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ON GENERALIZED SASAKIAN SPACE FORM

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ABSTRACT

In this article, we study conharmonically flat generalized Sasakian space forms. The eigenvalues of Ricci operator Q for such spaces are obtained. Under certain conditions we also find some relations between differentiable functions f_1, f_2 and f_3 in conharmonically flat generalized Sasakian space form. Three dimensional non-Sasakian contact metric generalized Sasakian space form are also studied. We have proved that in three dimensional conharmonically flat non-Sasakian contact metric generalized Sasakian space form $M(f_1, f_2, f_3)$ is flat. Projectively flat, quasi projectively flat, ξ-projectively flat and φ-projectively flat generalized Sasakian space forms are also studied under certain conditions.

Keywords: *generalized Sasakian space form, conharmonically flat, conformally flat, quasi projectively flat, ξ-projectively, φ-projectively, Ricci operator, curvature tensor, scalar curvature.*

1. INTRODUCTION:

The notion of generalized Sasakian space form was studied and introduced by Alegre et al [2]. For a given almost contact metric manifold M with almost contact metric structure (φ, ξ, η, g), we say that M is a generalized Sasakian space form if there exist three smooth functions f_1, f_2, f_3 on M such that

$$\begin{aligned} R(X,Y)Z = {} & f_1[g(Y,Z)X - g(X,Z)Y] \\ & + f_2[g(X,\varphi Z)\varphi Y - g(Y,\varphi Z)\varphi X + 2g(X,\varphi Y)\varphi Z] \\ & + f_3[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ & + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi], \end{aligned} \tag{1.1}$$

for all vector fields X, Y, Z on M. We shall write generalized Sasakian space form in such a case as $M(f_1, f_2, f_3)$. This type of manifold appears as a natural generalization of the well known Sasakian space form $M(c)$, which can be obtain as a particular case of generalized Sasakian space form by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$. More ever cosymplectic space form, Kenmotsu space form are also particular cases of generalized Sasakian space forms$M(f_1, f_2, f_3)$.

It is well known that the conformal transformations on the Riemannian manifolds do not change the angle between two vectors at a point, but in general the harmonicity of functions, vectors and forms are not preserve by conformal transformations. Ishii studied and calculated conharmonic curvature tensor which is geometric invariant under conharmonic transformation [8]. A Riemannian manifold M is said to be Ricci Symmetric if its Ricci tensor S is semi symmetric, that is its curvature tensor R satisfies $R(X,Y).S = 0$; $X, Y \in TM$, where $R(X,Y)$ acts on S as a derivative. In [12], M. Tarafdar and A. Bhattacharya and in [13], M. Tarafdar and A. Marya satisfied a type of Riemannian manifold which is conformally flat and conformally flat Kenmotsu manifold with $R(X,Y).S = 0$. A $(2n+1)$ −dimensional Riemannian manifold M is conformally flat if $C = 0$, where C is conharmonic tensor defined as

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$$C(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z) + g(Y,Z)QX - g(X,Z)QY], \tag{1.2}$$

for all $X, Y, Z \in TM$, where R, S and Q are Curvature-tensor, Ricci-tensor and Ricci-operator respectively.

Since Q is linear symmetric endomorphism on the tangent space over field of real numbers. So Q is symmetric linear operator over the field of real numbers hence matrix of Q is real symmetric matrix of order $(2n+1)$. We know that every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements i.e. every real symmetric matrix is diagonalizable and have real eigenvalues. Every real symmetric matrix of order m has m mutually orthogonal real eigenvectors. If algebraic multiplicity of an eigenvalue λ of a real symmetric matrix is p, then it can have p but not more then p mutually orthogonal real eigenvectors corresponding to λ. A square matrix is diagonalizable iff the geometric multiplicity of each of its eigenvalue is equal to its algebraic multiplicity. It is also known that the geometric multiplicity of an eigenvalue can not exceed with its algebraic multiplicity. A square matrix of order m is diagonalizable iff it possesses m linearly independent eigenvectors. If the eigenvalues of a square matrix of order m are all distinct then it is always diagonalizable but if the eigenvalues of square matrix are not all distinct then it may or may not be diagonalizable. In other words we say T is diagonalizable when the eigenvectors of T spans . A linear operator T on a finite dimensional vector space V is diagonalizable iff there is a basis for , each vector of which is an eigenvector of T .

