International Journal of Mathematical Archive-1(3), Dec. - 2010, Page: 81-85
Available online through www.ijma.info ISSN 2229 – 5046

A COMMON FIXED POINT THEOREM IN D*- METRIC SPACES

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(Received on: 29-11-10; Accepted on: 08-12-10)

ABSTRACT

In this paper we prove a common fixed point theorem for a class of weakly commuting mappings in complete D-metric spaces. This theorem is general extension of results of Shaban Sedghi and Nabi Shobe and Haiyun Zhou [11].*

***Mathematics Subject Classification**: 54E40; 54E35; 54H25*

***Keywords :** D*- metric contractive mapping , Complete D*- metric spaces, Common fixed point theorem.*

1. INTRODUCTION:

Many authors have studied the fixed point theorem in metric space, general metric space, fuzzy metric space, probabilistic fuzzy metric space. One of such generalizations is generalized metric space (or D-metric space) initiated by Dhage in 1992. Shaban Sedghi and Nabi Shobe and Haiyun Zhou [11] give new definition D* - metric space which is some modification of Dhage metric space (D-metric space).

In this paper, we establish a common fixed point theorem for a class of weakly commuting mappings in complete D* - metric space.

Definition 1.1. Let X be a nonempty set. A generalized metric (or D* -metric) on X is a function, $D^*: X^3 \to [0,\infty)$ that satisfies the following conditions for each $x, y, z \in X$.

(1) $D^*(x, y, z) \geq 0$,
(2) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
(3) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.
The pair (X, D*) is called a generalized metric (or D*-metric) space.

Examples 1.2
(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

(c) If $X = \mathbb{R}^n$ then we define

$D^*(x, y, z) = (\|x - y\|^p + \|y - z\|^p + \|z - x\|^p)^{1/p}$ for every $p \in \mathbb{R}^+$
(d) If X = R then we define

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z, \\ \max\{x, y, z\} & \text{otherwise}, \end{cases}$$

Remark 1.3. In a D*-metric space, we have $D^*(x, x, y) = D^*(x, y, y)$.

Proof: We have triangular inequality

(i) $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and similarly

(ii) $D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$.

Hence by (i), (ii) we get $D^*(x, x, y) = D^*(x, y, y)$.
Open ball with centre x and radius r : Let (X, D*) be a D*-metric space. For $r > 0$ define

$$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}$$

Example 1.4. Let $X = \mathbb{R}$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in \mathbb{R}$.

Then $B_{D^*}(1, 2) = \{y \in \mathbb{R}: D^*(1, y, y) < 2\}$
$= \{y \in \mathbb{R}: |y - 1| + |y - 1| < 2\}$
$= \{y \in \mathbb{R}: |y - 1| < 1\} = (0, 2)$.

Definition 1.5. Let (X, D*) be a D*- metric space and $A \subset X$.
(1) If for every $x \in A$, there exists $r > 0$ such that $B_{D^*}(x, r) \subset A$. We say that A is open in X

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(2) A subset A of X is said to be D*-bounded if there exists $r > 0$ such that $D^*(x, y, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $D^*(x_n, x_n, x) = D^*(x, x, x_n) \to 0$ as $n \to \infty$. That is for each $\in > 0$, there exists $n_0 \in N$ such that for all

$$n \geq n_0 \to D^*(x, x, x_n) < \in \quad (*)$$

This is equivalent with, for each $\in > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$.
$D^*(x, x_n, x_n) < \in$.

(4) Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $D^*(x_n, x_n, x_m) < \in$ for each $n, m \geq n_0$. The D*-metric space (X, D*) is said to be complete if every Cauchy sequence is convergent.

(5) Every convergent sequence in D*-metric space is Cauchy sequence.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that

$B_{D^*}(x, r) \subset A$. Then τ is a topology on X induced by the D*-metric.

Lemma 1.6. [11] Let (X, D*) be a D*-metric space. If $r > 0$, then ball $B_{D^*}(x, r)$ with center $x \in X$ and radius r is open set.

