



$\theta\ddot{g}$ -CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we offer a new class of sets called $\theta\ddot{g}$ -closed sets in topological spaces and we study some of its basic properties. The family of $\theta\ddot{g}$ -closed sets of a topological space forms a topology and is denoted by $\tau\theta\ddot{g}$. Notice that this class of sets lies between the class of θ -closed sets and the class of $\theta\omega$ -closed sets. Using these sets, we obtain a decomposition of θ -continuity and we introduce new spaces called $T\theta\ddot{g}$ and ${}_gT\theta\ddot{g}$. Using these spaces we obtain another decomposition of $T_{1/2}$ -spaces.

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1. INTRODUCTION:

In 1963 Levine [17] introduced the notion of semi-open sets. Velicko [30] introduced the notion of θ -closed sets and it is well known that the collection of all θ -closed sets of a topological space forms a topology and is denoted by $\tau\theta$. Levine [16] also introduced the notion of g-closed sets and investigated its fundamental properties. This notion was shown to be productive and very useful. Dontchev and Maki [11] introduced the notion of θ -generalized closed sets.

After the advent of g-closed sets, Arya and Nour [4], Sheik John [25], Ravi and Ganesan [23] and Dontchev [10] introduced gs-closed sets, ω -closed sets, \ddot{g} -closed sets and gsp-closed sets respectively.

Quite recently, Ganesan et al. [13] have introduced the notion of $\theta\omega$ -closed sets which lies between the θ -closed sets and the $\theta\ddot{g}$ -closed sets.

In this paper, we introduce a new class of sets called $\theta\ddot{g}$ -closed sets in topological spaces. This class lies between the class of θ -closed sets and the class of $\theta\omega$ -closed sets. We study some of its basic properties and characterizations. Interestingly it turns out that the family of $\theta\ddot{g}$ -closed sets of a topological space forms a topology. This collection is denoted by $\tau\theta\ddot{g}$. From the definitions, it follows immediately that $\tau_\theta \subseteq \tau\theta\ddot{g}$. Using these sets, we obtain a decomposition of θ -continuity and we introduce new type of spaces called $T\theta\ddot{g}$ -spaces and ${}_gT\theta\ddot{g}$ -spaces. Using these spaces, we obtain another decomposition of $T_{1/2}$ -spaces.

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2. PRELIMINARIES:

Throughout this paper (X, τ) and (Y, σ) (or X and Y) represents topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ) , $\text{cl}(A)$, $\text{int}(A)$ and A^c or $X \setminus A$ denote the closure of A , the interior of A and the complement of A respectively.

We recall the following definitions which are useful in the sequel.

Definition: 2.1 A subset A of a space (X, τ) is called:

- (i) semi-open set [17] if $A \subseteq \text{cl}(\text{int}(A))$;
- (ii) preopen set [19] if $A \subseteq \text{int}(\text{cl}(A))$;
- (iii) α -open set [20] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$;
- (iv) β -open set [1] (= semi-preopen set [2]) if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$;
- (v) regular open set [26] if $A = \text{int}(\text{cl}(A))$.

The complements of the above mentioned open sets are called their respective closed sets.

The preclosure [21] (resp. semi-closure [8], α -closure [20], semi-pre-closure [2]) of a subset A of X , denoted by $\text{pcl}(A)$ (resp. $\text{scl}(A)$, $\alpha \text{cl}(A)$, $\text{spcl}(A)$), is defined to be the intersection of all preclosed (resp. semi-closed, α -closed, semi-preclosed) sets of (X, τ) containing A . It is known that $\text{pcl}(A)$ (resp. $\text{scl}(A)$, $\alpha \text{cl}(A)$, $\text{spcl}(A)$) is a preclosed (resp. semi-closed, α -closed, semi-preclosed) set.

Definition: 2.2 [30]

A point x of a space X is called a θ -adherent point of a subset A of X if $\text{cl}(U) \cap A \neq \emptyset$, for every open set U containing x . The set of all θ -adherent points of A is called the θ -closure of A and is denoted by $\text{cl}_{\theta}(A)$. A subset A of a space X is called θ -closed if and only if $A = \text{cl}_{\theta}(A)$. The complement of a θ -closed set is called θ -open. Similarly, the θ -interior of a set A in X , written $\text{int}_{\theta}(A)$, consists of those points x of A such that for some open set U containing x , $\text{cl}(U) \subseteq A$. A set A is θ -open if and only if $A = \text{int}_{\theta}(A)$, or equivalently, $X \setminus A$ is θ -closed.

