lambda^{10} + 29883812\mu^{16}\lambda^{11} + 47637606\mu^{15}\lambda^{12} + 64890424\mu^{14}\lambda^{13} + 75880318\mu^{13}\lambda^{14} + 76373696\mu^{12}\lambda^{15} + 66208590\mu^{11}\lambda^{16} + 49364373\mu^{10}\lambda^{17} + 31533592\mu^9\lambda^{18} + 17142140\mu^8\lambda^{19} + 7847452\mu^7\lambda^{20} + 2978104\mu^6\lambda^{21} + 915120\mu^5\lambda^{22} + 219552\mu^4\lambda^{23} + 38720\mu^3\lambda^{24} + 4480\mu^2\lambda^{25} + 256\mu\lambda^{26}\}$$

$$U_{53} = \frac{1}{K_4}\{\mu^{25}\lambda^2 + 32\mu^{24}\lambda^3 + 480\mu^{23}\lambda^4 + 4496\mu^{22}\lambda^5 + 29563\mu^{21}\lambda^6 + 145477\mu^{20}\lambda^7 + 557822\mu^{19}\lambda^8 + 1713564\mu^{18}\lambda^9 + 4302677\mu^{17}\lambda^{10} + 8964529\mu^{16}\lambda^{11} + 15673674\mu^{15}\lambda^{12} + 23189892\mu^{14}\lambda^{13} + 29204809\mu^{13}\lambda^{14} + 31418543\mu^{12}\lambda^{15} + 28911330\mu^{11}\lambda^{16} + 22732948\mu^{10}\lambda^{17} + 15218222\mu^9\lambda^{18} + 8614699\mu^8\lambda^{19} + 4079190\mu^7\lambda^{20} + 1589372\mu^6\lambda^{21} + 497080\mu^5\lambda^{22} + 120080\mu^4\lambda^{23} + 21024\mu^3\lambda^{24} + 2368\mu^2\lambda^{25} + 128\mu\lambda^{26}\}$$

$$U_{54} = \frac{1}{K_4}\{\mu^{25}\lambda^2 + 33\mu^{24}\lambda^3 + 512\mu^{23}\lambda^4 + 4976\mu^{22}\lambda^5 + 34059\mu^{21}\lambda^6 + 175040\mu^{20}\lambda^7 + 703299\mu^{19}\lambda^8 + 2271386\mu^{18}\lambda^9 + 6016241\mu^{17}\lambda^{10} + 13267206\mu^{16}\lambda^{11} + 24638203\mu^{15}\lambda^{12} + 38863566\mu^{14}\lambda^{13} + 52394701\mu^{13}\lambda^{14} + 60623352\mu^{12}\lambda^{15} + 60329873\mu^{11}\lambda^{16} + 51644278\mu^{10}\lambda^{17} + 37951170\mu^9\lambda^{18} + 23832921\mu^8\lambda^{19} + 12693889\mu^7\lambda^{20} + 5668562\mu^6\lambda^{21} + 2086452\mu^5\lambda^{22} + 617160\mu^4\lambda^{23} + 141104\mu^3\lambda^{24} + 23392\mu^2\lambda^{25} + 2496\mu\lambda^{26} + 128\lambda^{27}\}$$

$$U_{55} = \frac{1}{K_1}\{512\mu^{13} + 5376\mu^{12}\lambda + 24960\mu^{11}\lambda^2 + 68672\mu^{10}\lambda^3 + 126368\mu^9\lambda^4 + 165616\mu^8\lambda^5 + 159784\mu^7\lambda^6 + 115084\mu^6\lambda^7 + 61852\mu^5\lambda^8 + 24464\mu^4\lambda^9 + 6910\mu^3\lambda^{10} + 1318\mu^2\lambda^{11} + 152\mu\lambda^{12} + 8\lambda^{13}\}$$

$$U_{56} = \frac{1}{K_1}\{256\mu^{13} + 2944\mu^{12}\lambda + 15424\mu^{11}\lambda^2 + 48352\mu^{10}\lambda^3 + 100592\mu^9\lambda^4 + 146248\mu^8\lambda^5 + 152812\mu^7\lambda^6 + 116314\mu^6\lambda^7 + 64612\mu^5\lambda^8 + 25926\mu^4\lambda^9 + 7320\mu^3\lambda^{10} + 1380\mu^2\lambda^{11} + 156\mu\lambda^{12} + 8\lambda^{13}\}$$

$$U_{57} = \frac{1}{2K_2}\{1728\mu^{14} + 14976\mu^{13}\lambda + 59664\mu^{12}\lambda^2 + 144912\mu^{11}\lambda^3 + 239860\mu^{10}\lambda^4 + 286384\mu^9\lambda^5 + 254487\mu^8\lambda^6 + 171053\mu^7\lambda^7 + 87388\mu^6\lambda^8 + 33751\mu^5\lambda^9 + 9705\mu^4\lambda^{10} + 2020\mu^3\lambda^{11} + 288\mu^2\lambda^{12} + 25\mu\lambda^{13} + \lambda^{14}\}$$

$$U_{58}=\frac{1}{2K_2}\{576\mu^{14}+5568\mu^{13}\lambda+24688\mu^{12}\lambda^2+65376\mu^{11}\lambda^3+114588\mu^{10}\lambda^4+140188\mu^9\lambda^5$$
$$+123289\mu^8\lambda^6+79076\mu^7\lambda^7+37072\mu^6\lambda^8+12574\mu^5\lambda^9+3005\mu^4\lambda^{10}+480\mu^3\lambda^{11}+46\mu^2\lambda^{12}$$
$$+2\mu\lambda^{13}\}$$

$$U_{59}=\frac{1}{2K_2}\{1728\mu^{13}\lambda+13248\mu^{12}\lambda^2+46416\mu^{11}\lambda^3+98496\mu^{10}\lambda^4+141364\mu^9\lambda^5+145020\mu^8\lambda^6$$
$$+109467\mu^7\lambda^7+61586\mu^6\lambda^8+25830\mu^5\lambda^9+7977\mu^4\lambda^{10}+1763\mu^3\lambda^{11}+264\mu^2\lambda^{12}+24\mu\lambda^{13}+\lambda^{14}\}$$