Definition 1.7. Let (X, D*) be a D*-metric space. D*- is said to be a continuous function on X^3 if

$$\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$$

Whenever a sequence $\{(x_n, y_n, z_n)\}$ in X^3 converges to a point $(x, y, z) \in X^3$, that is,

$$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ \lim_{n\to\infty} z_n = z.$$

Lemma 1.8. [11] Let (X, D*) be a D*- metric space. Then D*- is a continuous function on X^3.

Lemma 1.9. [11] Let (X, D*) be a D*-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 1.10. [11] Let (X, D*) be a D*-metric space. If sequence $\{x_n\}$ in X is convergent to x, then sequence $\{x_n\}$ is a Cauchy sequence.

Definition 1.11. Let A and S be two mappings from a D*-metric space (X, D*) into itself. Then {A, S} is said to be weakly commuting pair if $D^*(SAx, ASx, SAx) \leq D^*(Ax, Sx, Sx)$, for all $x \in X$. Clearly, a commuting pair is weakly commuting, but not conversely as shown in the following example.

Example 1.12. Let (X, D*) be a D*-metric space, where $X = [0, 1]$ and $D^*(x, y, z) = |x - y| + |y - z| + |x - z|$.

Define self-maps A and S on X as follows:

$Ax = \frac{x}{2}$ and $Sx = \frac{x+3}{5}$ for all $x \in X$. Then for all x in X one gets

$$D^*(SAx, ASx, ASx) = \left|\frac{x+3}{10} - \frac{x+6}{10}\right| + \left|\frac{x+6}{10} - \frac{x+6}{10}\right| + \left|\frac{x+3}{10} - \frac{x+6}{10}\right|$$

$$= 2\left|\frac{-3}{10}\right|$$

$$\leq 2\left|\frac{x+6}{10}\right|, \text{ for all } x \in X.$$

$$D^*(Sx, Ax, Ax) = \left|\frac{x+3}{5} - \frac{x}{2}\right| + 0 + \left|\frac{x}{2} - \frac{x+3}{5}\right|$$

$$= 2\left|\frac{x}{2} - \frac{x+3}{5}\right|$$

$$= 2\left|\frac{x+6}{10}\right|$$

Hence

$D^*(SAx, ASx, SAx) \leq D^*(Ax, Sx, Sx)$,

{A - S} is a weakly commuting pair.

However, for any nonzero $x \in X$ we have

$$ASx = \frac{x+3}{10} < \frac{x+6}{10} = SAx.$$

Thus A and S are not commuting mappings.

2. THE MAIN RESULTS

Throughout this section (X, D*) denotes a D*-metric space and Φ denotes a family of mappings such that each $\phi \in \Phi$, $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$, and ϕ is continuous and increasing in each co-ordinate variable.

Example 2.1. Let $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ be defined by $\phi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{7}(t_1 + t_2 + t_3 + t_4 + t_5)$.

Also $\gamma(t) = \phi(t, t, a_1 t, a_2 t, t) < t$ for every $t \in \mathbb{R}^+$ where $a_1 + a_2 = 3$.

The following lemma is the key in proving our result.

Lemma 2.2. [11] For every $t > 0$, $\gamma(t) < t$ if and only if $\lim_{n\to\infty} \gamma^n(t) = 0$, where γ^n denotes the composition of γ with itself n times. Our main result, for a complete D*-metric space X, reads follows:

Theorem 2.3. Let A and B be any two a self- mappings of a complete D*- metric space (X, D*) and
S, T be continuous self-mappings on X satisfying the following conditions:

(i) {A, S} and {B, T} are weakly commuting pairs such that $B(X) \subset S(X)$ & $A(X) \subset T(X)$,