In differential geometry projective curvature tensor is also important. On m −dimensional Riemannian manifold M, the projective curvature tensor is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{m-1}[g(Y,Z)QX - g(X,Z)QY], \tag{1.3}$$

for all $X, Y, Z \in TM$. For $m \geq 3$, M is locally projectively flat if and only if P vanishes. Moreover M is projectively flat ($P = 0$) if and only if it is of constant curvature [15].

2. PRELIMINARIES:

A $(2n+1)$-dimensional Riemannian manifold (M, g) is said to be an almost contact metric manifold if there exist a $(1,1)$-tensor field φ, a unique global non-vanishing structural vector field ξ (called the vector field) and a 1-form η such that:

$$\eta(\xi) = 1,\ \varphi^2(X) = -X + \eta(X)\xi,\ \varphi\xi = 0 \tag{2.1}$$

$$d\eta(X,\xi) = 0,\ g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

$$d\eta(X,Y) = g(X,\varphi Y), \qquad \eta o \varphi = 0, \tag{2.4}$$

such a manifold is called contact manifold if $\eta \wedge (d\eta)^n \neq 0$, where n is n^{th} exterior power. For contact manifold we also have $d\eta = \Phi$, where $\Phi(X,Y) = g(\varphi X, Y)$ is called fundamental 2-form on *M*. If ξ is Killing vector field, then *M* is said to be *K*-contact manifold. The almost contact metric structure (φ, ξ, η, g) of *M* is said to be normal if:

$$[\varphi,\varphi](X,Y) + 2d\eta(X,Y)\xi = 0, \tag{2.5}$$

for any vector field *X*, *Y* on *M*, where [φ, φ] denotes the Nijenhuis torsion of φ given by:

$$[\varphi,\varphi](X,Y = \varphi^2(X,Y) + (\varphi X, \varphi Y) - \varphi(\varphi X, Y) - \varphi(X, \varphi Y). \tag{2.6}$$

An almost contact metric manifold is called trans-Sasakian manifold of type (α , β) if:

$$(\nabla \times \varphi)Y = \alpha(g(X,Y)\xi - \eta(X)Y) + \beta(g(\varphi X, Y)\xi - \eta(X)\varphi Y), \tag{2.7}$$

for all vector field *X*, *Y* on *M*, where α and β are some smooth real valued functions. A trans-Sasakian manifold of type (1, 0) and (0, 1) is called Sasakian and Kenmotsu manifold respectively.

An almost contact metric manifold M is said to be η −Einstein if its Ricci-tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.8}$$

where a and b are smooth functions on M. An η −Einstein manifold becomes Einstein if $b = 0$. If $\{e_1, e_2, \ldots\ldots, e_{2n}, \xi\}$ is a local orthonormal basis of vector fields in a $(2n+1)$ −dimensional almost contact manifold M, then $\{\varphi e_1, \varphi e_2, \ldots\ldots, \varphi e_{2n}, \xi\}$ is also a local orthonormal basis. It is easy to verify that

$$\sum_{i=1}^{2n} g(e_i, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi e_i) = 2n, \tag{2.9}$$

$$\sum_{i=1}^{2n} g(e_i, Y)S(X, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, Y)S(X, \varphi e_i) = S(X,Y) - S(X,\xi)\eta(Y), \tag{2.10}$$

or all $X, Y \in TM$. In view of (2.4) and (2.10), we get
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$$\sum_{i=1}^{2n} g(e_i, \varphi Y)S(\varphi X, e_i) = \sum_{i=1}^{2n} g(\varphi e_i, \varphi Y)S(\varphi X, \varphi e_i) = S(\varphi X, \varphi Y). \tag{2.11}$$

