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A generalization of fuzzy Boundary

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ABSTRACT

The concept of fuzzy $\mathcal{C}$-boundary is introduced by using the arbitrary complement function $\mathcal{C}$ and by using fuzzy $\mathcal{C}$-closure of a fuzzy topological space where $\mathcal{C}$: [0, 1]→[0, 1] is a function. Let A be a fuzzy subset of a fuzzy topological space X and let $\mathcal{C}$ be a complement function. Then the fuzzy $\mathcal{C}$-boundary of A is defined as $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)$, where $Cl_{\mathcal{C}} A$ is the fuzzy $\mathcal{C}$-closure of A and $\mathcal{C}A(x) = \mathcal{C}(A(x))$, $0 \le x \le 1$. In this paper we discuss the basic properties of fuzzy $\mathcal{C}$-boundary.

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***Key words:** Fuzzy $\mathcal{C}$-boundary, fuzzy $\mathcal{C}$-closed sets and fuzzy topology.*

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1. INTRODUCTION AND PRELIMINARIES

Pu and Liu[7] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980. Following this, Athar and Ahmad [1] studied the properties of fuzzy boundary. The authors introduced the concept of fuzzy $\mathcal{C}$-closed sets in fuzzy topological spaces, where $\mathcal{C}$: [0, 1]→ [0, 1] is an arbitrary complement function.

In this paper, we generalize the concept of fuzzy boundary by using the arbitrary complement function $\mathcal{C}$, instead of the usual fuzzy complement function, and by using fuzzy $\mathcal{C}$-closure instead of fuzzy closure.

Such a generalized fuzzy boundary is defined as $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)$, called the fuzzy $\mathcal{C}$-boundary of A, where $Cl_{\mathcal{C}} A$ is the intersection of all fuzzy $\mathcal{C}$-closed sets containing A and $\mathcal{C}A(x) = \mathcal{C}(A(x))$, $0 \le x \le 1$.

For the basic concepts and notations, one can refer Chang[4]. The following definitions and lemmas are useful in studying the properties of fuzzy $\mathcal{C}$-boundary.

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Definition 1.1 [Definition 2.1, [3]]

Let $\mathcal{C}$: [0, 1]→[0, 1] be a complement function. If A is a fuzzy subset of (X,τ) with membership function μ_A then the complement $\mathcal{C}A$ of a fuzzy set A is a fuzzy subset with membership function defined by $\mu_{\mathcal{C}A}(x) = \mathcal{C}(\mu_A(x))$ for all $x \in X$.

For detailed discussion on complement function $\mathcal{C}$ and $\mathcal{C}A$, please refer [5] and [3]. The following properties are established in [3] for the complement function that satisfies the monotonic and involutive properties.

Lemma 1.2 [Lemma 2.9, [3]]

Let $\mathcal{C}$: [0, 1] → [0, 1] be a complement function that satisfies the involutive and monotonic properties. Then for any family $\{A_\alpha : \alpha \in \Delta\}$ of fuzzy subsets of X, we have

(i) $\mathcal{C}(\sup\{\mu_{A\alpha}(x): \alpha \in \Delta\}) = \inf\{\mathcal{C}(\mu_{A\alpha}(x)): \alpha \in \Delta\}$
$= \inf\{(\mu_{\mathcal{C}A\alpha}(x)): \alpha \in \Delta\}$ and

(ii) $\mathcal{C}(\inf\{\mu_{A\alpha}(x): \alpha \in \Delta\}) = \sup\{\mathcal{C}(\mu_{A\alpha}(x)): \alpha \in \Delta\}$
$= \sup\{(\mu_{\mathcal{C}A\alpha}(x)): \alpha \in \Delta\}$.

Lemma 1.3 [Demorgan's law, Lemma 2.10, [3]]
Let $\mathcal{C}$: [0, 1] → [0, 1] be a complement function that satisfies the involutive and monotonic properties. Then for any family $\{A_\alpha: \alpha\in\Delta\}$ of fuzzy subsets of X, we have

(i) $\mathcal{C}(\cup\{A_\alpha : \alpha\in\Delta\}) = \cap\{\mathcal{C}A_\alpha : \alpha\in\Delta\}$ and
(ii) $\mathcal{C}(\cap\{A_\alpha : \alpha\in\Delta\}) = \cup\{\mathcal{C}A_\alpha : \alpha\in\Delta\}$.