$$U_{61}=\frac{1}{K_4}\{\mu^{26}\lambda+32\mu^{25}\lambda^2+479\mu^{24}\lambda^3+4466\mu^{23}\lambda^4+29143\mu^{22}\lambda^5+141824\mu^{21}\lambda^6+535647\mu^{20}\lambda^7$$
$$+1613478\mu^{19}\lambda^8+3953201\mu^{18}\lambda^9+7994532\mu^{17}\lambda^{10}+13491405\mu^{16}\lambda^{11}+19152558\mu^{15}\lambda^{12}$$
$$+22996885\mu^{14}\lambda^{13}+23425400\mu^{13}\lambda^{14}+20253293\mu^{12}\lambda^{15}+14829226\mu^{11}\lambda^{16}+9144962\mu^{10}\lambda^{17}$$
$$+4704900\mu^9\lambda^{18}+1989256\mu^8\lambda^{19}+675344\mu^7\lambda^{20}+177504\mu^6\lambda^{21}+33984\mu^5\lambda^{22}+4224\mu^4\lambda^{23}$$
$$+256\mu^3\lambda^{24}\}$$

$$U_{62}=\frac{1}{K_4}\{\mu^{26}\lambda+34\mu^{25}\lambda^2+541\mu^{24}\lambda^3+5366\mu^{23}\lambda^4+37289\mu^{22}\lambda^5+193480\mu^{21}\lambda^6+780169\mu^{20}\lambda^7$$
$$+2512602\mu^{19}\lambda^8+6592147\mu^{18}\lambda^9+14299064\mu^{17}\lambda^{10}+25930611\mu^{16}\lambda^{11}+39643074\mu^{15}\lambda^{12}$$
$$+51399019\mu^{14}\lambda^{13}+56727760\mu^{13}\lambda^{14}+53376811\mu^{12}\lambda^{15}+42783190\mu^{11}\lambda^{16}+29111080\mu^{10}\lambda^{17}$$
$$+16704366\mu^9\lambda^{18}+7997180\mu^8\lambda^{19}+3142552\mu^7\lambda^{20}+988848\mu^6\lambda^{21}+239776\mu^5\lambda^{22}+42048\mu^4\lambda^{23}$$
$$+4736\mu^3\lambda^{24}+256\mu^2\lambda^{25}\}$$

$$U_{63}=\frac{1}{K_4}\{\mu^{25}\lambda^2+32\mu^{24}\lambda^3+479\mu^{23}\lambda^4+4466\mu^{22}\lambda^5+29143\mu^{21}\lambda^6+141824\mu^{20}\lambda^7+535647\mu^{19}\lambda^8$$
$$+1613478\mu^{18}\lambda^9+3953201\mu^{17}\lambda^{10}+7994532\mu^{16}\lambda^{11}+13491405\mu^{15}\lambda^{12}+19152558\mu^{14}\lambda^{13}$$
$$+22996885\mu^{13}\lambda^{14}+23425400\mu^{12}\lambda^{15}+20253293\mu^{11}\lambda^{16}+14829226\mu^{10}\lambda^{17}+9144962\mu^9\lambda^{18}$$
$$+4704900\mu^8\lambda^{19}+1989256\mu^7\lambda^{20}+675344\mu^6\lambda^{21}+177504\mu^5\lambda^{22}+33984\mu^4\lambda^{23}+4224\mu^3\lambda^{24}$$
$$+256\mu^2\lambda^{25}\}$$

$$U_{64}=\frac{1}{K_4}\{\mu^{25}\lambda^2+33\mu^{24}\lambda^3+511\mu^{23}\lambda^4+4945\mu^{22}\lambda^5+33609\mu^{21}\lambda^6+170967\mu^{20}\lambda^7+677471\mu^{19}\lambda^8$$
$$+2149125\mu^{18}\lambda^9+5566679\mu^{17}\lambda^{10}+11947733\mu^{16}\lambda^{11}+21485937\mu^{15}\lambda^{12}+32643963\mu^{14}\lambda^{13}$$
$$+42149443\mu^{13}\lambda^{14}+46422285\mu^{12}\lambda^{15}+43678693\mu^{11}\lambda^{16}+35082519\mu^{10}\lambda^{17}+23974188\mu^9\lambda^{18}$$
$$+13849862\mu^8\lambda^{19}+6694156\mu^7\lambda^{20}+2664600\mu^6\lambda^{21}+852848\mu^5\lambda^{22}+211488\mu^4\lambda^{23}+38208\mu^3\lambda^{24}$$
$$+4480\mu^2\lambda^{25}+256\mu\lambda^{26}\}$$

$$U_{65}=\frac{1}{K_1}\{512\mu^{13}+5376\mu^{12}\lambda+24448\mu^{11}\lambda^2+64064\mu^{10}\lambda^3+108448\mu^9\lambda^4+125680\mu^8\lambda^5+102760\mu^7\lambda^6$$
$$+59980\mu^6\lambda^7+24892\mu^5\lambda^8+7184\mu^4\lambda^9+1372\mu^3\lambda^{10}+156\mu^2\lambda^{11}+8\mu\lambda^{12}\}$$

$$U_{66}=\frac{1}{K_1}\{256\mu^{13}+2944\mu^{12}\lambda+15424\mu^{11}\lambda^2+48352\mu^{10}\lambda^3+100592\mu^9\lambda^4+146248\mu^8\lambda^5+152812\mu^7\lambda^6$$
$$+116314\mu^6\lambda^7+64612\mu^5\lambda^8+25926\mu^4\lambda^9+7320\mu^3\lambda^{10}+1380\mu^2\lambda^{11}+156\mu\lambda^{12}+8\lambda^{13}\}$$

