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gπ - Compactness and gπ-Connectedness Topological Spaces

S. Saranya* & A. Parvathi

Department of Mathematics, Avinashilingam University, Coimbatore-641043, India

E-mail: saranya.subbaiyan@gmail.com, aparvathi.s@gmail.com

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ABSTRACT

This paper deals with gπ-compact spaces and their properties by using nets, filter base and gπ-complete accumulation points. The notion of gπ-connectedness in topological spaces is also introduced and their properties are studied.

Mathematics Subject Classification: *54D05, 54D30*

Keywords: *gπ-open set, gπ-closed sets, gπ-compact spaces, gπ-complete accumulation point, gπ-connectedness.*

1. Introduction:

The notions of compactness and connectedness are useful and fundamental notions of not only general topology but also of other advanced branches of mathematics. Many researchers have investigated the basic properties of compactness and connectedness. The productivity and fruitfulness of these notions of compactness and connectedness motivated mathematicians to generalize these notions. In the course of these attempts many stronger and weaker forms of compactness and connectedness have been introduced and investigated.

In 1970, Levine [6] initiated the study of so called g-closed sets, that is, a subset A of a topological space (X, τ) is said to be g-closed if the closure of A is included in every open superset of A and defined a T_{12} space to be one in which the closed sets and g-closed sets coincide. Recently, Dontchev and Noiri [4] introduced the notion of πg-closed sets and used this notion to obtain a characterizations and some preservation theorems for quasi normal spaces.

More recently, Park [7] has introduced and studied the notion of πgp-closed sets which is implied by that of gp-closed sets. Park and Park [8] continued the study of πgp-closed sets and associated functions and introduced the concepts of πGP-compactness and πGPconnectedness. Also Aslim, Guler and Noiri [3] introduced the concept of πgs-closed sets and studied it's basic properties. Moreover they also introduced the notions of πgs-$T_{1/2}$ spaces and πgs-continuity in topological spaces. The aim of this paper is to introduce the concept of gπ-compactness and gπ-connectedness in topological spaces and is to give some characterizations of gπ-compact spaces in terms of nets and filter bases. The notion of gπ-complete accumulation points is introduce and is used to characterize gπ-compactness. Further it is proved that gπ-connectedness is preserved under gπ-irresolute surjections.

2 Preliminary Notes:

Throughout this paper (X, τ), (Y, σ) are topological spaces with no separation axioms assumed unless otherwise stated. Let A ⊆ X. The closure of A and the interior of A will be denoted by Cl(A) and Int(A) respectively.

Definition: 2.1 A subset A of a topological space (X, τ) is said to be gπ-closed [9] if πcl(A) ⊆ U whenever A ⊆ U and U is open in X.

Definition: 2.2 The gπ-closure [9] of a set A, denoted by gπCl(A), is the intersection of all gπ-closed sets containing A.

Definition: 2.3 The gπ-interior [9] of a set A, denoted by gπInt(A), is the union of all gπ-open sets contained in A.

Remark: 2.4 [9] Every π-closed set is gπ-closed.

***Corresponding author:** *S. Saranya*,* ***E-mail:** *saranya.subbaiyan@gmail.com*

3 gπ - Compactness:

Definition: 3.1 A collection $\{A_i: i \in \wedge\}$ of gπ-open sets in a topological space X is called a gπ-open cover of a subset B of X if $B \subset \{A_i: i \in \wedge\}$ holds.

Definition 3.2 A topological space X is gπ-compact if every gπ-open cover of X has a finite sub-cover.

Definition: 3.3 A subset B of a topological space X is said to be gπ-compact relative to X if, for every collection $\{A_i : i \in \wedge\}$ of gπ-open subsets of X such that $B \subset \{A_i : i \in \wedge\}$ there exists a finite subset $\wedge_0$ of $\wedge$ such that $B \subset \{A_i : i \in \wedge\}$.

Definition: 3.4 A subset B of a topological space X is said to be gπ-compact if B is gπ-compact as a subspace of X.

Theorem: 3.5 Every gπ-closed subset of a gπ-compact space is gπ-compact relative to X.

Proof: Let A be gπ-closed subset of gπ-compact space X. Then A^C is gπ-open in X.