Definition 1.4 [Definition 3.1, [3]]
Let (X,τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function. Then a fuzzy subset A of X is fuzzy $\mathcal{C}$ - closed in (X,τ) if $\mathcal{C}A$ is fuzzy open in (X,τ).

It has been proved in [3] that the arbitrary intersection of fuzzy $\mathcal{C}$ - closed sets is fuzzy $\mathcal{C}$ - closed and the finite union of fuzzy $\mathcal{C}$ - closed sets is fuzzy $\mathcal{C}$ - closed.

Definition 1.5 [Definition 4.1, [3]]
Let (X, τ) be a fuzzy topological space. Then for a fuzzy subset A of X, the fuzzy $\mathcal{C}$-closure of A is defined as the intersection of all fuzzy $\mathcal{C}$ - closed sets B containing A. The fuzzy $\mathcal{C}$ - closure of A is denoted by $Cl_{\mathcal{C}}A$ that is equal to $\cap\{B: B \supseteq A, \mathcal{C}B\in\tau\}$.

Lemma 1.6 [Lemma 4.2, [3]]
If the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties, then for any fuzzy subset A of X, (i) $\mathcal{C}(Int\ A) = Cl_{\mathcal{C}}(\mathcal{C}A)$ and (ii) $\mathcal{C}(Cl_{\mathcal{C}}A) = Int\ (\mathcal{C}A)$, where Int A is the union of all fuzzy open sets B contained in A.

Lemma 1.7 [Theorem 4.3, [3]]
Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any two fuzzy subsets A and B of a fuzzy topological space, we have

(i) $A \subseteq Cl_{\mathcal{C}} A$;
(ii) A is $\mathcal{C}$ - closed $\Leftrightarrow$ $Cl_{\mathcal{C}} A = A$;
(iii) $Cl_{\mathcal{C}} (Cl_{\mathcal{C}} A) = Cl_{\mathcal{C}} A$;
(iv) If $A \subseteq B$ then $Cl_{\mathcal{C}} A \subseteq Cl_{\mathcal{C}} B$;
(v) $Cl_{\mathcal{C}} (A\vee B) = Cl_{\mathcal{C}} A \vee Cl_{\mathcal{C}}$ and
(vi) $Cl_{\mathcal{C}} (A \wedge B) \leq Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} B$.

Lemma 1.8 [Lemma 5.2, [3]]
Let X and Y be two fuzzy topological spaces. Suppose f is a function from X to Y. Then $f^{-1}(\mathcal{C} B) = \mathcal{C} (f^{-1}(B))$ for any fuzzy subset B of Y.

Lemma 1.9 [Theorem 5.3, [3]]
A function $f :(X, \tau_x) \to (Y, \tau_y)$ be a mapping from a fuzzy space X to another fuzzy space Y. Then f is fuzzy continuous if and only if $f^{-1}(A)$ is a fuzzy $\mathcal{C}$ - closed subset of X for each fuzzy $\mathcal{C}$ - closed subset B of Y, where $\mathcal{C}$ satisfies the monotonic and involutive properties.

2. FUZZY $\mathcal{C}$- BOUNDARY

In this section, the concept of fuzzy $\mathcal{C}$ - boundary is introduced. The properties of fuzzy $\mathcal{C}$ - boundary are discussed analogously with the properties of fuzzy boundary.

Definition 2.1
Let A be a fuzzy subset of a fuzzy topological space X and let $\mathcal{C}$ be a complement function. Then the fuzzy $\mathcal{C}$ - boundary of A is defined as $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)$.

If the complement function $\mathcal{C}$ is taken as the usual complement function, $\mathcal{C}(x) = 1-x$, then the concept of fuzzy $\mathcal{C}$ - boundary coincides with the concept of fuzzy boundary. Since the arbitrary intersection of fuzzy $\mathcal{C}$ - closed sets is fuzzy $\mathcal{C}$ - closed, $Bd_{\mathcal{C}}$ A is fuzzy $\mathcal{C}$ - closed.