$$U_{67}=\frac{1}{2K_2}\{1728\mu^{14}+14976\mu^{13}\lambda+58512\mu^{12}\lambda^2+136656\mu^{11}\lambda^3+212996\mu^{10}\lambda^4+234720\mu^9\lambda^5$$
$$+187983\mu^8\lambda^6+110977\mu^7\lambda^7+48314\mu^6\lambda^8+15248\mu^5\lambda^9+3344\mu^4\lambda^{10}+426\mu^3\lambda^{11}+41\mu^2\lambda^{12}+2\mu\lambda^{13}\}$$

$$U_{68}=\frac{1}{2K_2}\{576\mu^{14}+5568\mu^{13}\lambda+24688\mu^{12}\lambda^2+65376\mu^{11}\lambda^3+114588\mu^{10}\lambda^4+140188\mu^9\lambda^5+123289\mu^8\lambda^6$$
$$+79076\mu^7\lambda^7+37072\mu^6\lambda^8+12574\mu^5\lambda^9+3005\mu^4\lambda^{10}+480\mu^3\lambda^{11}+46\mu^2\lambda^{12}+2\mu\lambda^{13}\}$$

$$U_{69}=\frac{1}{2K_2}\{1728\mu^{13}\lambda+13248\mu^{12}\lambda^2+45264\mu^{11}\lambda^3+91392\mu^{10}\lambda^4+121748\mu^9\lambda^5+112972\mu^8\lambda^6$$
$$+75011\mu^7\lambda^7+35966\mu^6\lambda^8+12376\mu^5\lambda^9+2984\mu^4\lambda^{10}+479\mu^3\lambda^{11}+46\mu^2\lambda^{12}+2\mu\lambda^{13}\}$$

$$U_{71}=\frac{1}{K_4}\{\mu^{26}\lambda+32\mu^{25}\lambda^2+480\mu^{24}\lambda^3+4495\mu^{23}\lambda^4+29536\mu^{22}\lambda^5+145139\mu^{21}\lambda^6+555210\mu^{20}\lambda^7$$
$$+1699563\mu^{19}\lambda^8+4247206\mu^{18}\lambda^9+8795467\mu^{17}\lambda^{10}+15266334\mu^{16}\lambda^{11}+22398705\mu^{15}\lambda^{12}$$
$$+27948376\mu^{14}\lambda^{13}+29771105\mu^{13}\lambda^{14}+27116934\mu^{12}\lambda^{15}+21105537\mu^{11}\lambda^{16}+13991129\mu^{10}\lambda^{17}$$
$$+7850129\mu^9\lambda^{18}+3690194\mu^8\lambda^{19}+1430996\mu^7\lambda^{20}+447176\mu^6\lambda^{21}+108592\mu^5\lambda^{22}+19296\mu^4\lambda^{23}$$
$$+2240\mu^3\lambda^{24}+128\mu^2\lambda^{25}\}$$

$$U_{72}=\frac{1}{K_4}\{\mu^{26}\lambda+34\mu^{25}\lambda^2+542\mu^{24}\lambda^3+5396\mu^{23}\lambda^4+37710\mu^{22}\lambda^5+197160\mu^{21}\lambda^6+802682\mu^{20}\lambda^7$$
$$+2615300\mu^{19}\lambda^8+6955624\mu^{18}\lambda^9+15324532\mu^{17}\lambda^{10}+28281942\mu^{16}\lambda^{11}+44087748\mu^{15}\lambda^{12}$$
$$+58398130\mu^{14}\lambda^{13}+65977336\mu^{13}\lambda^{14}+63682286\mu^{12}\lambda^{15}+52481308\mu^{11}\lambda^{16}+36811751\mu^{10}\lambda^{17}$$
$$+21841258\mu^9\lambda^{18}+10851684\mu^8\lambda^{19}+4445576\mu^7\lambda^{20}+1466800\mu^6\lambda^{21}+375776\mu^5\lambda^{22}+70336\mu^4\lambda^{23}$$
$$+8576\mu^3\lambda^{24}+512\mu^2\lambda^{25}\}$$

$$U_{73}=\frac{1}{K_4}\{\mu^{25}\lambda^2+32\mu^{24}\lambda^3+480\mu^{23}\lambda^4+4495\mu^{22}\lambda^5+29536\mu^{21}\lambda^6+145139\mu^{20}\lambda^7+555210\mu^{19}\lambda^8$$
$$+1699563\mu^{18}\lambda^9+4247206\mu^{17}\lambda^{10}+8795467\mu^{16}\lambda^{11}+15266334\mu^{15}\lambda^{12}+22398705\mu^{14}\lambda^{13}$$
$$+27948376\mu^{13}\lambda^{14}+29771105\mu^{12}\lambda^{15}+27116934\mu^{11}\lambda^{16}+21105537\mu^{10}\lambda^{17}+13991129\mu^9\lambda^{18}$$
$$+7850129\mu^8\lambda^{19}+3690194\mu^7\lambda^{20}+1430996\mu^6\lambda^{21}+447176\mu^5\lambda^{22}+108592\mu^4\lambda^{23}+19296\mu^3\lambda^{24}$$
$$+2240\mu^2\lambda^{25}+128\mu\lambda^{26}\}$$

$$U_{74}=\frac{1}{K_4}\{\mu^{25}\lambda^2+33\mu^{24}\lambda^3+512\mu^{23}\lambda^4+4975\mu^{22}\lambda^5+34031\mu^{21}\lambda^6+174675\mu^{20}\lambda^7+700349\mu^{19}\lambda^8$$
$$+2254773\mu^{18}\lambda^9+5946769\mu^{17}\lambda^{10}+13042673\mu^{16}\lambda^{11}+24061801\mu^{15}\lambda^{12}+37665039\mu^{14}\lambda^{13}$$
$$+50347081\mu^{13}\lambda^{14}+57719481\mu^{12}\lambda^{15}+56888039\mu^{11}\lambda^{16}+48222471\mu^{10}\lambda^{17}+35096666\mu^9\lambda^{18}$$
$$+21841258\mu^8\lambda^{19}+11540323\mu^7\lambda^{20}+5121190\mu^6\lambda^{21}+1878172\mu^5\lambda^{22}+555768\mu^4\lambda^{23}+127888\mu^3\lambda^{24}$$
$$+21536\mu^2\lambda^{25}+2368\mu\lambda^{26}+128\lambda^{27}\}$$