(ii) there exists a $\phi \in \Phi$ such that for all x, y, z $\in$ X,
$D^*(Ax, By, Bz) \le \phi$ { ($a_1 D^*$ (Sx, Ty, Tz) + $a_2 D^*$ (Sx, Ty, By) + $a_3 D^*$(Sx, Ty, Bz) +$a_4 D^*$ (Sx, Ty, Ax) + $a_5 D^*$(Tz, Ax, Ax) + $a_6 D^*$(Tz, By, By), D^*(Sx, Ax, Ax),

D^*(Sx, By, By) , D^*(Ty, Ax, Ax) , D^*(Ty, By, By)}

where $a_1, a_2, a_3, a_4, a_5, a_6 \ge 0$ and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = ½$.
Then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point in X. Then $Bx_0 \in X$. Since BX is contained in SX . There exists a point $x_1 \in X$ such that $Sx_1 = Bx_0 = y_0$.

Since AX is contained in TX . There exists a point $x_2 \in X$ such that $Tx_2 = Ax_1 = y_1$.

Continuing this way, we define by induction a sequence { y_n} in X such that
$Sx_{2n+1} = Bx_{2n} = y_{2n}$, n = 0, 1 , 2 …
$Tx_{2n+2} = Ax_{2n+1} = y_{2n+1}$, n = 0, 1 , 2 …

Let $d_n = D^*(y_n , y_{n+1} , y_{n+1})$, n = 0, 1 , 2 …

We prove that $d_{2n+1} \le d_{2n}$, for all n > 0.
Now if $d_{2n+1} > d_{2n}$, for some $n \in \mathbb{N}$,

Since ϕ is increasing function, then

$d_{2n+1} = D^*(y_{2n+1} , y_{2n+2} , y_{2n+2})$, n = 0, 1 , 2 …
$= D^*(Ax_{2n+1} , Bx_{2n+2} , Bx_{2n+2})$
$\le \phi$ { ($a_1 D^*$ ($Sx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}$)
+ $a_2 D^*$ ($Sx_{2n+1}, Tx_{2n+2,2n+2}$) +$a_3 D^*$(Sx_{2n+1}, T x_{2n+2}, Bx_{2n+2})
+ $a_4 D^*$ ($Sx_{2n+1}, Tx_{2n+2}, Ax_{2n+1}$) +$a_5 D^*$($Tx_{2n+2}, Ax_{2n+1}, Ax_{2n+1}$) + $a_6 D^*$($Tx_{2n+2}, Bx_{2n+2}, Bx_{2n+2}$),

D^*($Sx_{2n+1}, Ax_{2n+1}, Ax_{2n+1}$), D^*($Sx_{2n+1}, Bx_{2n+2}, Bx_{2n+2}$) , D^*($Tx_{2n+2}, Ax_{2n+1}, Ax_{2n+1}$) , D^*($Tx_{2n+2}, Bx_{2n+2}, Bx_{2n+2}$)}
$\le \phi$ { ($a_1 D^*$ ($y_{2n}, y_{2n+1}, y_{2n+1}$) + $a_2 D^*$ ($y_{2n}, y_{2n+1}, y_{2n+2}$)
+ $a_3 D^*$($y_{2n}, y_{2n+1}, y_{2n+2}$) +$a_4 D^*$ ($y_{2n}, y_{2n+1}, y_{2n+1}$)
+ $a_5 D^*$($y_{2n+1}, y_{2n+1}, y_{2n+1}$) + $a_6 D^*$($y_{2n+1}, y_{2n+2}, y_{2n+2}$),

D^*($y_{2n}, y_{2n+1}, y_{2n+1}$), D^*(y_{2n} , y_{2n+2}, y_{2n+2}) , D^*($y_{2n+1}, y_{2n+1}, y_{2n+1}$) , D^*($y_{2n+1}, y_{2n+2}, y_{2n+2}$)}

