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ANNIHILATOR INJECTIVE MODULES

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ABSTRACT

Annihilator Injective Module is a generalization of Injective Modules, in this paper we give several properties of Annihilator Injective Modules and discus the question when Annihilator Injective is Injective Module.

Key words: *Simple Module, Semi simple, Injective Module, Annihilator injective Modules,* $l_M(R)$ *and* $r_R(X)$.

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INTRODUCTION:

Throughout this paper R represent an associative ring with identity $1 \neq 0$ and all modules are unitary R-module we write M_R (res, ${}_RM$) to indicate that M is a right (res, left) R-module, we also write J, Z_r, S_r for the Jacobson radical, the right singular ideal and the right socal respectively of R and $E(M_R)$ the injective hull of M_R, If X is a subset of M then the right annihilator of X in R denoted by $r_R(X)$ and if A is subset of R then the set $l_M(A)$ is called left annihilator of A, If N is a submodule of M (res, proper submodule) we denoted by $N \leq M$ (res, $N \prec M$) moreover we write $N \leq^e M, N \ll M, N \leq^{\oplus} M$, and $N \leq^{\max} M$ to indicate that N is an essential submodule, a small submodule, a direct summand and a maximal submodule of M respectively, A module M is called uniform if $M \neq 0$ and every non zero submodule of M is essential in M, Let M and N be R-Modules N is called M-Injective Module if for every submodule A of M and the inclusion map $i: A \to M$ any homomorphism $\alpha: A \to N$ can be extended to a homomorphism $\beta: M \to N$ such that the diagram is commutative.

1. PRELIMINARIES:

Definition: 1.1 Let M be an R-Module and $X \subseteq M$ then set $r_R(X) = \{r \in R : xr = 0 \forall x \in X\}$ is called the right annihilator of X in R.

Definition: 1.2 For each $A \subseteq R$ the set $l_M(A) = \{m \in M : ma = 0 \forall a \in A\}$ called left annihilator of A in M, for a singleton set $\{x\}$ and $\{a\}$ we denote the annihilator as $r_R(x)$ and $l_M(a)$ respectively.

Proposition: 1.3 Let M be an R-Module and $X \subseteq M$ and $A \subseteq R$ then,

1. $r_R(X)$ is a right ideal of R
2. $l_M(A)$ is a submodule of M

Definition: 1.4 R modules N are called injective if it is M injective for every R-Module.

Definition: 1.5 R modules N is called quasi injective if N is N injective.

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Definition: 1.6 M and N be R - modules, then N is called M **Annihilator Injective Module** if the inclusion map $i: l_M(R) \to M$ and any homomorphism $\alpha: l_M(R) \to N$ can be extended to a homomorphism $\beta: M \to N$ such that the diagram is commutative i.e. $\alpha = \beta i$.

Definition: 1.7 N is called **Annihilator Injective Module** if for all R-Module M the inclusion map $i: l_M(R) \to M$ and any homomorphism $\alpha: l_M(R) \to N$ can be extended to a homomorphism $\beta: M \to N$ such that the diagram is commutative i.e. $\alpha = \beta i$.

1. ANNIHILATOR INJECTIVE MODULS:

Proposition: 2.1 Let N and N' be R - Modules if $N \cong N'$ and N is Annihilator Injective then N' is also Annihilator Injective.

Proof: Since N is annihilator Injective then for any R - module M $i: l_M(R) \to M$ be an injective map and $\alpha: l_M(R) \to N$ be a homomorphism then there exist a homomorphism $\beta: M \to N$ the map is commute i.e $\alpha = \beta i$, let $f: N \to N'$ be an isomorphism now define a homomorphism $\alpha': l_M(R) \to N'$ and $\beta': M \to N'$ where $\alpha' = f\alpha$ and $\beta' = f\beta$ then $\alpha' = f\alpha = f\beta i = \beta' i$ therefore N' is annihilator injective module.

Proposition: 2.2 If $N_1 \oplus N_2$ is Annihilator Injective module then N_1 and N_2 is relatively Annihilator Injective Modules.

Proof: Let $N_1 \oplus N_2$ be Annihilator Injective Module then we have to show that N_1 is N_2 Annihilator Injective Modue,
Let $i: l_{N_2}(R) \to N_2$ be the inclusion map and $\alpha: l_{N_2}(R) \to N_1 \oplus N_2$ be a homomorphism where $\alpha(n) = (\alpha'(n), n)$ so α is monomorphism since $N_1 \oplus N_2$ is Annihilator Injective Module then there exist a homomorphism $\beta: N_2 \to N_1 \oplus N_2$ such that $\alpha = \beta i$, Let π_1 be natural projection from $\pi_1: N_1 \oplus N_2 \to N_1$ and let $\beta': N_2 \to N_1$ be a homomorphism where $\beta' = \pi_1 \beta$ now define a homomorphism $\alpha': l_{N_2}(R) \to N_1$ where $\alpha' = \pi_1 \alpha = \pi_1 \beta i = \beta' i$ therefore N_1 is N_2 Annihilator Injective Modules similarly we can show that N_2 is N_1 Annihilator Injective Module.

Proposition: 2.3 Let $\{N_\lambda\}_{\lambda \in \wedge}$ be a family of Annihilator Injective modules, then $\prod_{\lambda \in \wedge} N_\lambda$ is Annihilator Injective Module if and only if each N_λ is Annihilator Injective Modules.