$$U_{75}=\frac{1}{K_1}\{512\mu^{13}+5376\mu^{12}\lambda+24704\mu^{11}\lambda^2+65984\mu^{10}\lambda^3+114656\mu^9\lambda^4+137168\mu^8\lambda^5+116312\mu^7\lambda^6$$
$$+70676\mu^6\lambda^7+30632\mu^5\lambda^8+9258\mu^4\lambda^9+1856\mu^3\lambda^{10}+222\mu^2\lambda^{11}+12\mu\lambda^{12}\}$$

$$U_{76}=\frac{1}{K_1}\{256\mu^{13}+2944\mu^{12}\lambda+15424\mu^{11}\lambda^2+48352\mu^{10}\lambda^3+100592\mu^9\lambda^4+146248\mu^8\lambda^5+152812\mu^7\lambda^6$$
$$+116314\mu^6\lambda^7+64612\mu^5\lambda^8+25926\mu^4\lambda^9+7320\mu^3\lambda^{10}+1380\mu^2\lambda^{11}+156\mu\lambda^{12}+8\lambda^{13}\}$$

$$U_{77}=\frac{1}{2K_2}\{1728\mu^{14}+15552\mu^{13}\lambda+64080\mu^{12}\lambda^2+160192\mu^{11}\lambda^3+271412\mu^{10}\lambda^4+329692\mu^9\lambda^5$$
$$+296123\mu^8\lambda^6+199810\mu^7\lambda^7+101798\mu^6\lambda^8+38978\mu^5\lambda^9+11044\mu^4\lambda^{10}+2246\mu^3\lambda^{11}+310\mu^2\lambda^{12}$$
$$+26\mu\lambda^{13}+\lambda^{14}\}$$

$$U_{78}=\frac{1}{2K_2}\{576\mu^{14}+5568\mu^{13}\lambda+24688\mu^{12}\lambda^2+65376\mu^{11}\lambda^3+114588\mu^{10}\lambda^4+140188\mu^9\lambda^5$$
$$+123289\mu^8\lambda^6+79076\mu^7\lambda^7+37072\mu^6\lambda^8+12574\mu^5\lambda^9+3005\mu^4\lambda^{10}+480\mu^3\lambda^{11}+46\mu^2\lambda^{12}+2\mu\lambda^{13}\}$$

$$U_{79}=\frac{1}{2K_2}\{1728\mu^{13}\lambda+13824\mu^{12}\lambda^2+50256\mu^{11}\lambda^3+109936\mu^{10}\lambda^4+161476\mu^9\lambda^5+168216\mu^8\lambda^6$$
$$+127907\mu^7\lambda^7+71903\mu^6\lambda^8+29895\mu^5\lambda^9+9083\mu^4\lambda^{10}+1961\mu^3\lambda^{11}+285\mu^2\lambda^{12}+25\mu\lambda^{13}+\lambda^{14}\}$$

$$U_{81}=\frac{1}{K_4}\{\mu^{26}\lambda+31\mu^{25}\lambda^2+449\mu^{24}\lambda^3+4046\mu^{23}\lambda^4+25490\mu^{22}\lambda^5+119649\mu^{21}\lambda^6+435561\mu^{20}\lambda^7$$
$$+1264002\mu^{19}\lambda^8+2983204\mu^{18}\lambda^9+5812263\mu^{17}\lambda^{10}+9454071\mu^{16}\lambda^{11}+12944634\mu^{15}\lambda^{12}$$
$$+15003742\mu^{14}\lambda^{13}+14767363\mu^{13}\lambda^{14}+12349571\mu^{12}\lambda^{15}+8755966\mu^{11}\lambda^{16}+5235163\mu^{10}\lambda^{17}$$
$$+2614966\mu^9\lambda^{18}+1075228\mu^8\lambda^{19}+355768\mu^7\lambda^{20}+91408\mu^6\lambda^{21}+17184\mu^5\lambda^{22}+2112\mu^4\lambda^{23}$$
$$+128\mu^3\lambda^{24}\}$$

$$U_{82}=\frac{1}{K_4}\{\mu^{26}\lambda+33\mu^{25}\lambda^2+509\mu^{24}\lambda^3+4887\mu^{23}\lambda^4+32823\mu^{22}\lambda^5+164337\mu^{21}\lambda^6+638345\mu^{20}\lambda^7$$
$$+1976955\mu^{19}\lambda^8+4978669\mu^{18}\lambda^9+10345863\mu^{17}\lambda^{10}+17936079\mu^{16}\lambda^{11}+26151669\mu^{15}\lambda^{12}$$
$$+32246461\mu^{14}\lambda^{13}+33730875\mu^{13}\lambda^{14}+29951411\mu^{12}\lambda^{15}+22529897\mu^{11}\lambda^{16}+14281854\mu^{10}\lambda^{17}$$
$$+7559404\mu^9\lambda^{18}+3292280\mu^8\lambda^{19}+1153296\mu^7\lambda^{20}+313504\mu^6\lambda^{21}+62272\mu^5\lambda^{22}+8064\mu^4\lambda^{23}$$
$$+512\mu^3\lambda^{24}\}$$

$$U_{83}=\frac{1}{K_4}\{\mu^{25}\lambda^2+31\mu^{24}\lambda^3+449\mu^{23}\lambda^4+4046\mu^{22}\lambda^5+25490\mu^{21}\lambda^6+119649\mu^{20}\lambda^7+435561\mu^{19}\lambda^8$$
$$+1264002\mu^{18}\lambda^9+2983204\mu^{17}\lambda^{10}+5812263\mu^{16}\lambda^{11}+9454071\mu^{15}\lambda^{12}+12944634\mu^{14}\lambda^{13}+15003742\mu^{13}\lambda^{14}$$
$$+14767363\mu^{12}\lambda^{15}+12349571\mu^{11}\lambda^{16}+8755966\mu^{10}\lambda^{17}+5235163\mu^9\lambda^{18}+2614966\mu^8\lambda^{19}+1075228\mu^7\lambda^{20}$$
$$+355768\mu^6\lambda^{21}+91408\mu^5\lambda^{22}+17184\mu^4\lambda^{23}+2112\mu^3\lambda^{24}+128\mu^2\lambda^{25}\}$$