$\le \phi$ { ($a_1 d_{2n} + a_2(d_{2n} + d_{2n+1}) + a_3(d_{2n} + d_{2n+1}) + a_4 (d_{2n})$
+ $a_5 (0) + a_6 d_{2n+1}$, d_{2n} ,($d_{2n} + d_{2n+1}$) , 0 , d_{2n+1} }
$\le \phi$ {($a_1 + a_2 + a_3 + a_4$) d_{2n} + ($a_2 + a_3 + a_6$) d_{2n+1} , d_{2n} ,
$d_{2n} + d_{2n+1}$, 0, d_{2n+1} }
$\le \phi$ {½ d_{2n+1} + ½ d_{2n+1} , d_{2n+1} , $2d_{2n+1}$, 0, d_{2n+1} }
$\le \phi$ {d_{2n+1} , d_{2n+1} , $2d_{2n+1}$, 0, d_{2n+1} }
$\le \phi$ {d_{2n+1} , d_{2n+1} , $2d_{2n+1}$, d_{2n+1} , d_{2n+1} } (since $d_{2n+1} \ge 0$)
$< d_{2n+1}$, a contradiction.

Hence $d_{2n+1} \le d_{2n}$

Similarly, we can prove that $d_{2n+2} \le d_{2n+1}$, for n = 0,1,2…

Now if $d_{2n+2} > d_{2n+1}$ for some $n \in \mathbb{N}$,
$d_{2n+2} = D^*(y_{2n+2} , y_{2n+3} , y_{2n+3})$, n = 0, 1 , 2 …
$= D^*(Bx_{2n+2} , Ax_{2n+3} , Ax_{2n+3})$
$\le \phi$ { ($a_1 D^*$ ($Sx_{2n+2}, Tx_{2n+3}, Tx_{2n+3}$)
+ $a_2 D^*$ ($Sx_{2n+2}, Tx_{2n+3}, Bx_{2n+3}$) +$a_3 D^*$(Sx_{2n+2}, T x_{2n+3}, Bx_{2n+3})
+ $a_4 D^*$ ($Sx_{2n+2}, Tx_{2n+3}, Ax_{2n+2}$) +$a_5 D^*$($Tx_{2n+3}, Ax_{2n+2}, Ax_{2n+2}$) + $a_6 D^*$($Tx_{2n+3}, Bx_{2n+3}, Bx_{2n+3}$),

D^*($Sx_{2n+2}, Ax_{2n+2}, Ax_{2n+2}$), D^*($Sx_{2n+2}, Bx_{2n+3}, Bx_{2n+3}$) , D^*($Tx_{2n+3}, Ax_{2n+2}, Ax_{2n+2}$) , D^*($Tx_{2n+3}, Bx_{2n+3}, Bx_{2n+3}$)}
$\le \phi$ { ($a_1 D^*$ ($y_{2n+1}, y_{2n+2}, y_{2n+2}$) + $a_2 D^*$($y_{2n+1}, y_{2n+2}, y_{2n+3}$) + $a_3 D^*$($y_{2n+1}, y_{2n+2}, y_{2n+3}$) +$a_4 D^*$ ($y_{2n+1}, y_{2n+2}, y_{2n+2}$)
+ $a_5 D^*$($y_{2n+2}, y_{2n+2}, y_{2n+2}$) + $a_6 D^*$($y_{2n+2}, y_{2n+3}, y_{2n+3}$),

D^*($y_{2n+1}, y_{2n+2}, y_{2n+2}$), D^*($y_{2n+1}, y_{2n+3}, y_{2n+3}$) ,
D^*($y_{2n+2}, y_{2n+2}, y_{2n+2}$) ,
D^*($y_{2n+2}, y_{2n+3}, y_{2n+3}$)}
$\le \phi$ { ($a_1 d_{2n+1} + a_2(d_{2n+1} + d_{2n+2}) + a_3(d_{2n+1} + d_{2n+2})$
+ $a_4 (d_{2n+1}) + a_5 (0) + a_6 d_{2n+2}$,d_{2n+1} , $d_{2n+1} + d_{2n+2}$
+ d_{2n+2} ,0, d_{2n+2} }
$\le \phi$ {($a_1 + a_2 + a_3 + a_4$) d_{2n+1} + ($a_2 + a_3 + a_6$) d_{2n+2} , d_{2n+1} , $d_{2n+1} + 2d_{2n+2}$, 0, d_{2n+2} }
$\le \phi$ {½ d_{2n+2} + ½ d_{2n+2} , d_{2n+2} , $3d_{2n+2}$, 0, d_{2n+1} }
$\le \phi$ {d_{2n+2} , d_{2n+2} , d_{2n+2} , $3d_{2n+2}$ d_{2n+2} , d_{2n+2} }
$< d_{2n+2}$, a contradiction.