We identify $Bd_{\mathcal{C}}$ A with Bd(A) when $\mathcal{C}(x) = 1-x$, the usual complement function. Athar and Ahmad [1] established that Bd(A) = Bd(1 − A) [Proposition 3.1.1, [1]]. The analogous result is true for fuzzy $\mathcal{C}$ - boundary when $\mathcal{C}$ satisfies the involutive property as shown in the next proposition.

Proposition 2. 2
Let (X,τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function that satisfies the involutive property. Then for any fuzzy subset A of X, $Bd_{\mathcal{C}} A = Bd_{\mathcal{C}}\mathcal{C}A$.

Proof: By using Definition 2.1, $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} (\mathcal{C}A)$.
Since $\mathcal{C}$ satisfies the
involutive property $\mathcal{C}(\mathcal{C}A) = A$, that implies $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} (\mathcal{C}A) \wedge Cl_{\mathcal{C}}\mathcal{C}(\mathcal{C}A)$.
Again by using Definition 2.1, $Bd_{\mathcal{C}} A = Bd_{\mathcal{C}} (\mathcal{C}A)$. This completes the proof.

Athar and Ahmad [1] established that the following results [Proposition 3.1, [1]]

(i) If A is fuzzy closed, then $Bd(A) \leq A$.
(ii) If A is fuzzy open, then $Bd(A) \leq (1-A)$.
(iii) Let $A \leq B$ and B be fuzzy closed. Then $Bd(A) \leq B$.
(iv) Let $A \leq B$ and B be fuzzy open. Then $Bd(A) \leq (1-B)$.

The analogous results are true for fuzzy $\mathcal{C}$ - boundary when $\mathcal{C}$ satisfies the monotonic and involutive properties as shown in the following propositions.

Proposition 2. 3
Let (X,τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. If A is fuzzy $\mathcal{C}$ - closed, then $Bd_{\mathcal{C}} A \leq A$.

Proof: Let A be fuzzy $\mathcal{C}$ - closed. By using Definition 2.1, $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} (\mathcal{C}A)$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(ii), we have

$Cl_{\mathcal{C}} A = A$. Hence $Bd_{\mathcal{C}} A \leq Cl_{\mathcal{C}} A = A$. This completes the proof.

Proposition 2. 4

Let (X,τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. If A is fuzzy open, then $Bd_{\mathcal{C}}A \leq \mathcal{C}A$.

Proof: Let A be fuzzy open. Since $\mathcal{C}$ satisfies the involutive property $\mathcal{C}(\mathcal{C}A) = A$. This implies that $\mathcal{C}(\mathcal{C}A)$ is fuzzy open. By using Definition 1.4, $\mathcal{C}A$ is fuzzy $\mathcal{C}$ - closed. Since $\mathcal{C}$ satisfies the monotonic and the involutive properties, by using Proposition 2.3, we get $Bd_{\mathcal{C}} (\mathcal{C}A) \leq \mathcal{C}A$. Also by using Proposition 2.2, we get $Bd_{\mathcal{C}} (A) \leq \mathcal{C} A$. This completes the proof.

Proposition 2.5

Let (X, τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. If $A \leq B$ and B is fuzzy $\mathcal{C}$ - closed then $Bd_{\mathcal{C}}A \leq B$.

Proof: Let $A \leq B$ and B be fuzzy $\mathcal{C}$ - closed. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(iv), we have $A \leq B$ implies $Cl_{\mathcal{C}} A \leq Cl_{\mathcal{C}} B$. By using Definition 2.1, $Bd_{\mathcal{C}}A = Cl_{\mathcal{C}}A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)$. Since $Cl_{\mathcal{C}}A \leq Cl_{\mathcal{C}}B$, we have

$Bd_{\mathcal{C}}A \leq Cl_{\mathcal{C}}B \wedge Cl_{\mathcal{C}}(\mathcal{C}A) \leq Cl_{\mathcal{C}}B$. Again by using Lemma 1.7 (ii), we have $Cl_{\mathcal{C}} B = B$. This implies that $Bd_{\mathcal{C}} A \leq B$. This completes the proof.

Proposition 2. 6

Let (X,τ) be a fuzzy topological space and $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. If $A \leq B$ and B is fuzzy open then $Bd_{\mathcal{C}}A \leq \mathcal{C}B$.