Let $M = \{G\alpha : \alpha \in \wedge\}$ be a cover of A by gπ-open sets in X. Then $M_* = M \cup A^c$ is a gπ-open cover of X. Since X is gπ-compact M_* is reducible to a finite subcover of X, say

$X = G_{\alpha 1} \cup G_{\alpha 2} \cup \cdots \cup G_{\alpha m} \cup A^c$, $G_{\alpha k} \in M$. But A and A^C are disjoint hence

$A \subset G_{\alpha 1} \cup \cdots \cup G_{\alpha m}$, $G_{\alpha k} \in M$, which implies that any gπ-open cover M of A contains a finite sub-cover. Therefore A is gπ-compact relative to X. Thus every gπ-closed subset of a gπ-compact space X is gπ-compact.

Definition: 3.6 A function f: X → Y is said to be gπ-continuous [5] if f^{-1} (V) is gπ-closed in X for every closed set V of Y.

Definition: 3.7 A function f: X → Y is said to be gπ-irresolute [5] if f^{-1} (V) is gπ-closed in X for every gπ-closed set V of Y.

Theorem: 3.8 A gπ-continuous image of a gπ-compact space is compact.

Proof: Let f: X → Y be a gπ-continuous map from a gπ-compact space X onto a topological space Y. Let $\{A_i: i \in \wedge\}$ be an open cover of Y. Then $\{f^{-1}(A_i) : i \in \wedge\}$ is a gπ-open cover of X. Since X is gπ-compact it has a finite sub-cover say $\{f^{-1} (A_1), f^{-1}(A_2), \cdots , f^{-1}(A_n)\}$. Since f is onto $\{A_1, \cdots, A_n\}$ is a cover of Y, which is finite. Therefore Y is compact.

Theorem: 3.9 If a map f: X → Y is gπ-irresolute and a subset B of X is gπ-compact relative to X, then the image f (B) is gπ-compact relative to Y.

Proof: Let $\{A_\alpha : \alpha \in \wedge\}$ be any collection of gπ-open subsets of Y such that $f(B) \subset \cup \{A_\alpha : \alpha \in \wedge\}$. Then $B \subset \{f^{-1} (A_\alpha) : \alpha \in \wedge\}$ holds. Since by hypothesis B is gπ-compact relative to X there exists a finite subset $\wedge_0$ of $\wedge$ such that $B \subset \cup \{f^{-1} (A_\alpha) : \alpha \in \wedge_0\}$.

Therefore we have $f (B) \subset \cup \{A_\alpha : \alpha \in \wedge_0\}$, which shows that f(B) is gπ-compact relative to Y.

Definition: 3.10 Let $\wedge$ be a directed set. A net $\xi = \{x_\alpha : \alpha \in \wedge\}$ gπ accumulates at a point $x \in X$ if the net is frequently in every $U \in g\pi O(X, x)$, i.e., for each $U \in g\pi O(X, x)$ and for each $\alpha_0 \in \wedge$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. The net ξ gπ-converges to a point x of X if it is eventually in every $U \in g\pi O(X, x)$.

Definition: 3.11 We say that a filter base $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ gπ-accumulates at a point $x \in X$ if $x \in D \cap_{\alpha\in\Gamma} g\pi Cl (F_\alpha)$. A filter base $\Theta = \{F_\alpha : \alpha \in \Gamma\}$ gπ-converges to a point x in X if for each $U \in g\pi O(X, x)$, there exists an F_α in Θ such that $F_\alpha \subset U$.

Definition: 3.12 A point x in a space X is said to be gπ-complete accumulation point of a subset S of X if $Card(S \cap U) = Card(S)$ for each $U \in g\pi O(X, x)$, where Card(S) denotes the cardinality of S.

Definition: 3.13 In a topological space X, a point x is said to be a gπ-adherent point of a filter base Θ on X if it lies in the gπ-closure of all sets of Θ.

Theorem: 3.14 A space X is gπ-compact if and only if each infinite subset of X has a gπ-complete accumulation point.

Proof: Let the space X be gπ-compact and let S be an infinite subset of X. Let K be the set of points x in X which are not gπ-complete accumulation points of S. Now it is obvious that for each x in K, we are able to find U(x) ∈ gπO(X, x) such that Card(S ∩ U(x)) ≠ Card(S). If K is the whole space X, then Θ = {U(x): x ∈ X} is a gπ-cover of X. By the hypothesis X is gπ-compact, so there exists a finite sub-cover ψ = {U (x_i)}, where i = 1, 2, ⋯, n such that S ⊂ {U (x_i) ∩ S : i = 1, 2, · · ·, n}. Then Card(S) = max {Card (U (x_i) ∩ S)}, where i = 1, 2, · ·, n which does not agree with what we assumed. This implies that S has gπ-complete accumulation point.