Proof: Let M be an R-Module and put $N = \prod_{\lambda \in \wedge} N_\lambda$ Let $\phi_\lambda: N_\lambda \to N$ and $\pi_\lambda: N \to N_\lambda$ be injection and projection respectively.

Let each N_λ be M Annihilator Injective Module, let $\alpha: l_M(R) \to N$ be a homomorphism and $i: l_M(R) \to M$ be an injection map then for each λ there exist $\beta_\lambda: M \to N_\lambda$ such that the diagram is commutative, therefore $\pi_\lambda \alpha = \beta_\lambda i$ now define $\beta: M \to N$ where $\beta(m) = \{\beta_\lambda(m)\}_{\lambda \in \wedge} (m \in M)$ then β is an R-homomorphism and if $x \in l_M(R)$ then $\beta i(x) = \{\beta_\lambda i(x)\} = \{\pi_\lambda \alpha(x)\} = \alpha(x)$ which shows that N is Annihilator Injective.

Conversely:

Suppose that N is Annihilator Injective Module.
Consider $i: l_M(R) \to M$ and $\gamma: l_M(R) \to N$ be a homomorphism since N is Annihilator Injective then for any Hom$(l_M(R), N)$ there exist a homomorphism $\mu: M \to N$ and the map commute i.e $\mu i = \phi_\lambda \gamma$ now define a map

$\mu':M\to N_\lambda$ where $\mu'(m)=\pi_\lambda\mu(m)$ then μ' is an R homomorphism and if $a\in l_M(R)$ then $\mu' i(a)=\pi_\lambda\mu i(a)=\pi_\lambda\phi_\lambda\gamma(a)=\gamma(a)$ this shows that each N_λ is Annihilator Injective Module.

Proposition: 2.4 let N_λ be a family of R-modules.

1. If $\underset{\lambda\in\wedge}{\oplus} N_\lambda$ is Annihilator Injective then each N_λ is Annihilator Injective.
2. If the Index set $\wedge$ is finite and each N_λ is Annihilator Injective then $\underset{\lambda\in\wedge}{\oplus} N_\lambda$ is Annihilator Injective.

Proof: Proof is similar for the proposition (2.3).

Note: 2.5 Every Injective Modules is Annihilator Injective Modules but converse is nee not true.

Example: 2.6 $Z(Z)$ is Annihilator Injective Module but it is not Injective Module.

Proposition: 2.7 If N is Annihilator Injective Module then any monomorphism $f:N\to M$ split.

Proof: Let $f:N\to M$ be a monomorphism then $f^{-1}:f(N)\to N$ be inverse of f since N is Annihilator Injective then there exist a homomorphism $\beta:M\to N$ that extends f^{-1} set $g=\beta f$ then g is clearly an identity map of N hence f splits in M.

Proposition: 2.8 If N is Annihilator Injective module then N is Annihilator K-Injective module for any submodule K of M.

Proof: Let X be submodule of K and $f:X\to N$ be a homomorphism then X be also a submodule of M and by Annihilator M-Injective module of N, f extends to a homomorphism $f^*:M\to N$ the restriction f^*/K of f^* on K is a homomorphism $K\to N$ which extends f hence N is Annihilator K-Injective module.

Proposition: 2.9 Every direct summand of Annihilator Injective Modules is also Annihilator Injective Modules.

Proof: Assume N is Annihilator Injective module and $N=K\oplus K'$ let $l_M(R)$ be a submodule of M and $f:l_M(R)\to K$ be a homomorphism define $g:l_M(R)\to N=K\oplus K'$ by $g(m)=(f(m),0)$ thus g is monomorphism and by Annihilator Injective Module of N, g extends to a homomorphism $\beta:M\to N$, let π_K be the natural projection of $N=K\oplus K'$ in to K then $\pi_K\beta:M\to K$ is a homomorphism which extends f ,therefore K is Annihilator Injective.

Definition: - let M be an R-Module an element m of M is said to be divisible if for every r of R which is not a right zero divisor there exist $m'\in M$ such that $m=rm'$, if every element of M is divisible then M is said to be a divisible module. An abelian group is said to be divisible if it is divisible as a Z - module. Alternatively M is divisible if $M=rM$ whenever r is an element of R which is not a right zero divisor.

Note 2.10 Every Injective module is divisible but every Annihilator Injective modules need not be divisible.

Lemma 2.11 Let M be an R-Module such that every submodule of M and let M' be a submodule of M then every submodule of M' is a direct summand of M.

Proof: Let A be a submodule of M' then A is also a submodule of M so there is a submodule B of M such that $M=A+(M'\cap B)$ (d. s) it follows that $M'=A+(M'\cap B)=A\cap B=0$.

Proposition: 2.12 Let M be an R-module then the following statements are equivalent.

1. M has a family $\{S_i\}_{i\in I}$ of simple submodule such that $M=\sum_{i\in I} S_i$ (d. s)

2. M has a family of simple submodules whose sum is M itself.

3. Every submodule of M is a direct summand of M.

Proof: [5, Proposition 3.2]

Proposition: 2.13 Every simple Annihilator Injective modules is Injective Modules.

Proof: Simple

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