Hence $d_{2n+2} \le d_{2n+1} \le d_{2n}$

Consequently {d_n} is non increasing sequence of non – negative real's.

Now $d_1 = D^*(y_1 , y_2 , y_2) = D^*(Ax_1 , Bx_2 , Bx_2)$
$\le \phi$ { ($a_1 D^*$ (Sx_1, Tx_2, Tx_2) + $a_2 D^*$ (Sx_1, Tx_2, Bx_2) + $a_3 D^*$(Sx_1, T x_2, B x_2) +$a_4 D^*$ (Sx_1, Tx_2, Ax_1) + $a_5 D^*$(Tx_2, Ax_1, A x_1) + $a_6 D^*$(T x_2, B x_2, B x_2),

D^*(S x_1, A x_1, A x_1), D^*(S x_1, B x_2, B x_2) , D^*(T x_2, A x_1, A x_1) , D^*(T x_2, B x_2, B x_2)}

$\le \phi$ { ($a_1 D^*$ (y_0, y_1, y_1) + $a_2 D^*$ (y_0, y_1, y_2) + $a_3 D^*$(y_0, y_1, y_2) + $a_4 D^*$ (y_0, y_1, y_1)+ $a_5 D^*$(y_1, y_1, y_1) + $a_6 D^*$(y_1, y_2, y_2), D^*(y_0, y_1, y_1), D^*(y_0, y_2, y_2) , D^*(y_1, y_1, y_1) ,D^*(y_1, y_2, y_2)}

$\le \phi$ { ($a_1 d_0 + a_2 (d_0 + d_1) + a_3(d_0 + d_1) + a_4 d_0 + a_5 (0) + a_6 d_1$, d_0 , $d_0 + d_1$, 0 , d_1 }

$\le \phi$ {($a_1 + a_2 + a_3 + a_4$) d_0 + ($a_2 + a_3 + a_6$) d_1 , d_0 , $d_0 + d_1$, 0, d_1 }
$\le \phi$ {½ d_0 + ½ d_1 , d_0 , $d_0 + d_1$, 0 , d_1 }
$= \phi${d_1 , d_1 , d_1, $2d_1$, 0 , d_1 }
$\le \phi${ d_1 , d_1 , d_1, $2d_1$, d_1 , d_1 }
$= \gamma (d_0)$

Similarly $d_2 \le \gamma (d_1) \le \gamma (\gamma d_0) = \gamma^2 (d_0)$

In general we have $d_n \le \gamma^n (d_0)$
so, if $d_0 > 0$, then lemma 2.2 gives
$\lim_{n\to\infty} d_n = 0$, for $d_0 = 0$. We have $d_n = 0$, for each
Now we prove that { y_n} is a Cauchy sequence in X.
For $m > n \ge n_0$ for some $n_0 \in N$

We have
$D^*(y_n, y_n, y_m) \le D^*(y_n, y_n, y_{n+1}) + D^*(y_{n+1}, y_m, y_m)$
$\le D^*(y_n, y_n, y_{n+1}) + D^*(y_{n+1}, y_{n+1}, y_m)$

$\le D^*(y_n, y_n, y_{n+1}) + D^*(y_{n+1}, y_{n+1}, y_{n+2}) + . . .$

$+ D^*(y_{m-1}, y_{m-1}, y_m)$

$= d_n + d_{n-1} + \ldots + d_{m-1}$

$\to 0$ as $m, n \to \infty$.