A point x of a space X is called a δ -adherent point of a subset A of X if $\text{int}(\text{cl}(U)) \cap A \neq \emptyset$, for every open set U containing x . The set of all δ -adherent points of A is called the δ -closure of A and is denoted by $\text{cl}_{\delta}(A)$. A subset A of a space X is called δ -closed if and only if $A = \text{cl}_{\delta}(A)$. The complement of a δ -closed set is called δ -open. Similarly, the δ -interior of a set A in X , written $\text{int}_{\delta}(A)$, consists of those points x of A such that for some regularly open set U containing x , $U \subseteq A$. A set A is δ -open if and only if $A = \text{int}_{\delta}(A)$, or equivalently, $X \setminus A$ is δ -closed.

The family of all θ -open (resp. δ -open) subsets of (X, τ) forms a topology on X and is denoted by τ_{θ} (resp. τ_{δ}). From the definitions it follows immediately that $\tau_{\theta} \subseteq \tau_{\delta} \subseteq \tau$. [7].

Definition: 2.3 A point $x \in X$ is called a semi θ -cluster [9] point of A if $A \cap \text{scl}(U) \neq \emptyset$ for each semi-open set U containing x .

The set of all semi θ -cluster points of A is called the semi- θ -cluster of A and is denoted by $\text{scl}_{\theta}(A)$. Hence, a subset A is called semi- θ -closed if $\text{scl}_{\theta}(A) = A$. The complement of a semi- θ -closed set is called semi- θ -open set.

Recall that a subset A of a space (X, τ) is said to be δ -semi-open [22] if $A \subseteq \text{cl}(\text{int}_{\delta}(A))$.

Definition: 2.4 A subset A of a space (X, τ) is called:

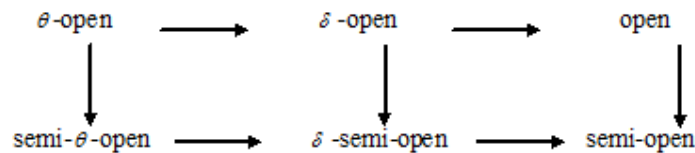
- (i) a generalized closed (briefly, g-closed) set [16] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (ii) a generalized semi-closed (briefly, gs-closed) set [4] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (iii) a semi-generalized closed (briefly, sg-closed) set [5] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ) .
- (iv) an α -generalized closed (briefly, α g-closed) set [18] if $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .

- (v) a generalized semi-preclosed (briefly, gsp-closed) set [10] if $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (vi) a generalized preclosed (briefly, gp-closed) set [21] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (vii) a \ddot{g} -closed set [27] (= ω -closed set [25]) if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ) .
- (viii) a \ddot{g} -closed set [23] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ) .
- (ix) ψ -closed set [29] if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ) .
- (x) a θ -generalized closed set (briefly, θg -closed) [11] if $\text{cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- (xi) a $\theta\omega$ -closed set [13] if $\text{cl}_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in (X, τ) .

Remark: 2.5 The collection of all θg -closed (resp. \ddot{g} -closed, $\theta\omega$ -closed, sg-closed, ψ -closed, ω -closed, g-closed, θ -closed, α -closed, semi-closed) sets of X is denoted by $\theta GC(X)$ (resp. $\ddot{G}C(X)$, $\theta\omega C(X)$, $SGC(X)$, $\psi C(X)$, $\omega C(X)$, $GC(X)$, $\theta C(X)$, $\alpha C(X)$, $SC(X)$).

We denote the power set of X by $P(X)$.

Remark: 2.6 [6] We have the following diagram in which the converses of the implications need not be true.



Remark: 2.7 [25]

- (i) Every θ -closed set is θg -closed.
- (ii) θg -closed sets and ω -closed sets are independent.

Remark: 2.8 [7] (X, τ) is regular if and only if $\tau_\theta = \tau$.

Remark: 2.9 [24] A space X is called $\tau\ddot{g}$ if \ddot{g} -closed set in X is closed.

Definition: 2.10 A topological space (X, τ) is called a R_1 -space [12] if every two different points with distinct closures have disjoint neighborhoods.

Proposition: 2.11 [7] Let (X, τ) be a space. Then,

- (i) if $A \subseteq X$ is preopen then $\text{cl}(A) = \text{cl}_\theta(A)$.
- (ii) (X, τ) is R_1 if and only if $\text{cl}(\{x\}) = \text{cl}_\theta(\{x\})$ for each $x \in X$.