Conversely, suppose that X is not gπ-compact and that every infinite subset S ⊂ X has a gπ-complete accumulation point in X. It follows that there exists a gπ-cover E with no finite sub-cover. Set δ = min {Card(Φ) : Φ ⊂ E}, where Φ is a gπ-cover of X. Fix ψ ⊂ E for which Card(ψ) = δ and ∪{U : U ∈ ψ} =X. Let N denote the set of natural numbers. Then by hypothesis δ ≥ Card (N). By well ordering of ψ by some minimal well ordering ~ suppose that U is any member of ψ. By minimal well ordering ~ we have Card ({V: V ∈ ψ, V ~ U}) < Card ({V: V ∈ ψ}). Since ψ can not have any sub-cover with cardinality less than δ, then for each U ∈ ψ we have X ≠ ∪{V: V □ ψ, V ~ U}. For each U ∈ ψ choose a point x(U) ∈ X − U {V ∪{x(V)}: V ∈ ψ, V ~ U}. We are always able to do this if not one can choose a cover of smaller cardinality from ψ. If H = {x(U) : U ∈ ψ}, then to finish the proof we will show that H has no gπ-accumulation points in X. Suppose that z is a point of X. Since ψ is a gπ-cover of X, then z is a point of some set W in ψ. By the fact that U~W, we have x(U) ∈W. But Card (T) < δ. Therefore, Card (H ∩ W) < δ. But Card (H) = δ ≥ Card (N), since for two distinct points U and W in ψ, we have x(U) ≠ x(W).

This means that H has no gπ-complete accumulation point in X which contradicts our assumptions. Therefore X is gπ-compact.

Theorem: 3.15 For a space X the following are equivalent.

(1) X is gπ-compact.
(2) Every net in X with a well ordered directed set as its domain gπ-accumulates to some point of X.

Proof: (1) ⇒ (2):
Suppose that (X, τ) is gπ-compact and ξ = {x_α : α ∈ Λ} a net with a well ordered directed set Λ as domain. Assume that ξ has no gπ adherent point in X. Then for each point x in X, there exist V (x) ∈ gπO(X, x) and an α(x) ∈Λ such that V (x) ∩ {x_α : α ≥ α(x)} = φ. This implies that {x_α : α ≥ α(x)} is a subset of X − V (x). Then the collection C = {V (x): x ∈ X} is a gπ-cover of X. By hypothesis of theorem, X is gπ-compact and so C has a finite subfamily {V (x_i)}, where i = 1, 2, · · ·, n such that X = ∪{V (x_i)}. Suppose that the corresponding elements of Λ are {α (x_i)}, where i = 1, 2,.., n. Since Λ is well ordered and {α(x_i)}, where i = 1, 2, · ·, n is finite, the largest element of {α(x_i)} exists. Suppose it is {α(x_l)}, then for γ ≥ {α(x_l)}. We have {x_δ: δ ≥ γ} ⊂ D$\bigcup_{i=1}^{n}(X -$V (x_i)) = X− $\bigcup_{i=1}^{n} V$(x_i) = φ which is impossible.

This shows that ξ has at least one gπ-adherent point in X.

(2) ⇒ (1): Now by the last Theorem 3.14, it is enough to prove that each infinite subset has a gπ-complete accumulation point. Suppose that S ⊂ X is an infinite subset of X. According to Zorn's lemma, the infinite set S can be well ordered. This means that we can assume S to be a net with a domain, which is a well ordered index set. It follows that S has a gπ-adherent point z. Therefore z is a gπ-complete accumulation point of S. This shows that X is gπ-compact.

Theorem: 3.16 A space X is gπ-compact if and only if each family of gπ-closed subsets of X with the finite intersection property has a non-empty intersection.

Proof: Given a collection 𝔸 of subsets of X, let ℂ = {X − A : A ∈ 𝔸} be the collection of their complements. Then the following statements hold.

(a) 𝔸 is a collection of gπ-open sets if and only if ℂ is a collection of gπ –closed sets.

(b) The collection 𝔸 covers X if and only if the intersection $\cap_{c\in c}$C of all the elements of ℂ is non-empty.

(c) The finite sub collection {A_1 · · ·A_n} of A covers X if and only if the intersection of the corresponding elements C_i = X − Ai of ℂ is empty. The statement (a) is trivial, while the (b) and (c) follow from DeMorgan's law.