$$U_{84}=\frac{1}{K_4}\{\mu^{25}\lambda^2+32\mu^{24}\lambda^3+480\mu^{23}\lambda^4+4495\mu^{22}\lambda^5+29536\mu^{21}\lambda^6+145139\mu^{20}\lambda^7+555210\mu^{19}\lambda^8$$
$$+1699563\mu^{18}\lambda^9+4247206\mu^{17}\lambda^{10}+8795467\mu^{16}\lambda^{11}+15266334\mu^{15}\lambda^{12}+22398705\mu^{14}\lambda^{13}$$
$$+27948376\mu^{13}\lambda^{14}+29771105\mu^{12}\lambda^{15}+27116934\mu^{11}\lambda^{16}+21105537\mu^{10}\lambda^{17}+13991129\mu^9\lambda^{18}$$
$$+7850129\mu^8\lambda^{19}+3690194\mu^7\lambda^{20}+1430996\mu^6\lambda^{21}+447176\mu^5\lambda^{22}+108592\mu^4\lambda^{23}+19296\mu^3\lambda^{24}$$
$$+2240\mu^2\lambda^{25}+128\mu\lambda^{26}\}$$

$$U_{85} = \frac{1}{K_1}\{512\mu^{13} + 4864\mu^{12}\lambda + 19840\mu^{11}\lambda^2 + 46144\mu^{10}\lambda^3 + 68512\mu^9\lambda^4 + 68656\mu^8\lambda^5 + 47656\mu^7\lambda^6 + 23020\mu^6\lambda^7 + 7612\mu^5\lambda^8 + 1646\mu^4\lambda^9 + 210\mu^3\lambda^{10} + 12\mu^2\lambda^{11}\}$$

$$U_{86} = \frac{1}{K_1}\{256\mu^{13} + 2688\mu^{12}\lambda + 12736\mu^{11}\lambda^2 + 35616\mu^{10}\lambda^3 + 64976\mu^9\lambda^4 + 81272\mu^8\lambda^5 + 71540\mu^7\lambda^6 + 44774\mu^6\lambda^7 + 19838\mu^5\lambda^8 + 6088\mu^4\lambda^9 + 1232\mu^3\lambda^{10} + 148\mu^2\lambda^{11} + 8\mu\lambda^{12}\}$$

$$U_{87} = \frac{1}{2K_2}\{1728\mu^{14} + 13824\mu^{13}\lambda + 50256\mu^{12}\lambda^2 + 109936\mu^{11}\lambda^3 + 161476\mu^{10}\lambda^4 + 168216\mu^9\lambda^5 + 127907\mu^8\lambda^6 + 71903\mu^7\lambda^7 + 29811\mu^6\lambda^8 + 8887\mu^5\lambda^9 + 1800\mu^4\lambda^{10} + 229\mu^3\lambda^{11} + 18\mu^2\lambda^{12} + \mu\lambda^{13}\}$$

$$U_{88} = \frac{1}{2K_2}\{576\mu^{14} + 5568\mu^{13}\lambda + 24688\mu^{12}\lambda^2 + 65376\mu^{11}\lambda^3 + 114588\mu^{10}\lambda^4 + 140188\mu^9\lambda^5 + 123289\mu^8\lambda^6 + 79076\mu^7\lambda^7 + 37072\mu^6\lambda^8 + 12574\mu^5\lambda^9 + 3005\mu^4\lambda^{10} + 480\mu^3\lambda^{11} + 46\mu^2\lambda^{12} + 2\mu\lambda^{13}\}$$

$$U_{89} = \frac{1}{2K_2}\{1728\mu^{13}\lambda + 12096\mu^{12}\lambda^2 + 38160\mu^{11}\lambda^3 + 71776\mu^{10}\lambda^4 + 89700\mu^9\lambda^5 + 78516\mu^8\lambda^6 + 49391\mu^7\lambda^7 + 22512\mu^6\lambda^8 + 7383\mu^5\lambda^9 + 1700\mu^4\lambda^{10} + 261\mu^3\lambda^{11} + 24\mu^2\lambda^{12} + \mu\lambda^{13}\}$$

$$U_{91} = \frac{1}{K_4}\{\mu^{26}\lambda + 32\mu^{25}\lambda^2 + 479\mu^{24}\lambda^3 + 4467\mu^{23}\lambda^4 + 29171\mu^{22}\lambda^5 + 142189\mu^{21}\lambda^6 + 538597\mu^{20}\lambda^7 + 1630091\mu^{19}\lambda^8 + 4022673\mu^{18}\lambda^9 + 8219065\mu^{17}\lambda^{10} + 14067807\mu^{16}\lambda^{11} + 20351085\mu^{15}\lambda^{12} + 25044505\mu^{14}\lambda^{13} + 26329271\mu^{13}\lambda^{14} + 23695127\mu^{12}\lambda^{15} + 18251033\mu^{11}\lambda^{16} + 11999466\mu^{10}\lambda^{17} + 6696563\mu^9\lambda^{18} + 3142822\mu^8\lambda^{19} + 1222716\mu^7\lambda^{20} + 385784\mu^6\lambda^{21} + 95376\mu^5\lambda^{22} + 17440\mu^4\lambda^{23} + 2112\mu^3\lambda^{24} + 128\mu^2\lambda^{25}\}$$