Proposition: 2.12 [12, 14] Let (X, τ) be a space. If $A \subseteq X$ is preopen then $\text{cl}(A) = \alpha \text{cl}(A) = \text{cl}_\delta(A)$.

Definition: 2.13 [16] A space (X, τ) is called $T_{1/2}$ -space if every g-closed set is closed.

Lemma: 2.14 [13] In any space, if a singleton is θ -open then it is regular open.

Lemma: 2.15 [13] In a regular space, singleton is θ -open if and only if it is regular open.

Lemma: 2.16 [13] If A is both closed and preopen of a topological space X , then the following are equivalent.

- (i) A is θ -closed.
- (ii) A is δ -closed.
- (iii) A is α -closed.

Lemma: 2.17 [13] If a subset A of a space (X, τ) is clopen, then the following are equivalent.

- (i) A is θ -closed.
- (ii) A is δ -closed.
- (iii) A is α -closed.
- (iv) A is regular closed.

3. $\theta\ddot{g}$ -CLOSED SETS:

We introduce the following definition.

Definition: 3.1 A subset A of X is called a $\theta\ddot{g}$ -closed set if $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open in (X, τ) . The complement of $\theta\ddot{g}$ -closed set is called $\theta\ddot{g}$ -open set.

The collection of all $\theta\ddot{g}$ -closed sets of X is denoted by $\theta\ddot{G}C(X)$.

Proposition: 3.2 Every θ -closed set is $\theta\ddot{g}$ -closed.

Proof: Let A be a θ -closed set and G be any sg-open set containing A in (X, τ) . Since A is θ -closed, $cl_\theta(A) = A$ for every subset A of X. Therefore $cl_\theta(A) \subseteq G$ and hence A is $\theta\ddot{g}$ -closed set.

The converse of Proposition 3.2 need not be true as seen from the following example.

Example: 3.3 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$. Then $\theta\ddot{G}C(X) = \{\emptyset, \{b, c\}, X\}$ and $\theta C(X) = \{\emptyset, X\}$. Here, $A = \{b, c\}$ is $\theta\ddot{g}$ -closed but not θ -closed set in (X, τ) .

Proposition: 3.4 Every $\theta\ddot{g}$ -closed set is θg -closed.

Proof: Let A be a $\theta\ddot{g}$ -closed set and G be any open set containing A in (X, τ) . Since every open set is sg-open and A is $\theta\ddot{g}$ -closed, $cl_\theta(A) \subseteq G$. Therefore $cl_\theta(A) \subseteq G$ and G is open. Hence A is θg -closed.

The converse of Proposition 3.4 need not be true as seen from the following example.

Example: 3.5 Let X and τ be as in the Example 3.3. Then $\theta\ddot{G}C(X) = \{\emptyset, \{b, c\}, X\}$ and $\theta g C(X) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here, $A = \{a, c\}$ is θg -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.6 Every $\theta\ddot{g}$ -closed set is $\theta\omega$ -closed.

Proof: Let A be a $\theta\ddot{g}$ -closed set and G be any semi open set containing A in (X, τ) . Since every semi open set is sg-open and A is $\theta\ddot{g}$ -closed, $cl_\theta(A) \subseteq G$. Hence A is $\theta\omega$ -closed.

The converse of Proposition 3.6 need not be true as seen from the following example.

Example: 3.7 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\theta\ddot{G}C(X) = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\theta\omega C(X) = P(X)$. Here, $A = \{a, b\}$ is $\theta\omega$ -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.8 Every $\theta\ddot{g}$ -closed set is g-closed.

Proof: Let A be a $\theta\ddot{g}$ -closed set and G be any open set containing A in (X, τ) . Since every open set is sg-open and A is $\theta\ddot{g}$ -closed, $cl_\theta(A) \subseteq G$. Since $cl(A) \subseteq cl_\theta(A) \subseteq G$, $cl(A) \subseteq G$ and hence A is g-closed.

The converse of Proposition 3.8 need not be true as seen from the following example.

Example: 3.9 Let X and τ be as in the Example 3.3. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and $GC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Here, $A = \{a, b\}$ is g -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.10 Every $\theta\ddot{g}$ -closed set is ω -closed.