Thus $\{ y_n \}$ is a Cauchy sequence in X.

In the same way we can show that $\{ y_{2n} \}$ and $\{ y_{2n+1} \}$ are Cauchy sequence in X.

Since X is D*-complete $y_n \to z$ in X and $\{ y_{2n} \}$, $\{ y_{2n+1} \}$ are subsequences of $\{ y_n \}$.

We have $y_{2n} \to z$ and $y_{2n+1} \to z$ as $n \to \infty$.

As S and T are continuous, we have $STx_{2n+2} = SAx_{2n+1} = Sy_{2n+1} \to Sz$ (as $y_{2n+1} \to z$)
$TSx_{2n+1} = TBx_{2n} = Ty_{2n} \to Tz$ (as $y_{2n} \to z$)

Now we prove that $Sz = Tz$.

Suppose $Sz \neq Tz$. Now consider

$$D^*(Sz, Tz, Tz) = \lim_{n\to\infty} D^*(STx_{2n+2}, TSx_{2n+1}, TSx_{2n+1})$$
$$= \lim_{n\to\infty} D^*(SAx_{2n+1}, TBx_{2n}, TBx_{2n})$$
$$= \lim_{n\to\infty} D^*(SAx_{2n+1}, SAx_{2n+1}, TBx_{2n})$$
$$\leq \lim_{n\to\infty} \{D^*(SA\ x_{2n+1}, SAx_{2n+1}, ASx_{2n+1}) + D^*(ASx_{2n+1}, TBx_{2n}, TBx_{2n})\}$$
$$\leq \lim_{n\to\infty} \{D^*(SAx_{2n+1}, SAx_{2n+1}, ASx_{2n+1}) + D^*(BTx_{2n}, TBx_{2n}, TBx_{2n}) + D^*(BTx_{2n}, ASx_{2n+1}, ASx_{2n+1})\}$$
$$\leq \lim_{n\to\infty} \{D^*(SAx_{2n+1}, SAx_{2n+1}, ASx_{2n+1}) + D^*(BTx_{2n}, TBx_{2n}, TBx_{2n}) + D^*(ASx_{2n+1}, BTx_{2n}, BTx_{2n})\}$$
$$= \lim_{n\to\infty} \{D^*(Sx_{2n+1}, Sx_{2n+1}, Ax_{2n+1}) + D^*(Bx_{2n}, Tx_{2n}, Tx_{2n}) + D^*(ASx_{2n+1}, BTx_{2n}, BTx_{2n})\}$$

{ there fore {A, S} &{B, T } are weakly commuting}

$$\leq \lim_{n\to\infty} \phi\ \{d_{2n}, d_{2n}, 2d_{2n}, 0, d_{2n}\}$$
$$\leq \lim_{n\to\infty} d_{2n}$$
$$\leq \gamma D^*(y_{2n}, y_{2n+1}, y_{2n+1})$$
$$\leq D^*(Sz, z, z)$$

$= 0$. Which is a contradiction.

Therefore $Tz = Sz$.
Now we prove that $Sz = Az$.
Suppose $Sz \neq Az$.