$$U_{92} = \frac{1}{K_4}\{\mu^{26}\lambda + 34\mu^{25}\lambda^2 + 541\mu^{24}\lambda^3 + 5367\mu^{23}\lambda^4 + 37318\mu^{22}\lambda^5 + 193872\mu^{21}\lambda^6 + 783457\mu^{20}\lambda^7 + 2531827\mu^{19}\lambda^8 + 6675620\mu^{18}\lambda^9 + 14579068\mu^{17}\lambda^{10} + 26676075\mu^{16}\lambda^{11} + 41248941\mu^{15}\lambda^{12} + 54237826\mu^{14}\lambda^{13} + 60888064\mu^{13}\lambda^{14} + 58466083\mu^{12}\lambda^{15} + 47999393\mu^{11}\lambda^{16} + 33592995\mu^{10}\lambda^{17} + 19923122\mu^9\lambda^{18} + 9915316\mu^8\lambda^{19} + 4078920\mu^7\lambda^{20} + 1355504\mu^6\lambda^{21} + 351072\mu^5\lambda^{22} + 66752\mu^4\lambda^{23} + 8320\mu^3\lambda^{24} + 512\mu^2\lambda^{25}\}$$

$$U_{93} = \frac{1}{K_4}\{\mu^{25}\lambda^2 + 32\mu^{24}\lambda^3 + 479\mu^{23}\lambda^4 + 4467\mu^{22}\lambda^5 + 29171\mu^{21}\lambda^6 + 142189\mu^{20}\lambda^7 + 538597\mu^{19}\lambda^8 + 1630091\mu^{18}\lambda^9 + 4022673\mu^{17}\lambda^{10} + 8219065\mu^{16}\lambda^{11} + 14067807\mu^{15}\lambda^{12} + 20351085\mu^{14}\lambda^{13} + 25044505\mu^{13}\lambda^{14} + 26329271\mu^{12}\lambda^{15} + 23695127\mu^{11}\lambda^{16} + 18251033\mu^{10}\lambda^{17} + 11999466\mu^9\lambda^{18} + 6696563\mu^8\lambda^{19} + 3142822\mu^7\lambda^{20} + 1222716\mu^6\lambda^{21} + 385784\mu^5\lambda^{22} + 95376\mu^4\lambda^{23} + 17440\mu^3\lambda^{24} + 2112\mu^2\lambda^{25} + 128\mu\lambda^{26}\}$$

$$U_{94} = \frac{1}{K_4}\{ \mu^{25}\lambda^2 + 33\mu^{24}\lambda^3 + 511\mu^{23}\lambda^4 + 4946\mu^{22}\lambda^5 + 33638\mu^{21}\lambda^6 + 171360\mu^{20}\lambda^7 + 680786\mu^{19}\lambda^8$$
$$+2168688\mu^{18}\lambda^9 + 56522764\mu^{17}\lambda^{10} + 12241738\mu^{16}\lambda^{11} + 22286872\mu^{15}\lambda^{12} + 34418892\mu^{14}\lambda^{13}$$
$$+45395590\mu^{13}\lambda^{14} + 51373776\mu^{12}\lambda^{15} + 50024398\mu^{11}\lambda^{16} + 41946160\mu^{10}\lambda^{17} + 30250499\mu^{9}\lambda^{18}$$
$$+18696029\mu^{8}\lambda^{19} + 9839385\mu^{7}\lambda^{20} + 4365538\mu^{6}\lambda^{21} + 1608500\mu^{5}\lambda^{22} + 481160\mu^{4}\lambda^{23} + 112816\mu^{3}\lambda^{24}$$
$$+19552\mu^{2}\lambda^{25} + 2240\mu\lambda^{26} + 128\lambda^{27}\}$$

$$U_{95} = \frac{1}{K_1}\{ 512\mu^{13} + 5376\mu^{12}\lambda + 24448\mu^{11}\lambda^2 + 64320\mu^{10}\lambda^3 + 110112\mu^{9}\lambda^4 + 130224\mu^{8}\lambda^5$$
$$+109704\mu^{7}\lambda^6 + 66588\mu^{6}\lambda^7 + 28980\mu^{5}\lambda^8 + 8836\mu^{4}\lambda^9 + 1794\mu^{3}\lambda^{10} + 218\mu^{2}\lambda^{11} + 12\mu\lambda^{12}\}$$

$$U_{96} = \frac{1}{K_1}\{ 256\mu^{13} + 2944\mu^{12}\lambda + 15424\mu^{11}\lambda^2 + 48352\mu^{10}\lambda^3 + 100592\mu^{9}\lambda^4 + 146248\mu^{8}\lambda^5 + 152812\mu^{7}\lambda^6$$
$$+116314\mu^{6}\lambda^7 + 64612\mu^{5}\lambda^8 + 25926\mu^{4}\lambda^9 + 7320\mu^{3}\lambda^{10} + 1380\mu^{2}\lambda^{11} + 156\mu\lambda^{12} + 8\lambda^{13}\}$$

$$U_{97} = \frac{1}{2K_2}\{1728\mu^{14} + 14976\mu^{13}\lambda + 59088\mu^{12}\lambda^2 + 141648\mu^{11}\lambda^3 + 231684\mu^{10}\lambda^4 + 274448\mu^{9}\lambda^5$$
$$+243227\mu^{8}\lambda^6 + 163873\mu^{7}\lambda^7 + 84279\mu^{6}\lambda^8 + 32879\mu^{5}\lambda^9 + 9562\mu^{4}\lambda^{10} + 2007\mu^{3}\lambda^{11} + 287\mu^{2}\lambda^{12}$$
$$+25\mu\ \lambda^{13} + \lambda^{14}\}$$

$$U_{98} = \frac{1}{2K_2}\{576\mu^{14} + 5568\mu^{13}\lambda + 24688\mu^{12}\lambda^2 + 65376\mu^{11}\lambda^3 + 114588\mu^{10}\lambda^4 + 140188\mu^{9}\lambda^5$$
$$+123289\mu^{8}\lambda^6 + 79076\mu^{7}\lambda^7 + 37072\mu^{6}\lambda^8 + 12574\mu^{5}\lambda^9 + 3005\mu^{4}\lambda^{10} + 480\mu^{3}\lambda^{11} + 46\mu^{2}\lambda^{12} + 2\mu\ \lambda^{13}\}$$