Proof: Let A be a $\theta\ddot{g}$ -closed set and G be any semi open set containing A in (X, τ) . Since every semi open set is sg -open and A is $\theta\ddot{g}$ -closed, $cl_{\theta}(A) \subseteq G$. Since $cl(A) \subseteq cl_{\theta}(A) \subseteq G$, $cl(A) \subseteq G$ and hence A is ω -closed.

The converse of Proposition 3.10 need not be true as seen from the following example.

Example: 3.11 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and $\omega C(X) = \{\phi, \{c\}, \{b, c\}, X\}$. Here, $A = \{c\}$ is ω -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.12 Every $\theta\ddot{g}$ -closed set is \ddot{g} -closed.

Proof: Let A be a $\theta\ddot{g}$ -closed and G be any sg -open set containing A . Since $cl(A) \subseteq cl_{\theta}(A) \subseteq G$ and hence A is \ddot{g} -closed.

The converse of Proposition 3.12 need not be true as seen from the following example.

Example: 3.13 Let X and τ be as in the Example 3.11. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and $\ddot{G}C(X) = \{\phi, \{c\}, \{b, c\}, X\}$. Here, $A = \{c\}$ is \ddot{g} -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.14 Every $\theta\ddot{g}$ -closed set is sg -closed.

Proof: Let A be a $\theta\ddot{g}$ -closed and G be any semi open set containing A in (X, τ) . Since every semi open set is sg -open, $cl_{\theta}(A) \subseteq G$. Since $scl(A) \subseteq cl_{\theta}(A) \subseteq G$, $scl(A) \subseteq G$ and hence A is sg -closed.

The converse of Proposition 3.14 need not be true as seen from the following example.

Example: 3.15 Let X and τ be as in the Example 3.7. Then $\theta\ddot{G}C(X) = \{\phi, \{a\}, \{b, c\}, X\}$ and $SGC(X) = P(X)$. Here, $A = \{a, b\}$ is sg -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 3.16 Every $\theta\ddot{g}$ -closed set is ψ -closed.

Proof: It is true that $scl(A) \subseteq cl_{\theta}(A)$ for every subset A of (X, τ) .

The converse of Proposition 3.16 need not be true as seen from the following example.

Example: 3.17 Let X and τ be as in the Example 3.3. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and $\psi C(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Here, $A = \{b\}$ is ψ -closed but not $\theta\ddot{g}$ -closed set in (X, τ) .

Remark: 3.18 The following examples show that $\theta\ddot{g}$ -closedness is independent of closedness, semi-closedness and α -closedness.

Example: 3.19 Let X and τ be as in the Example 3.3. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and $\alpha C(X) = SC(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$. Here, $A = \{b\}$ is α -closed as well as semi-closed in (X, τ) but it is not $\theta\ddot{g}$ -closed set in (X, τ) .

Example: 3.20 Let $X = \{a, b, c\}$ with $\tau = \{\phi, \{a, b\}, X\}$. Then $\theta\ddot{G}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and $\alpha C(X) = SC(X) = \{\phi, \{c\}, X\}$. Here, $A = \{a, c\}$ is $\theta\ddot{g}$ -closed but it is neither α -closed set nor semi-closed set in (X, τ) .

Example: 3.21 In Example 3.11, $\{c\}$ is closed but not $\theta\ddot{g}$ -closed set.

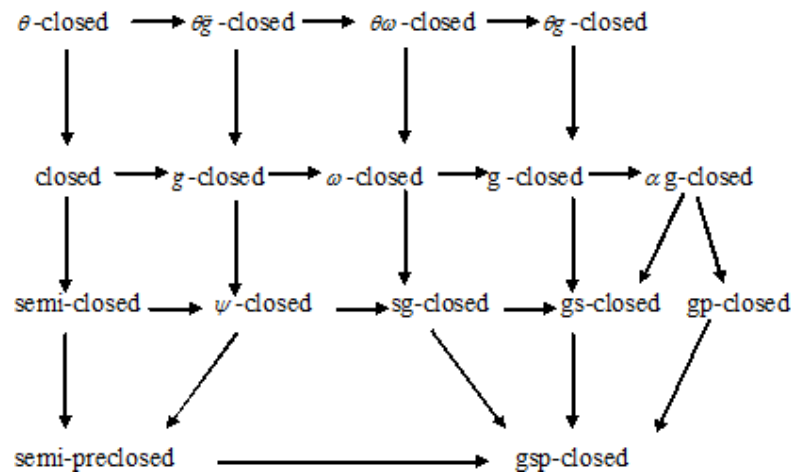
In Example 3.20, $\{b, c\}$ is $\theta\ddot{g}$ -closed but not closed set.