3. SOME BASIS RESULTS ON GENERALIZED SASAKIAN SPACE FORM:

For a $(2n + 1)$ −dimensional generalized Sasakian space forms $M(f_1, f_2, f_3)$, we have

$$R(X,Y)\xi = (f_1 - f_3)(\eta(Y)X - \eta(X)Y), \tag{3.1}$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)\, g(X,Y) - (3f_2 + (2n-1)f_3)\,\eta(X)\eta(Y). \tag{3.2}$$

From (3.1), we have

$$R(X,\xi)\xi = (f_1 - f_3)(X - \eta(X)\xi), \tag{3.3}$$

$$R(X,\xi)Y = (f_1 - f_3)(\eta(Y)X - g(X,Y)\xi), \tag{3.4}$$

$$Q(X) = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X).\xi, \tag{3.5}$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \tag{3.6}$$

where Q is the Ricci operator and r is scalar curvature of $M(f_1, f_2, f_3)$,. Now from (3.2) and (3.5), we have

$$S(X,\xi) = 2n(f_1 - f_3)\,\eta(X) \tag{3.7}$$

and

$$Q\xi = 2n(f_1 - f_3)\,\xi. \tag{3.8}$$

From (3.7), we get

$$\sum_{i=1}^{2n} S(e_i, e_i) = \sum_{i=1}^{2n} S(\varphi e_i, \varphi e_i) = r - 2n(f_1 - f_3), \tag{3.9}$$

where r is scalar curvature. In generalized Sasakian space form $M(f_1, f_2, f_3)$, we also have

$$R(X,\xi,\xi,Y) = R(\xi,X,Y,\xi) = (f_1 - f_3)\, g(\varphi X, \varphi Y) \tag{3.10}$$

and

$$\sum_{i=1}^{2n} R(e_i, X, Y, e_i) = \sum_{i=1}^{2n} R(\varphi e_i, X, Y, \varphi e_i) = S(X,Y) - (f_1 - f_3)g(\varphi X, \varphi Y), \tag{3.11}$$

for all $X, Y \in TM$.

Proposition: 3.1. In a $(2n + 1)$ −dimensional generalized Sasakian space form $M(f_1, f_2, f_3)$, the Ricci-operator Q commutes with φ.

Proof: On replacing X by φX in equation (3.5), we have

$$Q\varphi X = (2nf_1 + 3f_2 - f_3)\,\varphi X. \quad (3.12)$$

Again operating φ on equation (3.5), we get

$$\varphi QX = (2nf_1 + 3f_2 - f_3)\varphi X. \quad (3.13)$$

From (3.12) and (3.13), we have the result.

Proposition: 3.2 Let $M(f_1, f_2, f_3)$, be a $(2n + 1)$ −dimensional generalized Sasakian space form. Then the Ricci-operator Q has two distinct non-zero eigenvalues $(2nf_1 + 3f_2 - f_3)$ and $2n\,(f_1 - f_3)$.

Proof: The equation (3.8) implies that $2n\,(f_1 - f_3)$ is an eigenvalue of Q corresponding to eigenvector ξ. On other hand from equation (3.5), we have

$$Q\,(X) = (2nf_1 + 3f_2 - f_3)X$$

for all vector fields X orthogonal to ξ. Hence $(2nf_1 + 3f_2 - f_3)$ is eigenvalue of Q corresponding to eigenvector $X \in D$, where $D = ker\,(\eta)$the 2n-dimensional contact distribution is orthogonal to ξ.