$X - (D\cup_{\alpha\in J} A_\alpha) = D\cap_{\alpha\in J} (X - A_\alpha)$ The proof of the theorem now proceeds in two steps, taking the contrapositive of the theorem and then the complement.

The statement X is gπ-compact is equivalent to: Given any collection $\mathcal{A}$ of gπ-open subsets of X, if $\mathcal{A}$ covers X, then some finite sub collection of $\mathcal{A}$ covers X. This statement is equivalent to its contrapositive, which is the following.

Given any collection $\mathcal{A}$ of gπ-open sets, if no finite sub-collection of $\mathcal{A}$ covers X, then $\mathcal{A}$ does not cover X. Letting $\mathcal{C}$ be as earlier, the collection $\{X - A : A \in \mathcal{A}\}$ and applying (a) to (c), we see that this statement is in turn equivalent to the following:

Given any collection $\mathcal{C}$ of gπ-closed sets, if every finite intersection of elements of $\mathcal{C}$ is non-empty, then the intersection of all the elements of $\mathcal{C}$ is non-empty. This is just the condition of our theorem.

Theorem: 3.17 A space X is gπ-compact if and only if each filter base in X has at least one gπ-adherent point.

Proof: Suppose that X is gπ-compact and $\theta = \{F_\alpha : \alpha \in \Gamma\}$ a filter base in it. Since all finite intersections of F_α's are non-empty, it follows that all finite intersections of gπCl (F_α)'s are also non-empty. Now it follows from Theorem 3.16 that $D\cap_{\alpha\in\Gamma}$ gbCl (F_α) is non-empty. This means that θ has at least one gπ-adherent point.

Conversely, suppose θ is any family of gπ-closed sets. Let each finite intersection be non-empty. The sets (F_α) with their finite intersection establish a filter base θ. Therefore θ gπ-accumulates to some point z in X. It follows that $z \in D\cap_{\alpha\in\Gamma} (F_\alpha)$. Now by Theorem 3.16, we have that X is gπ-compact.

Theorem: 3.18 A space X is gπ-compact if and only if each filter base on X with at most one gπ-adherent point is gπ-convergent.

Proof: Suppose that X is gπ-compact, x a point of X and a filter base on X. The gπ-adherence of θ is a subset of {x}. Then the gπ-adherence of θ is equal to {x} by Theorem 3.17. Assume that there exists $V \in g\pi O(X, x)$ such that for all $F \in \theta$, $F \cap (X - V)$ is non-empty. Then

$\psi = \{F - V: F \in \theta\}$ is a filter base on X. It follows that the gπ-adherence of θ is non-empty. However,

$D\cap_{F\in\theta}\, g\pi Cl(F - V) \subset (D\cap_{F\in\theta}\, g\pi Cl(F) \cap (X - V)) = \{x\} \cap \{X - V\} = \varphi$.

But this is a contradiction. Hence for each $V \in g\pi O(X, x)$ there exists an $F \in \theta$ with $F \subset V$. This shows that θ gπ-converges to x.

To prove the converse, it suffices to show that each filter base in X has at least one gπ-accumulation point. Assume that θ is a filter base on X with no gπ-adherent point. By hypothesis, θ gπ-converges to some point z in X. Suppose F_α is an arbitrary element of θ. Then for each $V \in g\pi O(X, z)$, there exists $F_\beta \in \theta$ such that $F_\beta \subset V$. Since θ is a filter base, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where F_γ non-empty. This means that $F_\alpha \cap V$ is non-empty for every $V \in g\pi O(X, z)$ and correspondingly for each α, z is a point of gπCl (F_α) it follows that $z \in D\cap_\alpha$ gπCl (F_α). Therefore z is a gπ-adherent point of θ which is contradiction.

This shows that X is gπ-compact.

4 gπ - Connectedness:

Definition: 4.1 A topological space X is said to be gπ-connected if X can not be expressed as a disjoint union of two non-empty gπ-open sets. A subset of X is gπ-connected if it is gπ-connected as a subspace.

Example: 4.2 Let X = {a, b} and let τ = {X, φ, {a}}. Then it is gπ-connected.

Remark: 4.3 Every gπ-connected space is connected but the converse need not be true in general, which follows from the following example.

Example: 4.4 Let X = {a, b} and let τ = {X, φ}. Clearly (X, τ) is connected. The gπ-open sets of X are {X, φ, {a}, {b}}. Therefore (X, τ) is not a gπ-connected space, because X = {a} ∪ {b} where {a} and {b} are non-empty gπ-open sets.