$$U_{99} = \frac{1}{2K_2}\{2304\mu^{13}\lambda + 18240\mu^{12}\lambda^2 + 65536\mu^{11}\lambda^3 + 141488\mu^{10}\lambda^4 + 204784\mu^{9}\lambda^5 + 209852\mu^{8}\lambda^6$$
$$+6664\mu^{7}\lambda^7 + 86285\mu^{6}\lambda^8 + 35066\mu^{5}\lambda^9 + 10387\mu^{4}\lambda^{10} + 2180\mu^{3}\lambda^{11} + 307\mu^{2}\lambda^{12} + 26\mu\ \lambda^{13} + \lambda^{14}\}$$

$$K_\beta = K_1K_2K_4 + 2K_1K_2[\lambda(u_{81}+u_{82}+u_{83}+u_{84}) + \mu(u_{91}+u_{92}+u_{93}+u_{94})] + 2K_2K_4[\lambda(u_{85}+u_{86}) + \mu(u_{95}+u_{96})] + K_1K_4[\lambda(u_{87}+u_{88}+u_{89}) + \mu(u_{97}+u_{98}+u_{99})] \qquad \beta = \frac{K_\beta}{K_1K_2K_4}$$

The unique fixed probability vector W' is given by

$$W = (W_{010},\ W_{020},\ W_{112},\ W_{111},\ W_{122},\ W_{121},\ W_{212},\ W_{211},\ W_{221},\ W_{222})$$

$$= \frac{1}{K_\beta}[2K_1K_2(\lambda u_{81} + \mu u_{91}),\ 2K_1K_2(\lambda u_{82} + \mu u_{92}),\ 2K_1K_2(\lambda u_{83} + \mu u_{93}),\ 2K_1K_2(\lambda u_{84} + \mu u_{94}),$$
$$2K_2K_4(\lambda u_{85} + \mu u_{95}),\ 2K_2K_4(\lambda u_{86} + \mu u_{96}),\ 2K_1K_4(\lambda u_{87} + \mu u_{97}),\ 2K_1K_4(\lambda u_{88} + \mu u_{98}),$$
$$2K_1K_4(\lambda u_{89} + \mu u_{99}),\ K_1K_2K_4]$$

4. THE MEAN FIRST PASSAGE TIME MATRIX:

1. For 0 < i < 10, 0 < j < 10.

$$M = (m_{ij}) = \frac{1}{W_{j-1}}[(U_{jj} - U_{ij}) - W_{j-1}\{\sum_{k=1}^{9} U_{jk} - \sum_{k=1}^{9} U_{ik}\}]$$

2. For i = 10, 0 < j < 10

$$M = (m_{ij}) = (W_9 - W_{j-1})[\sum_{k=0}^{8} W_k (\sum_{l=1}^{9} U_{k+1,l})] + \sum_{k=0}^{8} W_k U_{k+1,j}$$

3. For I = 10, j = 10

$$M = (m_{ij}) = W_9[\sum_{k=0}^{8} W_k (\sum_{l=1}^{9} U_{k+1,l})]$$

Now we introduce, another notation for the probabilities ie, let

$$p_0 = W_{010} + W_{020}$$
$$p_1 = W_{112} + W_{111} + W_{122} + W_{121}$$
$$p_2 = W_{212} + W_{211} + W_{222} + W_{221}$$

Now we can obtain some performance measures related to this queuing model, they are

(1) The mean number of customers in the queue $L_q = \sum_{n=0}^{2} (n-1)\, p_n$

(2) The mean number of customers in the service facility $L_s = \frac{\lambda}{\mu}$

(3) The mean number of customers in the system $L = L_q + L_s$

(4) The effective rate of arrival $\lambda^{'} = \lambda(1 - p_N)$

(5) The system effectiveness $\alpha = \frac{\lambda^{'}}{\lambda}$

(6) The mean waiting time in the system is $W = \frac{L}{\lambda^{'}}$

(7) The mean waiting time in the queue is $W_q = \frac{L_q}{\lambda^{'}}$

(8) The mean waiting time in the service facility $W_s = \frac{L_s}{\lambda^{'}}$

5. SOME PARTICULAR MODELS:

In this section we analyse three particular models (denoted by Model I, Model II and Model III) related to model discussed in section 3 In this study we vary the values of λ and μ . For model I, λ = 0:5; μ = 0:4, Model II, λ = 0:5; μ = 0:3; Model III, λ = 0:6; μ = 0:3. The probability vector W0 and the operating characteristics are calculated, using the formulas in the sections 3 and 4. The values are presented in tables 5.1 and table 5.2 for models I - III. The mean first passage time matrices are also given for the above three models.

Probabilities	Model I	Model II	Model III
W_{010}	0.048	0.026	0.017
W_{020}	0.119	0.070	0.048
W_{112}	0.060	0.043	0.033
W_{111}	0.136	0.114	0.010
W_{122}	0.089	0.074	0.063
W_{121}	0.125	0.127	0.123
W_{212}	0.156	0.187	0.203
W_{211}	0.100	0.130	0.147
W_{221}	0.111	0.148	0.168
W_{222}	0.056	0.081	0.098

Table 5.1: Steady state probabilities

Operating characteristics	Model I	Model II	Model III
Mean number of customers in the queue	0.256	0.450	0.551
Mean number of customers in the service facility	1.250	1.670	2.000
Mean number of customers in the system	1.506	2.120	2.551
The effective rate of arrival	0.289	0.227	0.230
The system effectiveness	0.577	0.454	0.384
Mean waiting time in the system	5.220	9.339	11.072
Mean waiting time in the queue	0.887	1.982	2.392
Mean waiting time in the service facility	4.333	7.357	8.681