Remark: 3.22 $\theta\omega$ -closed sets and \ddot{g} -closed sets are independent.

Example: 3.23 In Example 3.7, $\{a, b\}$ is $\theta\omega$ -closed but not \ddot{g} -closed set.

In Example 3.11, $\{c\}$ is \ddot{g} -closed but not $\theta\omega$ -closed set.

Remark: 3.24 From the above discussions and known results in [10, 12, 13, 23, 25, 28, 29], we obtain the following diagram, where $A \rightarrow B$ (resp. $A \longleftrightarrow B$) represents A implies B but not conversely (resp. A and B are independent of each other).



4. PROPERTIES OF $\theta\ddot{g}$ -CLOSED SETS:

Definition: 4.1 [23] The intersection of all sg-open subsets of (X, τ) containing A is called the sg-kernel of A and is denoted by $\text{sg-ker}(A)$.

Lemma: 4.2 A subset A of (X, τ) is $\theta\ddot{g}$ -closed if and only if $cl_\theta(A) \subseteq \text{sg-ker}(A)$.

Proof: Suppose that A is $\theta\ddot{g}$ -closed. Then $cl_\theta(A) \subseteq U$ whenever $A \subseteq U$ and U is sg-open. Let $x \in cl_\theta(A)$. If $x \notin \text{sg-ker}(A)$, then there is a sg-open set U containing A such that $x \notin U$. Since U is a sg-open set containing A , we have $x \in cl_\theta(A)$ and this is a contradiction.

Conversely, let $cl_\theta(A) \subseteq \text{sg-ker}(A)$. If U is any sg-open set containing A , then $cl_\theta(A) \subseteq \text{sg-ker}(A) \subseteq U$. Therefore, A is $\theta\ddot{g}$ -closed.

Remark: 4.3 The collection of all $\theta\ddot{g}$ -closed sets of a topological space forms a topology and is denoted by $\tau_{\theta\ddot{g}}$.

Remark: 4.4 If A is a $\theta\ddot{g}$ -closed set and F is a θ -closed set, then $A \cap F$ is a $\theta\ddot{g}$ -closed set.

Proof: Since F is θ -closed, it is $\theta\ddot{g}$ -closed. Therefore by Remark 4.3, $A \cap F$ is also a $\theta\ddot{g}$ -closed set.

Proposition: 4.5 If a set A is $\theta\ddot{g}$ -closed in (X, τ) , then $cl_\theta(A) - A$ contains no nonempty sg-closed set in (X, τ) .

Proof: Suppose that A is $\theta\ddot{g}$ -closed. Let F be a sg-closed subset of $cl_\theta(A) - A$. Then $A \subseteq F^c$. Therefore $cl_\theta(A) \subseteq F^c$. Consequently, $F \subseteq (cl_\theta(A))^c$. We already have $F \subseteq cl_\theta(A)$. Thus $F \subseteq cl_\theta(A) \cap (cl_\theta(A))^c$ and F is empty.

The converse of Proposition 4.5 need not be true as seen from the following example.

Example: 4.6 Let X and τ be as in the Example 3.20. Then $\theta\ddot{g}C(X) = SG C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. If $A = \{c\}$, then $cl_{\theta}(A) - A = \{a, b\}$ does not contain any nonempty sg-closed set. But A is not $\theta\ddot{g}$ -closed in (X, τ) .

Proposition: 4.7 Let $A \subseteq Y \subseteq X$ where Y is open and suppose that A is $\theta\ddot{g}$ -closed in (X, τ) . Then A is $\theta\ddot{g}$ -closed relative to Y .

Proof: Let $A \subseteq Y \cap G$, where G is sg-open in (X, τ) . Then $A \subseteq G$ and hence $cl_{\theta}(A) \subseteq G$. This implies that $Y \cap cl_{\theta}(A) \subseteq Y \cap G$. Thus A is $\theta\ddot{g}$ -closed relative to Y .

Proposition: 4.8 If A is a sg-open and $\theta\ddot{g}$ -closed in (X, τ) , then A is θ -closed in (X, τ) .

Proof: Since A is sg-open and $\theta\ddot{g}$ -closed, $cl_{\theta}(A) \subseteq A$ and hence A is θ -closed in (X, τ) .