Theorem: 4.5 For a topological space X the following are equivalent.

(i) X is gπ-connected.
(ii) X and φ are the only subsets of X which are both gπ-open and gπ-closed.
(iii) Each gπ-continuous map of X into a discrete space Y with at least two points is a constant map.

Proof: (i) ⇒ (ii):
Let O be any gπ-open and gπ-closed subset of X. Then O^C is both gπ-open and gπ-closed. Since X is disjoint union of the gπ-open sets O and O^C implies from the hypothesis of (i) that either O = φ or O = X.

(ii) ⇒ (i):
Suppose that X = A ∪B where A and B are disjoint non-empty gπ-open subsets of X. Then A is both gπ-open and gπ-closed. By assumption A = φ or X.

Therefore X is gπ-connected.

(ii) ⇒ (iii):
Let f: X → Y be a gπ-continuous map. Then X is covered by gπ-open and gπ-closed covering $\{f^{-1}(Y): y \in (Y)\}$. By assumption $f^{-1}(y) = \varphi$ or X for each y ∈ Y. If $f^{-1}(y) = \varphi$ for all y ∈ Y, then f fails to be a map. Then there exists only one point y ∈ Y such that $f^{-1}(y) \neq \varphi$ and hence $f^{-1}(y) = X$. This shows that f is a constant map.

(iii) ⇒ (ii):
Let O be both gπ-open and gπ-closed in X. Suppose O ≠ φ. Let f: X → Y be a gπ-continuous map defined by f (O) = y and f (O^C) = {w} for some distinct points y and w in Y.

By assumption f is constant. Therefore we have O = X.

Theorem: 4.6 If f : X → Y is a gπ-continuous and X is gπ-connected, then Y is connected.

Proof: Suppose that Y is not connected. Let Y = A∪B where A and B are disjoint non-empty open set in Y. Since f is gπ-continuous and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is connected.

Theorem: 4.7 If f: X → Y is a gπ-irresolute surjection and X is gπ-connected, then Y is gπ-connected.

Proof: Suppose that Y is not gπ-connected. Let Y = A ∪ B where A and B are disjoint non-empty gπ-open set in Y. Since f is gπ-irresolute and onto, $X = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint non-empty gπ-open sets in X. This contradicts the fact that X is gπ-connected. Hence Y is gπ -connected.

Theorem: 4.8 In a topological space (X, τ) with at least two points, if πO(X, τ) = πCL(X, τ) then X is not gπ-connected.

Proof: By hypothesis we have πO(X, τ) = πCL(X, τ) and by Remark 2.4 we have every π closed set is gπ-closed, there exists some non-empty proper subset of X which is both gπ-open and gπ-closed in X. So by last Theorem 4.5 we have X is not gπ-connected.

Definition: 4.9 A topological space X is said to be $T_{g\pi}$-space if every gπ-closed subset of X is closed subset of X.

Theorem: 4.10 Suppose that X is a $T_{g\pi}$-space then X is connected if and only if it is gπ-connected.

Proof: Suppose that X is connected. Then X can not be expressed as disjoint union of two non-empty proper subsets of X. Suppose X is not a gπ-connected space. Let A and B be any two gπ-open subsets of X such that X = A∪B, where A ∩ B = φ and A ⊂ X, B ⊂ X. Since X is $T_{g\pi}$-space and A, B are gπ-open, A, B are open subsets of X, which contradicts that X is connected. Therefore X is gπ-connected.

Conversely, every open set is gπ-open. Therefore every gπ-connected space is connected.

Theorem: 4.11 If the gπ-open sets C and D form a separation of X and if Y is gπ-connected subspace of X, then Y lies entirely within C or D.

Proof: Since C and D are both gπ-open in X the sets C ∩ Y and D ∩ Y are gπ-open in Y these two sets are disjoint and their union is Y. If they were both non-empty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D.

Theorem: 4.12 Let A be a gπ-connected subspace of X. If A ⊂ B ⊂ gπCl(A) then B is also gπ-connected.

Proof: Let A be gπ-connected and let A ⊂ B ⊂ gπCl(A). Suppose that B = C∪D is a separation of B by gπ-open sets. Then by Theorem 4.11 above A must lie entirely in C or in D. Suppose that A ⊂ C, then gπCl(A) ⊆ gπCl(C). Since gπCl(C) and D are disjoint, B cannot intersect D. This contradicts the fact that D is non-empty subset of B. So D = φ which implies B is gπ-connected.

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