4. CONHARMONICALLY flAT GENERALIZED SASAKIAN SPACE FORM:

For a conharmonically flat Riemannian manifold of dimension-$(2n + 1)$ M, conharmonic curvature tensor $C(X, Y)Z = 0$, so from equation (1.2), we have:

$$R(X,Y)Z = \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \quad (4.1)$$

Taking $Z = \xi$ and using equations (3.1) & (3.5), we have

$$((f_1 - f_3)(\eta\,(Y)X - \eta\,(X)Y) = \frac{1}{2n-1}\,[2n\,(f_1 - f_3)(\eta\,(Y)X - \eta\,(X)Y)(\eta Q\,(Y)X - \eta\,(X)QY)]. \quad (4.2)$$

Again putting $Y = \xi$ and using equations (3.7) and (3.8), we get

$$QX = ((f_1 - f_3)\,((2n + 1)\,\eta\,(X)\,\xi - X)\,, \quad (4.3)$$

$$S\,(X,Y) = g\,(QX,Y) = (f_1 - f_3)\,[(2n + 1)\,\eta\,(X)\,\eta\,(Y) - g\,(X,Y)]\,. \quad (4.4)$$

Taking$X = \xi$, in equation (4.3), we get (3.8), which shows that ξ is an eigenvector corresponding to eigenvalue$2n\,(f_1 - f_3)$. If we take X ⊥ ξ, then equation (4.3) implies that

$$QX = -\,(f_1 - f_3)\,X, \quad (4.5)$$

which shows that $-(f_1 - f_3)$ is eigenvalue correspond to each eigenvector X ∈ D.

Let $\{e_1, e_2, \ldots\ldots, e_{2n}, \xi\}$ be a local orthonormal basis of vector fields in M, then $Q\xi = 2n(f_1 - f_3)\xi$ and $Qe_i = -(f_1 - f_3)e_i$, for all $i = 1, 2, \ldots. 2n$. Here eigenvalues of Q are $-(f_1 - f_3)$ and $2n(f_1 - f_3)$. The multiplicity of $-(f_1 - f_3)$ and $2n(f_1 - f_3)$ is $2n$ and 1 respectively. As Q is real symmetric linear operator, it is diagonalizable. Since scalar curvature is the trace of Q, so

$$r = 2n(f_1 - f_3) - 2n(f_1 - f_3) = 0. \quad (4.6)$$

Hence for conharmonically flat generalized Sasakian space form scalar curvature $r = 0$. [8]

Proposition: 4.1 For a conharmonically flat generalized Sasakian space form $M(f_1, f_2, f_3)$, the differentiable functions f_1, f_2 and f_3 are related by

$$(2n + 1)f_1 + 3f_2 - 2f_3 = 0. \quad (4.7)$$

Proof: In conharmonically flat generalized Sasakian space form scalar curvature $r = 0$, so from equation (3.6) we have equation (4.7).

For a three dimensional conharmonically generalized Sasakian space form, $3f_1 + 3f_2 - 2f_3 = 0$ and $2f_1 + 3f_2 - f_3 = 0$ [12]. Hence $f_1 = f_3$ and $f_1 + 3f_2 = 0$, which implies that M is flat manifold.

From equations (4.1), (4.3) and (4.4), we have

$$R(X,Y)Z = \frac{(f_1-f_3)}{(2n-1)}[2\{g(X,Z)Y - g(Y,Z)X\} + (2n+1)\{\eta(X)\eta(Y)(X-Y)\} + \{g(Y,Z)\,\eta(X) - g(X,Z)\eta(Y)\}\xi] \tag{4.8}$$

Proposition: 4.2 Let $M(f_1, f_2, f_3)$ be a three dimensional non-Sasakian contact metric generalized Sasakian space form, then

$$2f_1 + 3f_2 - f_3 = 0. \tag{4.9}$$

Proof: In [12] it is proved that the ξ-sectional curvature of a generalized Sasakian space form $M(f_1, f_2, f_3)$ is given by $(f_1 - f_3)$ and its φ-sectional curvature is equal to $(f_1 + 3f_2)$. It was proved in [5] that if $(M, \varphi, \xi, \eta, g)$ is a three dimensional contact metric manifold such that $Q\varphi = \varphi Q$, then M is Sasakian manifold, a flat manifold or a space of constant ξ-sectional curvatures $k < 1$ and constant φ −sectional curvature $-k$. So one of the following cases must hold.