Proof: Let $A \leq B$ and B is fuzzy open. Since $\mathcal{C}$ satisfies the monotonic property, $\mathcal{C}B \leq \mathcal{C}A$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(iv), we have $\mathcal{C}B \leq \mathcal{C}A$ that implies $Cl_{\mathcal{C}}\mathcal{C}B \leq Cl_{\mathcal{C}}\mathcal{C}A$. By using Definition 2.1, $Bd_{\mathcal{C}}A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}\mathcal{C}A$. Taking complement on both sides, we get

$\mathcal{C} (Bd_{\mathcal{C}} A) = \mathcal{C} (Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} (\mathcal{C}A))$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.3(ii), we have $\mathcal{C}(Bd_{\mathcal{C}}A) = \mathcal{C}(Cl_{\mathcal{C}}A) \vee \mathcal{C}(Cl_{\mathcal{C}}(\mathcal{C}A))$.

Since $Cl_{\mathcal{C}}\mathcal{C}B \leq Cl_{\mathcal{C}}\mathcal{C}A$, we get $\mathcal{C}(Bd_{\mathcal{C}}A) \geq \mathcal{C}(Cl_{\mathcal{C}}\mathcal{C}B) \vee \mathcal{C}(Cl_{\mathcal{C}}A)$. By using Lemma 1.6(ii), the above implies that $\mathcal{C}(Bd_{\mathcal{C}} A) \geq Int\, B \vee Int\, \mathcal{C}A \geq Int\, B$. Since B is fuzzy open, we have $\mathcal{C}(Bd_{\mathcal{C}}A) \geq B$. Since $\mathcal{C}$ satisfies the monotonic properties, $Bd_{\mathcal{C}}A \leq \mathcal{C}B$. This completes the proof.

Athar and Ahmad[1] established that " $1- Bd(A) = Int\, A \vee Int\, (1- A)$" [Proposition 3.1.5, [1]]. The same result is true for fuzzy $\mathcal{C}$ - boundary when $\mathcal{C}$ satisfies the monotonic and involutive properties as shown in the next proposition.

Proposition 2.7

Let (X,τ) be a fuzzy topological space. Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any fuzzy subset A of X, we have

$\mathcal{C}(Bd_{\mathcal{C}}A) = Int\, A \vee Int\, (\mathcal{C}A)$.

Proof: By using Definition 2.1, $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} (\mathcal{C}A)$. Taking complement on both sides, we get $\mathcal{C}(Bd_{\mathcal{C}}A) = \mathcal{C}(Cl_{\mathcal{C}}A \wedge Cl_{\mathcal{C}} (\mathcal{C}A))$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.3(ii), we have

$\mathcal{C}(Bd_{\mathcal{C}}A) = \mathcal{C}(Cl_{\mathcal{C}}A) \vee \mathcal{C}(Cl_{\mathcal{C}}(\mathcal{C}A))$. Also by using Lemma 1.6(ii), this implies that

$\mathcal{C} (Bd_{\mathcal{C}} A) = Int\, (\mathcal{C}A) \vee Int(\mathcal{C}(\mathcal{C}A))$. Since $\mathcal{C}$ satisfies the involutive property

$\mathcal{C}(\mathcal{C} A) = A$. Thus we have $\mathcal{C}(Bd_{\mathcal{C}}A) = Int\, A \vee Int\, (\mathcal{C}A)$. This completes the proof.

Proposition 2.8

Let (X,τ) be a fuzzy topological space. Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any fuzzy subset A of X, we have

$Bd_{\mathcal{C}}A = Cl_{\mathcal{C}}A \wedge \mathcal{C}(Int\, A)$.

Proof: By using Definition 2.1, we have $Bd_{\mathcal{C}}A = Cl_{\mathcal{C}}A \wedge Cl_{\mathcal{C}} (\mathcal{C}A)$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.6(ii), we have

$Bd_{\mathcal{C}} A = Cl_{\mathcal{C}}A \wedge \mathcal{C}(Int\, A)$.

Proposition 2.9

Let (X,τ) be a fuzzy topological space. Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then for any subset A of X, $Bd_{\mathcal{C}}\, Int\, A \leq Bd_{\mathcal{C}}A$.