Theorem: 4.9 Let A be a subset of a regular space (X, τ) . Then,

- (i) A is $\theta\ddot{g}$ -closed if and only if A is \ddot{g} -closed.
- (ii) if (X, τ) is $\tau\ddot{g}$, then A is $\theta\ddot{g}$ -closed if and only if A is closed.

Proof:

- (i) It follows from Remark 2.8.
- (ii) It follows from Remark 2.9.

Theorem: 4.10 Let A be a preopen subset of a topological space (X, τ) . Then the following conditions are equivalent.

- (i) A is $\theta\ddot{g}$ -closed.
- (ii) A is $\theta\omega$ -closed (or \ddot{g} -closed).
- (iii) A is θg -closed (or ω -closed).
- (iv) A is g-closed.
- (v) A is α g-closed.

Proof:

- (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v). It is obvious from Remark 3.24.
- (v) \Rightarrow (i). It follows from Propositions 2.11 and 2.12.

Recall that a partition space [12] is a topological space where every open set is closed.

Corollary: 4.11 Let A be a subset of the partition space (X, τ) . Then the following conditions are equivalent.

- (i) A is $\theta\ddot{g}$ -closed.
- (ii) A is $\theta\omega$ -closed (or \ddot{g} -closed).
- (iii) A is θg -closed (or ω -closed).
- (iv) A is g-closed.
- (v) A is α g-closed.

Proof: A topological space is a partition space if and only if every subset is preopen. Then the claim follows straight from Theorem 4.10.

Theorem: 4.12 For a singleton subset A of an R_1 topological space (X, τ) , the following conditions are equivalent.

- (i) A is $\theta\ddot{g}$ -closed.
- (ii) A is \ddot{g} -closed.

Proof: (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (i). Note that in R_1 -spaces, the concepts of closure and θ -closure coincide for singleton sets: see Proposition 2.11.

Theorem: 4.13 For a subset A of a topological space (X, τ) , the following conditions are equivalent.

- (i) A is clopen.
- (ii) A is $\theta\ddot{g}$ -closed, preopen and semi-closed.
- (iii) A is $\theta\ddot{g}$ -closed and (regular) open.
- (iv) A is α g-closed and (regular) open.

Proof: (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (i). It follows from Theorem 3.13 [12].

Definition: 4.14 A space (X, τ) is called locally sg- θ -indiscrete space if every sg-open set is θ -closed.

Theorem: 4.15 For a topological space (X, τ) , the following conditions are equivalent.

- (i) X is locally sg- θ -indiscrete.
- (ii) Every subset of X is $\theta\ddot{g}$ -closed.

Proof: (i) \Rightarrow (ii). Let $A \subseteq U$, where U is sg-open and A is an arbitrary subset of X. Since X is locally sg- θ -indiscrete, then U is θ -closed. We have $cl_{\theta}(A) \subseteq cl_{\theta}(U) = U$. Thus A is $\theta\ddot{g}$ -closed.

(ii) \Rightarrow (i). If $U \subseteq X$ is sg-open, then by (ii) $cl_{\theta}(U) \subseteq U$ or equivalently U is θ -closed. Hence X is locally sg- θ -indiscrete.

5. DECOMPOSITION OF θ -CONTINUITY:

In this section, we obtain a decomposition of continuity called θ -continuity in topological spaces.

To obtain a decomposition of θ -continuity, we first introduce the notion of $\theta\ddot{g}$ lc*-continuous functions in topological spaces and by using $\theta\ddot{g}$ -continuity, prove that a function is θ -continuous if and only if it is both $\theta\ddot{g}$ -continuous and $\theta\ddot{g}$ lc*-continuous.

We introduce the following definition.

Definition: 5.1 A subset A of a space (X, τ) is called $\theta\ddot{g}$ lc*-set if $A = M \cap N$, where M is sg-open and N is θ -closed in (X, τ) .

Example: 5.2 Let X and τ be as in the Example 3.3. Then $\{a, b\}$ is $\theta\ddot{g}$ lc*-set in (X, τ) .

Remark: 5.3 Every θ -closed set is $\theta\ddot{g}$ lc*-set but not conversely.

Example: 5.4 Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{b\}, X\}$. Then $\{b, c\}$ is $\theta\ddot{g}$ lc*-set but not θ -closed in (X, τ) .

Remark: 5.5 $\theta\ddot{g}$ -closed sets and $\theta\ddot{g}$ lc*-sets are independent of each other.

Example: 5.6 Let X and τ be as in the Example 3.20. Then $\{a, c\}$ is an $\theta\ddot{g}$ -closed set but not $\theta\ddot{g}$ lc*-set in (X, τ) .