Now consider

$$D^*(Sz, Az, Az) = \lim_{n\to\infty} D^*(ST\ x_{2n+2}, AS\ x_{2n+1}, AS\ x_{2n+1})$$
$$= \lim_{n\to\infty} D^*(SA\ x_{2n+1}, AB\ x_{2n}, ABx_{2n})$$
$$= \lim_{n\to\infty} D^*(SAx_{2n+1}, SAx_{2n+1}, ABx_{2n})$$
$$\leq \lim_{n\to\infty} \{D^*(SAx_{2n+1}, SAx_{2n+1}, ASx_{2n+1}) + D^*(BAx_{2n}, ABx_{2n}, ABx_{2n}) + D^*(BAx_{2n}, ASx_{2n+1}, ASx_{2n+1})\}$$
$$\leq \lim_{n\to\infty} \{D^*(SAx_{2n+1}, SAx_{2n+1}, ASx_{2n+1}) + D^*(BAx_{2n}, ABx_{2n}, ABx_{2n}) + D^*(BAx_{2n}, BAx_{2n}, ASx_{2n+1})\}$$
$$= \lim_{n\to\infty} \{D^*(Sx_{2n+1}, Sx_{2n+1}, Ax_{2n+1}) + D^*(Bx_{2n}, Ax_{2n}, Ax_{2n}) + D^*(BAx_{2n+1}, BAx_{2n}, ASx_{2n+1})\}$$
$$\leq \lim_{n\to\infty} \phi\ \{d_{2n}, d_{2n}, 2d_{2n}, 0, d_{2n}\}$$
$$\leq \lim_{n\to\infty} d_{2n}$$
$$\leq \gamma D^*(y_{2n}, y_{2n+1}, y_{2n+1})$$
$$\leq D^*(Az, z, z)$$

$= 0$. Which is a contradiction.

Therefore $Az = Sz$.

Now we prove that $Bz = Tz$.

Suppose $Bz \neq Tz$.

Now consider

$$D^*(Tz, Bz, Bz) = \lim_{n\to\infty} D^*(TSx_{2n+1}, BTx_{2n+2}, BTx_{2n+2})$$
$$= \lim_{n\to\infty} D^*(TBx_{2n}, STx_{2n-1}, STx_{2n-1})$$
$$\leq \lim_{n\to\infty} \{ D^*(STx_{2n+1}, STx_{2n+1}, TSx_{2n+1}) + D^*(TBx_{2n}, TBx_{2n}, STx_{2n}) + D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n-1})\}$$
$$\leq \lim_{n\to\infty} \{ D^*(Sx_{2n+1}, Sx_{2n+1}, Tx_{2n+1}) + D^*(Tx_{2n}, Tx_{2n}, STx_{2n}) + D^*(STx_{2n}, TSx_{2n+1}, TSx_{2n+1})\}$$
$$\leq \lim_{n\to\infty} \phi\ \{d_{2n}, d_{2n}, 2d_{2n}, 0, d_{2n}\}$$
$$\leq \lim_{n\to\infty} d_{2n}$$
$$\leq \gamma D^*(y_{2n}, y_{2n+1}, y_{2n+1})$$
$$\leq D^*(Bz, z, z)$$

$= 0$. Which is a contradiction.

Therefore $Tz = Bz = Sz = Az$.

It now follows that

$$D^*(Az, z, z) = \lim_{n\to\infty} D^*(Az, Ax_{2n}, Ax_{2n})$$
$$\leq \phi\ \{ (a_1 D^*(Sz, Tx_{2n}, Tx_{2n}) + a_2 D^*(Sz, Tx_{2n}, Bx_{2n}) + a_3 D^*(Sz, Tx_{2n}, Bx_{2n}) + a_4 D^*(Sz, Tx_{2n}, Az) + a_5 D^*(Tx_{2n}, Az, Az) + a_6 D^*(Tx_{2n}, Bx_{2n}, Bx_{2n}),$$
$$D^*(Sz, Az, Az), D^*(Sz, Bx_{2n}, Bx_{2n}), D^*(Tx_{2n}, Bx_{2n}, Bx_{2n}), D^*(T\ x_{2n}, B\ x_{2n}, B\ x_{2n})\}$$
$$\leq \phi\ \{ (a_1 D^*(Sz, z, z) + a_2 D^*(Sz, z, z) + a_3 D^*(Sz, z, z) + a_4 D^*(Sz, z, Az) + a_5 D^*(z, Az, Az) + a_6 D^*(z, z, z), D^*(Sz, Az, Az), D^*(Sz, z, z), D^*(z, z, z), D^*(z, z, z)\}$$
$$\leq \phi\ \{ (a_1 D^*(Sz, z, z) + a_2 D^*(Sz, z, z) + a_3 D^*(Sz, z, z) + a4\, D^*(Sz, z, Az) + a_5 D^*(z, Az, Az) + a_6 (0), D^*(Sz, Az, Az), D^*(Sz, z, z), 0, 0\}$$
$$\leq \gamma D^*(Az, z, z)$$
$$\leq D^*(Az, z, z)$$

. Which contradiction

There fore $Tz = Bz = Sz = Az = z$.