Proof: Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Proposition 2.12, we have $Bd_{\mathcal{C}}\, Int\, A = Cl_{\mathcal{C}}\, Int A \wedge \mathcal{C}(Int\, Int A)$. Since $Int\, A$ is fuzzy open $Bd_{\mathcal{C}}\, Int\, A = Cl_{\mathcal{C}}\, Int A \wedge \mathcal{C}(Int A)$. Since $Int\, A \leq A$, by using Lemma 1.7(iv), we get $Cl_{\mathcal{C}}\, Int A \leq Cl_{\mathcal{C}} A$. Thus, we have $Bd_{\mathcal{C}} Int\, A \leq Cl_{\mathcal{C}} A \wedge \mathcal{C}(Int A)$.

Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.6(ii), $Bd_{\mathcal{C}} Int\, A \leq Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} (\mathcal{C}A)$. By using Definition 2.1, we have $Bd_{\mathcal{C}}\, Int\, A \leq Bd_{\mathcal{C}}A$. This completes the proof.

Proposition 2.10
Let (X,τ) be a fuzzy topological space. Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then $Bd_{\mathcal{C}}\, Cl_{\mathcal{C}} A \leq Bd_{\mathcal{C}} A$.

Proof: Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Proposition 2.8,

we have $Bd_{\mathcal{C}}\, Cl_{\mathcal{C}} A = Cl_{\mathcal{C}}(Cl_{\mathcal{C}} A) \wedge \mathcal{C}(Int\ Cl_{\mathcal{C}} A)$. By using Lemma 1.7(iii), we have
$Cl_{\mathcal{C}}(Cl_{\mathcal{C}} A) = Cl_{\mathcal{C}} A$, that implies $Bd_{\mathcal{C}}\, Cl_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge \mathcal{C}(Int\ Cl_{\mathcal{C}} A)$. Since $A \leq Cl_{\mathcal{C}} A$. The above implies that $Int\ A \leq Int\ Cl_{\mathcal{C}} A$. Therefore, $Bd_{\mathcal{C}}\, Cl_{\mathcal{C}} A \leq Cl_{\mathcal{C}} A \wedge \mathcal{C}(Int\ A)$. By using Lemma 1.6(ii), and by using Definition 2.1, we get $Bd_{\mathcal{C}}\, Cl_{\mathcal{C}} A \leq Bd_{\mathcal{C}} A$.

This completes the proof.

Athar and Ahmad[1] established the following

(i) $Bd(A \vee B) \leq Bd(A) \vee Bd(B)$ [Theorem 3.2, [1]]
(ii) $Bd(A \wedge B) \leq (Bd(A) \wedge Cl\ B) \vee (Bd(B) \wedge Cl\ A)$ [Theorem 3.3, [1]].

The analogous results are true for fuzzy $\mathcal{C}$- boundary when $\mathcal{C}$ satisfies the monotonic and involutive properties as shown in the following theorems.

Theorem 2.11
Let (X,τ) be a fuzzy topological space. Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties. Then $Bd_{\mathcal{C}}(A \vee B) \leq Bd_{\mathcal{C}} A \vee Bd_{\mathcal{C}} B$.

Proof: By using Definition 2.1, $Bd_{\mathcal{C}}(A \vee B) = Cl_{\mathcal{C}}(A \vee B) \wedge Cl_{\mathcal{C}}(\mathcal{C}(A \vee B))$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(v), that implies

$Bd_{\mathcal{C}}(A \vee B) = (Cl_{\mathcal{C}} A \vee Cl_{\mathcal{C}} B) \wedge Cl_{\mathcal{C}}(\mathcal{C}(A \vee B))$. By using Lemma 1.3(ii), and also we have

$Bd_{\mathcal{C}}(A \vee B) \leq (Cl_{\mathcal{C}} A \vee Cl_{\mathcal{C}} B) \wedge (Cl_{\mathcal{C}}(\mathcal{C}A) \wedge Cl_{\mathcal{C}}(\mathcal{C}B))$.That is,

$Bd_{\mathcal{C}}(A \vee B) \leq (Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)) \vee (Cl_{\mathcal{C}} B) \wedge Cl_{\mathcal{C}}(\mathcal{C}B))$.