Example: 5.7 Let X and τ be as in the Example 5.4. Then $\{a, b\}$ is an $\theta\ddot{g}$ lc*-set but not $\theta\ddot{g}$ -closed set in (X, τ) .

Proposition: 5.8 Let (X, τ) be a topological space. Then a subset A of (X, τ) is θ -closed if and only if it is both $\theta\ddot{g}$ -closed and $\theta\ddot{g}$ lc*-set.

Proof: Necessity is trivial. To prove the sufficiency, assume that A is both $\theta\ddot{g}$ -closed and $\theta\ddot{g}$ lc*-set. Then $A = M \cap N$, where M is sg-open and N is θ -closed in (X, τ) . Therefore, $A \subseteq M$ and $A \subseteq N$ and so by hypothesis, $cl_{\theta}(A) \subseteq M$ and $cl_{\theta}(A) \subseteq N$. Thus $cl_{\theta}(A) \subseteq M \cap N = A$ and hence $cl_{\theta}(A) = A$ i.e., A is θ -closed in (X, τ) .

We introduce the following definition

Definition: 5.9 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\theta\ddot{g}$ lc*-continuous if for each closed set V of (Y, σ) , $f^{-1}(V)$ is a $\theta\ddot{g}$ lc*-set in (X, τ) .

Example: 5.10 Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $\theta\ddot{g}$ lc*-continuous function.

Definition: 5.11 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

- (i) θ -continuous [3] if for each closed set V of Y, $f^{-1}(V)$ is θ -closed in X.
- (ii) $\theta\ddot{g}$ -continuous if for each closed set V of Y, $f^{-1}(V)$ is $\theta\ddot{g}$ -closed in X.

Proposition: 5.12 Every θ -continuous function is $\theta\ddot{g}$ -continuous but not conversely.

Proof: It follows from Proposition 3.2.

Example: 5.13 Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. We have $\theta C(X) = \{\emptyset, X\}$ and $\theta\ddot{G} C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Define $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $\theta\ddot{g}$ -continuous but not θ -continuous, since $f^{-1}(\{a, c\}) = \{a, c\}$ is not θ -closed in (X, τ) .

Remark: 5.14 Every θ -continuous function is $\theta\ddot{g}$ lc*-continuous but not conversely.

Example: 5.15 Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $\theta\ddot{g}$ lc*-continuous function but not θ -continuous since for the closed set $\{b\}$ in (Y, σ) , $f^{-1}(\{b\}) = \{b\}$, which is not θ -closed in (X, τ) .

Remark: 5.16 $\theta\ddot{g}$ -continuity and $\theta\ddot{g}$ lc*-continuity are independent of each other.

Example: 5.17 Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{b\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $\theta\ddot{g}$ -continuous function but not $\theta\ddot{g}$ lc*-continuous.

Example: 5.18 Let $X = Y = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b, c\}, Y\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be the identity function. Then f is $\theta\ddot{g}$ lc*-continuous function but not $\theta\ddot{g}$ -continuous.

We have the following decomposition for continuity.

Theorem: 5.19 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is θ -continuous if and only if it is both $\theta\ddot{g}$ -continuous and $\theta\ddot{g}$ lc*-continuous.

Proof: Assume that f is θ -continuous. Then by Proposition 5.12 and Remark 5.14, f is both $\theta\ddot{g}$ -continuous and $\theta\ddot{g}$ lc*-continuous.

Conversely, assume that f is both $\theta\ddot{g}$ -continuous and $\theta\ddot{g}$ lc*-continuous. Let V be a closed subset of (Y, σ) . Then $f^{-1}(V)$ is both $\theta\ddot{g}$ -closed and $\theta\ddot{g}$ lc*-set. By Proposition 5.8, $f^{-1}(V)$ is a θ -closed set in (X, τ) and so f is θ -continuous.

6. DECOMPOSITION OF $T_{1/2}$ -SPACES:

We introduce the following definition:

Definition: 6.1 A space (X, τ) is called a $T_{\theta\ddot{g}}$ -space if every $\theta\ddot{g}$ -closed set in it is closed.

Example: 6.2 Let X and τ be as in the Example 3.3. Then $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$ and the sets in $\{\phi, \{b, c\}, X\}$ are closed. Thus (X, τ) is a $T_{\theta\ddot{g}}$ -space.