Case: 1 In a flat manifold M, both the ξ-sectional curvature and the φ-sectional curvature vanish i.e. $2f_1 + 3f_2 - f_3 = 0$.

Case: 2 If M is a space of constant ξ-sectional curvature $k < 1$ and φ-sectional curvature $-k$, then $2f_1 + 3f_2 - f_3 = 0$.

Theorem: 4.1 Let $M(f_1, f_2, f_3)$ be a three dimensional conharmonically flat, non-Sasakian contact metric generalized Sasakian space form, then M is flat.

Proof: For three dimensional conharmonically flat generalized Sasakian space form, we have from (4.7)

$$3f_1 + 3f_2 - 2f_3 = 0. \tag{4.10}$$

From equations (4.8), (4.9) and (4.10), we get $R(X,Y)Z = 0$, hence the theorem.

5. PROJECTIVELY flAT GENERALIZED SASAKIAN SPACE FORM:

In a $(2n + 1)$-dimensional almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ the projective curvature tensor P is defined as

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[g(Y,Z)\,QX - g(X,Z)\,QY], \quad X,Y,Z \in TM. \tag{5.1}$$

For a projectively flat Riemannian manifold M, projective curvature tensor $P(X,Y)Z = 0$, so from equation (5.1), we get

$$R(X,Y)Z = \frac{1}{2n}[g(Y,Z)\,QX - g(X,Z)\,QY], \quad X,Y,Z \in TM. \tag{5.2}$$

From equation (3.5), put the value of QX and QY in the above equation, we have

$$R(X,Y)Z = \frac{1}{2n}[(2nf_1 + 3f_2 - f_3)\{Xg(Y,Z) - Yg(X,Z)\} - (3f_2 + (2n-1)f_3)\{\eta(X)\,g(Y,Z) - \eta(Y)g(X,Z)\}],$$

for all $X, Y, Z \in TM$. Taking $Z = \xi$ in the above equation, we get

$$R(X,Y)\xi = \frac{1}{2n}[(2nf_1 + 3f_2 - f_3)\{\eta(Y)X - \eta(X)Y\}], \tag{5.3}$$

for all X, Y ∈ TM. For three-dimensional projectively flat generalized Sasakian space form

$$R(X,Y)\xi = \frac{1}{2}[(2f_1 + 3f_2 - f_3)\{\eta(Y)X - \eta(X)Y\}], \tag{5.4}$$

for all $X, Y \in TM$. But in a three-dimensional non-Sasakian contact metric generalized Sasakian space form $(2f_1 + 3f2-f3=0$. So from equation (5.4), we have

$$R(X,Y)\xi = 0, \tag{5.5}$$

for all $X, Y \in TM$, which implies $R(X, Y)Z = 0$, for all $X, Y, Z \in TM$ [3]. Hence we have the following proposition.

Proposition: 5.1 A three-dimensional non-Sasakian projectively flat generalized Sasakian space form is also flat.

6. SOME STRUCTURE THEOREMS ON GENERALIZED SASAKIAN SPACE FORM:

The Definition of quasi projectively flat, ξ-projectively flat and φ-projectively flat are as follows:

Definition: 6.1 An almost contact metric manifold M is said to be (pp.374 of [14]) quasi projectively flat if:

$$g(P(X,Y)Z, \varphi W) = 0, \qquad X, Y, Z, W \in TM, \tag{6.1}$$

ξ-projectively flat if

$$P(X,Y)\xi = 0, \qquad X, Y \in TM \tag{6.2}$$

and φ-projectively flat if

$$g(P(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0, \qquad X, Y, Z, W \in TM. \tag{6.3}$$

Lemma: 6.1 A $(2n+1)$-dimensional generalized Sasakian space form M is Einstein if $g(P(\varphi X, Y)Z, \varphi W) = 0$.