Now to prove uniqueness,

It possible $y \neq z$ be another common fixed point of A, B, S and T.
Then $D^*(y, z, z) = D^*(Ay, Bz, Bz)$
$\leq \phi\ \{(a_1D^* (Sy, Tz, Tz) + a_2 D^* (Sy, Tz, Bz)$
$+ a_3 D^*(Sy, Tz, Bz) + a_4 D^* (Sy, Tz, Ay) + a_5 D^*(Tz, Ay, Ay)$
$+ a_6 D^*(Tz, Bz, Bz),$

$D^*(Sy, Ay, Ay), D^*(Sy, Bz, Bz) , D^*(Tz, By, By) , D^*(Tz, Bz, Bz)\}$

$\leq \phi\ \{ (a_1D^* (y, z, z) + a_2 D^* (y, z, z) + a_3 D^*(y, z, z)$
$+ a_4 D^* (y, z, y) + a_5 D^*(z, y, y) + a_6 D^*(z, z, z), D^*(y, y, y),$
$D^*(y, z, z) , D^*(z, y, y) , D^*(z, z, z)\}$

$\leq \gamma D^*(y, z, z)$
$\leq D^*(y, z, z).$

Which contradiction. Therefore $y = z$.

Then z is a common fixed point of A, B, S and T.

This completes the proof of the theorem.

Remark 2.4
Putting $A = B$ and $a_2 = a_3 = a_4 = a_5 = a_6 = 0$. We get the following theorem as corollary.

Corollary 2.5. Let A be a self-mapping of complete D*-metric space (X, D*) and S, T be continuous self-mappings on X satisfying the following conditions:
(i) {A, S} and {A, T} are weakly commuting pairs such that $A(X) \subset S(X) \cap T(X)$;

(ii) there exist a $\phi \in \Phi$ such that for all $x, y, z \in X$,
$D^*(Ax, Ay, Az) \leq \phi\ \{D^*(Sx, Ty, Tz), D^*(Sx, Ax, Ax), D^*(Sx, Ay, Ay) , D^*(Ty, Ax, Ax) ,$
$D^*(Ty, Ay, Ay)\}$

Then A, S and T have a unique common fixed point in X.

Corollary 2.5. Let A_i and B_j be two a self-mapping of complete D*-metric space (X, D*) and S, T be continuous self-mappings on X satisfying the following conditions:

(i) $\{A_i, S\}$ and $\{B_j, T\}$ are weakly commuting pairs such that $B_j (X) \subset S(X)$ & $A_i (X) \subset T(X)$;

(ii) there exist a $\phi \in \Phi$ and $i, j, k \in N$ such that for all $x, y, z \in X$,

$D^*(A_i x, B_j y, B_k z) \leq \phi\ \{ (a_1D^* (Sx, Ty, Tz)$
$+ a_2 D^* (Sx, Ty, B_j y) + a_3 D^*(Sx, Ty, B_k z) + a_4 D^* (Sx, Ty, A_i x)$
$+ a_5 D^*(Tz, A_i x, A_i x) + a_6 D^*(Tz, B_j y, B_j y),$

$D^*(Sx, A_i x, A_i x), D^*(Sx, B_j y, B_j y) , D^*(Ty, A_i x, A_i x) , D^*(Ty, B_j y, B_j y)\}$
where $a_1, a_2 , a_3 , a_4, a_5 , a_6 \geq 0$
and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = ½$. Then A_i, B_j , S and T have a unique common fixed point in X.

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