Example: 6.3 Let X and τ be as in the Example 3.20. Then $\theta\ddot{G}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$ and the sets in $\{\phi, \{c\}, X\}$ are closed. Thus (X, τ) is not a $T_{\theta\ddot{g}}$ -space.

Theorem: 6.4 For a topological space (X, τ) , the following properties are equivalent:

- (i) (X, τ) is a $T_{\theta\ddot{g}}$ -space.
- (ii) Every singleton of (X, τ) is either open or sg-closed.

Proof: (i) \rightarrow (ii). If $\{x\}$ is not sg-closed, then $X - \{x\}$ is not sg-open. Hence X is only sg-open set containing $X - \{x\}$. Therefore $cl_{\theta}(X - \{x\}) \subseteq X$. Thus $X - \{x\}$ is $\theta\ddot{g}$ -closed. By (i) $X - \{x\}$ is closed, i.e. $\{x\}$ is open.

(ii) \rightarrow (i). Let $A \subseteq X$ be a $\theta\ddot{g}$ -closed. Let $x \in cl_{\theta}(A)$. We consider the following two cases:

Case: (a) Let $\{x\}$ be open. Since x belongs to the closure of A , then $\{x\} \cap A \neq \phi$. This shows that $x \in A$.

Case: (b) Let $\{x\}$ be sg-closed. If we assume that $x \notin A$, then we would have $x \in cl_{\theta}(A) - A$ which cannot happen according to Proposition 4.5. Hence $x \in A$.

So in both cases we have $cl_{\theta}(A) \subseteq A$. Since the reverse inclusion is trivial, then $A = cl_{\theta}(A)$ or equivalently A is θ -closed. It implies that A is closed.

Definition: 6.5 A space (X, τ) is called ${}_gT_{\theta\ddot{g}}$ -space if every g -closed set is $\theta\ddot{g}$ -closed.

Example: 6.6 Let X and τ be as in the Example 3.20. Then $GC(X) = \theta\ddot{G}C(X) = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus (X, τ) is a ${}_gT_{\theta\ddot{g}}$ -space.

Example: 6.7 Let X and τ be as in the Example 3.3. Then $GC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$. Thus (X, τ) is not a ${}_gT_{\theta\ddot{g}}$ -space.

Proposition: 6.8 Every $T_{1/2}$ -space is $T_{\theta\ddot{g}}$ -space but not conversely.

Proof: Follows from Proposition 3.8.

The converse of Proposition 6.8 need not be true as seen from the following example.

Example: 6.9 Let X and τ be as in the Example 3.3. Then $GC(X) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$ and $\theta\ddot{G}C(X) = \{\phi, \{b, c\}, X\}$. Thus (X, τ) is $T_{\theta\ddot{g}}$ -space but it is not a $T_{1/2}$ -space.

Proposition: 6.10 Every regular $T_{1/2}$ -space is ${}_gT_{\theta\ddot{g}}$ -space but not conversely.

Proof: Obvious.

The converse of Proposition 6.10 need not be true as seen from the following example.

Example: 6.11 Let X and τ be as in the Example 3.20. Then $G C(X) = \theta\ddot{g} C(X) = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$. Thus (X, τ) is a ${}_gT\theta\ddot{g}$ -space but not a $T_{1/2}$ -space.

Remark: 6.12 $T\theta\ddot{g}$ -spaces and ${}_gT\theta\ddot{g}$ -spaces are independent.

Example: 6.13 Let X and τ be as in the Example 3.20. Thus (X, τ) is a ${}_gT\theta\ddot{g}$ -space but it is not a $T\theta\ddot{g}$ -space.

Example: 6.14 Let X and τ be as in the Example 3.3. Thus (X, τ) is a $T\theta\ddot{g}$ -space but it is not a ${}_gT\theta\ddot{g}$ -space.

Theorem: 6.15 A regular space (X, τ) is $T_{1/2}$ if and only if it is both $T\theta\ddot{g}$ and ${}_gT\theta\ddot{g}$.

Proof: Necessity. Follows from Propositions 6.8 and 6.10.

Sufficiency. Assume that (X, τ) is both $T\theta\ddot{g}$ and ${}_gT\theta\ddot{g}$. Let A be a g -closed set of (X, τ) . Then A is $\theta\ddot{g}$ -closed, since (X, τ) is ${}_gT\theta\ddot{g}$. Again since (X, τ) is a $T\theta\ddot{g}$, A is closed set in (X, τ) and so (X, τ) is $T_{1/2}$.

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