Proof: Let M be a $(2n+1)$-dimensional generalized Sasakian space form. From equation (5.1) we have

$$g(P(\varphi X, Y)Z, \varphi W) = R(\varphi X, Y, Z, \varphi W) - \frac{1}{2n}[g(Y,Z)\,S(\varphi X, \varphi W) - g(\varphi X, Z)\,S(Y, \varphi W)], \tag{6.4}$$

for all X, Y, Z, W ∈ TM. If $\{e_1, e_2, \ldots\ldots, e_{2n}, \xi\}$ is a local orthonormal basis of the vector fields in M, then from equation (6.4) we get

$$\sum_{i=1}^{2n} g\,(P\,(\varphi e_i, Y)\,Z, \varphi e_i) = \sum_{i=1}^{2n} R\,(\varphi e_i, Y, Z, \varphi e_i) - \sum_{i=1}^{2n}[g\,(Y,Z)\,S\,(\varphi e_i, \varphi e_i) - g\,(\varphi e_i, Z)\,S\,(Y, \varphi e_i)],$$

for all Y, Z ∈ TM. Using equations (2.10), (3.9) and (3.11) in the above equation, we get

$$2n\sum_{i=1}^{2n} g\,(P\,(\varphi e_i, Y)Z, \varphi e_i) = (2n+1)S(Y,Z) - rg(Y,Z) + 2n(f_1 - f_3)\eta(Y)\eta(Z) - S(Y,\xi)\eta(Z). \tag{6.5}$$

If M is quasi projectively flat then, from equations (6.1) and (6.5), we get

$$(2n+1)\,S\,(Y,Z) = rg(Y,Z) - 2n(f_1 - f_3)\eta(Y)\eta(Z) + S\,(Y,\xi)\,\eta\,(Z). \tag{6.6}$$

Putting $Y = \xi$ in equation (6.6) and using equations (2.1) and (2.2), we have

$$(2n+1)\,S\,(\xi, Z) = r\eta(Z), \tag{6.7}$$

from (6.7) we get

$$r = 2n(2n+1)(f_1 - f_3). \tag{6.8}$$

From equations (6.6), (6.7) and (6.8), we have

$$S(Y,Z) = 2n(f_1 - f_3)g(Y,Z), \tag{6.9}$$

which is the condition of Einstein manifold.

In view of **lemma 6.1**, we have the following theorem.

Theorem: 6.1 A quasi projectively flat generalized Sasakian space form is Einstein.

Next, we have the following:

Theorem: 6.2 A generalized Sasakian space form M is quasi projectively flat then

$$R(X,Y,Z,\varphi W) = (f_1 - f_3)[g(Y,Z)\, g(X,\varphi W) - g(X,Z)\, g(Y,\varphi W)], \tag{6.10}$$

for all $X, Y, Z, W \in TM$.

Proof: If M is quasi projectively flat generalized Sasakian space form, using (6.9) in

$$g(P(X,Y)Z,\varphi W) = R(X,Y,Z,\varphi W) - \frac{1}{2n}[g(Y,Z)\, S(X,\varphi W) - g(X,Z)\, S(Y,\varphi W)],$$

we get (6.10).

Theorem: 6.3 A ξ-projectively flat generalized Sasakian space form M is Einstein.

Proof: Using equations (5.1) and (2.2), we get

$$g(P(X,Y)\xi, W) = R(X,Y,\xi,W) - \frac{1}{2n}[\eta(Y)S(X,W) - \eta(X)S(Y,W)], \tag{6.11}$$

for all $X, Y, W \in TM$. For a local orthonormal basis $\{e_1, e_2, \ldots\ldots, e_{2n}, \xi\}$ of the vector fields in M, from equation (6.11) we get

$$\sum_{i=1}^{2n} g(P(e_i,Y)\,\xi, e_i) = \sum_{i=1}^{2n} R(e_i,Y,\xi,e_i) - \frac{1}{2n}\sum_{i=1}^{2n}\eta(Y)S(e_i,e_i),$$

for all $Y \in TM$. Using equations (2.1), (3.9) and (3.11) in the above equation, we get

$$\sum_{i=1}^{2n} g(P(e_i,Y)\xi, e_i) = S(Y,\xi) - \frac{r-2n(f_1-f_3)}{2n}\eta(Y), \tag{6.12}$$