Table 5.2: Operating characteristics

Model: I

Mean first passage time matrix [M_1, M_2] where

$$M_1 = \begin{pmatrix} 20.752 & 1.000 & 8.385 & 2.000 & 6.578 \\ 19.752 & 8.381 & 7.386 & 1.000 & 5.578 \\ 12.367 & 4.147 & 16.602 & 2.839 & 3.479 \\ 18.752 & 7.381 & 6.385 & 7.378 & 4.578 \\ 21.261 & 5.665 & 8.894 & 2.509 & 11.247 \\ 22.860 & 8.968 & 10.492 & 4.108 & 4.457 \\ 21.469 & 9.198 & 9.102 & 2.717 & 5.785 \\ 23.138 & 10.609 & 10.771 & 4.386 & 7.022 \\ 22.643 & 9.651 & 10.275 & 3.891 & 5.750 \\ 23.473 & 10.739 & 11.106 & 4.721 & 7.012 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 4.673 & 6.445 & 8.743 & 9.103 & 15.373 \\ 3.673 & 5.445 & 7.743 & 8.103 & 14.373 \\ 4.764 & 5.073 & 8.834 & 8.056 & 15.464 \\ 2.673 & 4.445 & 6.743 & 7.103 & 13.373 \\ 3.837 & 2.975 & 7.907 & 6.219 & 14.537 \\ 8.015 & 2.942 & 4.070 & 5.340 & 10.700 \\ 2.968 & 6.419 & 7.038 & 3.712 & 13.668 \\ 3.946 & 1.915 & 9.974 & 3.637 & 6.630 \\ 2.205 & 1.863 & 6.274 & 8.984 & 12.905 \\ 3.727 & 2.448 & 3.344 & 2.576 & 17.954 \end{pmatrix}$$

Model: II

Mean first passage time matrix $M = [M_1, M_2]$ where

$$M_1 = \begin{pmatrix} 38.927 & 1.000 & 13.097 & 2.000 & 8.249 \\ 37.927 & 14.235 & 12.097 & 1.000 & 7.249 \\ 25.831 & 7.572 & 23.356 & 3.501 & 3.718 \\ 36.927 & 13.235 & 11.097 & 8.759 & 6.249 \\ 40.329 & 10.515 & 14.498 & 3.402 & 13.464 \\ 42.585 & 15.632 & 16.754 & 5.658 & 6.767 \\ 40.770 & 15.823 & 14.939 & 3.843 & 8.115 \\ 42.939 & 17.703 & 17.108 & 6.012 & 9.828 \\ 42.076 & 16.377 & 16.245 & 5.148 & 8.235 \\ 43.240 & 17.831 & 17.410 & 6.313 & 9.855 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 4.376 & 6.268 & 7.684 & 8.171 & 12.547 \\ 3.376 & 5.268 & 6.684 & 7.171 & 11.547 \\ 4.670 & 4.601 & 7.979 & 6.909 & 12.841 \\ 2.376 & 4.268 & 5.684 & 6.171 & 10.547 \\ 3.847 & 2.601 & 7.155 & 5.151 & 12.017 \\ 7.893 & 3.069 & 3.308 & 4.729 & 8.170 \\ 3.129 & 5.337 & 6.437 & 2.939 & 11.299 \\ 4.585 & 2.350 & 7.696 & 3.475 & 4.862 \\ 2.581 & 1.776 & 5.889 & 6.777 & 10.751 \\ 4.458 & 2.759 & 2.833 & 2.797 & 12.313 \end{pmatrix}$$

Model: III

Mean first passage time matrix M = [M_1, M_2] where

$$M_1 = \begin{pmatrix} 60.105 & 0.833 & 14.925 & 1.667 & 8.313 \\ 49.254 & 20.764 & 14.091 & 0.833 & 7.480 \\ 35.163 & 9.716 & 30.053 & 3.438 & 3.327 \\ 48.121 & 16.470 & 13.258 & 10.018 & 6.647 \\ 51.911 & 13.325 & 16.748 & 3.490 & 15.861 \\ 54.217 & 19.013 & 19.054 & 5.796 & 7.474 \\ 52.406 & 19.154 & 17.243 & 3.985 & 8.644 \\ 54.536 & 21.024 & 19.373 & 6.115 & 10.377 \\ 53.565 & 19.662 & 18.402 & 5.144 & 8.810 \\ 54.768 & 21.126 & 19.605 & 6.347 & 10.410 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 3.537 & 5.269 & 6.029 & 6.585 & 9.503 \\ 2.704 & 4.436 & 5.196 & 5.751 & 8.669 \\ 3.946 & 3.668 & 6.437 & 5.375 & 9.911 \\ 1.871 & 3.602 & 4.362 & 4.918 & 7.836 \\ 3.316 & 2.034 & 5.808 & 3.936 & 9.281 \\ 8.157 & 2.736 & 2.491 & 3.857 & 5.965 \\ 2.789 & 4.922 & 5.281 & 2.195 & 8.754 \\ 4.306 & 2.254 & 6.798 & 2.984 & 3.474 \\ 2.415 & 1.468 & 4.906 & 5.948 & 8.380 \\ 4.231 & 2.547 & 2.191 & 2.545 & 10.196 \end{pmatrix}$$

6. CONCLUSION:

In section 5, we analyse the model discussed in section 3 and 4 numerically by assigning particular values to λ and μ. The stationary distribution W and the expected values of the first passage times from one level to other levels are obtained using the formula given in section 5. Our computational experience shows that the analysis in this paper provides a practical and computational advantage.

REFERENCES:

[1] Adi-Ben-Israel and Greville T.N.E. Generalised inverses: Theory and Applications, Wiley-Inter science, New York, 1974.

[2] Boullian and Odell P.L. Generalised Inverse of matrices, Wiley–Inter science, New York, 1971.

[3] Campbell S.L. and Meyer Jr. C.D. Generalised Inverses of Linear Transformations, Pitman, London, 1979.

[4] Hunter J.J. On the moments of Markov renewal processes, Adv. Appl. Prob., vol. 1, 188-210, 1969.

[5] Kemeny J.G. and Snell J.L. Finite Markov chain Van Nortrand, Princeton, N.J, 1960.

[6] Meyer C.D. The role of the group generalised inverse in the theory of finite Markov Chain, SIAM Review, 19, 443-464, 1975.

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