for all $Y \in TM$. If M is ξ-projectively flat then from equation (6.12), we have

$$S\,(Y,\xi) = \frac{r-2n(f_1-f_3)}{2n}\eta(Y), \tag{6.13}$$

for all $Y \in TM$. Putting $Y = \xi$ in equation (6.13) and using equations (2.1) and (3.7), we get (6.8). In view of equation (6.8), equation (6.13) becomes

$$S(Y,\xi) = 2n(f_1 - f_3)\,\eta(Y). \tag{6.14}$$

for all $Y \in TM$. Since M is ξ-projectively flat, Putting $Y = \xi$ in equation (6.11) and using equations (3.10), (6.14) and (2.3), we get

$$S\,(X,W) = 2n(f_1 - f_3)\,\eta(X)\eta(W) \tag{6.15}$$

for all $X, W \in TM$. In view of equation (6.15), proof of Theorem is obvious.

Now we prove the following:

Theorem: 6.4 A generalized Sasakian space form M is φ-projectively flat if and only if

$$R(\varphi X,\varphi Y,\varphi Z,\varphi W) = (f_1 - f_3)[g\,(\varphi Y,\varphi Z)\; g\,(\varphi X,\varphi W) - g(\varphi X,\varphi Z)\; g(\varphi Y,\varphi W)]. \tag{6.16}$$

Proof: Let M be a $(2n+1)$-dimensional generalized Sasakian space form. From equation (5.1) we have

$$g(P(\varphi X,\varphi Y)\varphi Z,\varphi W)=R(\varphi X,\varphi Y,\varphi Z,\varphi W)-\frac{1}{2n}[g(\varphi Y,\varphi Z)S(\varphi X,\varphi W)-g(\varphi X,\varphi Z)S(\varphi Y,\varphi W)], \tag{6.17}$$

for all $X,Y,Z,W\in TM$. For a local orthonormal basis $\{e_1,e_2,\ldots\ldots,e_{2n},\xi\}$ of the vector fields in M, from equation (6.17) we get

$$\sum_{i=1}^{2n} g(P(\varphi e_i,\varphi Y)\varphi Z,\varphi e_i)=\sum_{i=1}^{2n} R(\varphi e_i,\varphi Y,\varphi Z,\varphi e_i)-\frac{1}{2n}\sum_{i=1}^{2n}\{g(\varphi Y,\varphi Z)\,S(\varphi e_i,\varphi e_i)\} +\frac{1}{2n}\sum_{i=1}^{2n}\{g(\varphi e_i,\varphi Z)S(\varphi Y,\varphi e_i)\},$$

for all $Y,Z\in TM$. In view of equations (3.11), (3.9) and (2.11), above equation becomes

$$\sum_{i=1}^{2n} g(P(\varphi e_i,\varphi Y)\,\varphi Z,\varphi e_i)=\frac{(1+2n)}{2n}S(\varphi Y,\varphi Z)-\frac{r}{2n}g(\varphi Y,\varphi Z), \tag{6.18}$$

for all $Y,Z\in TM$. If M is φ-projectively flat then from equation (6.18), we have

$$S(\varphi Y,\varphi Z)=\frac{r}{(2n+1)}g(\varphi Y,\varphi Z), \tag{6.19}$$

for all $Y,Z\in TM$. Using equations (3.9) and (2.9) in the equation (6.19), we get (6.8). In view of equation (6.8), equation (6.19) becomes

$$S(\varphi Y,\varphi Z)=2n\,(f_1-f_3)\,g(\varphi Y,\varphi Z), \tag{6.20}$$

for all $Y,Z\in TM$. Since M is φ-projectively flat, in view of equation (6.20), equation (6.17) yields (6.16). The converse is obvious.

Corollary: 6.1 In a quasi projectively flat, ξ-projectively flat and φ-projectively flat generalized Sasakian space form $3f_2+(2n-1)f_3=0$.

Proof: Using equations (3.6) and (6.8), proof is obvious.

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