Again by using Definition 2.1, $Bd_{\mathcal{C}}(A \vee B) \leq Bd_{\mathcal{C}} A \vee Bd_{\mathcal{C}} B$. This completes the proof.

Theorem 2.12
Let (X,τ) be a fuzzy topological space. Suppose the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties. Then for any two fuzzy subsets A and B of a fuzzy topological space X, we have $Bd_{\mathcal{C}}(A \wedge B) \leq (Bd_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} B) \vee (Bd_{\mathcal{C}} B \wedge Cl_{\mathcal{C}} A)$.

Proof: By using Definition 2.1, we have $Bd_{\mathcal{C}}(A \wedge B) = Cl_{\mathcal{C}}(A \wedge B) \wedge Cl_{\mathcal{C}}(\mathcal{C}(A \wedge B))$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(v) and Lemma 1.7 (vi), and also by using Lemma 1.3(ii), we get

$Bd_{\mathcal{C}}(A \wedge B) \leq (Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} B) \wedge (Cl_{\mathcal{C}}(\mathcal{C}A) \vee Cl_{\mathcal{C}}(\mathcal{C}B))$
$= [Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)] \wedge (Cl_{\mathcal{C}} B) \vee [Cl_{\mathcal{C}} B \wedge Cl_{\mathcal{C}}(\mathcal{C}B)] \wedge Cl_{\mathcal{C}} A$.
Again by using Definition 2.1, we get $Bd_{\mathcal{C}}(A \wedge B) \leq (Bd_{\mathcal{C}} A \wedge Cl_{\mathcal{C}} B) \vee (Bd_{\mathcal{C}} B \wedge Cl_{\mathcal{C}} A)$. This completes the proof.

Athar and Ahmad[1] established that

(i) $Bd\ Bd(A) \leq Bd(A)$ and (ii) $Bd\ Bd\ Bd(A) \leq Bd\ Bd(A)$ [Proposition 3.3, [1]]. The similar results are true for true fuzzy $\mathcal{C}$ - boundary when $\mathcal{C}$ satisfies the monotonic and involutive properties as shown in the following proposition.

Proposition 2.13
Let (X, τ) be a fuzzy topological space. Suppose the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties. Then for any two fuzzy subsets A and B of a fuzzy topological space X, we have
(i) $Bd_{\mathcal{C}} Bd_{\mathcal{C}} A \leq Bd_{\mathcal{C}} A$
(ii) $Bd_{\mathcal{C}} Bd_{\mathcal{C}} Bd_{\mathcal{C}} A \leq Bd_{\mathcal{C}} Bd_{\mathcal{C}} A$.

Proof: By using Definition 2.1, $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A)$. We have

$Bd_{\mathcal{C}} Bd_{\mathcal{C}} A = Cl_{\mathcal{C}}(Bd_{\mathcal{C}} A) \wedge Cl_{\mathcal{C}}[\mathcal{C}(Bd_{\mathcal{C}} A)] \leq Cl_{\mathcal{C}} Bd_{\mathcal{C}} A$. Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.7(ii), we have $Cl_{\mathcal{C}} A = A$, when A is fuzzy $\mathcal{C}$ - closed. Here $Bd_{\mathcal{C}} A$ is fuzzy $\mathcal{C}$ - closed. So, $Cl_{\mathcal{C}} Bd_{\mathcal{C}} A = Bd_{\mathcal{C}} A$. This implies that $Bd_{\mathcal{C}} Bd_{\mathcal{C}} A \leq Bd_{\mathcal{C}} A$. This proves (i). (ii) follows from (i). This completes the proof.

Definition 2.14 [Katsaras and Liu, [6]]
If A is a fuzzy subset of X and B is a fuzzy subset of Y, then $A \times B$ is a fuzzy subset of $X \times Y$ defined by $(A \times B)(x, y) = \min\{A(x), B(y)\}$, for each $(x, y) \in X \times Y$.

Definition 2.15
A fuzzy topological space (X, τ_x) is $\mathcal{C}$ - product related to another fuzzy topological space (Y, τ_y) if for any fuzzy subset P of X and Q of Y, whenever $\mathcal{C}A \ngeq P$ and $\mathcal{C}B \ngeq Q$ imply $\mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B \geq P \times Q$, where $A \in \tau_x$ and $B \in \tau_y$, there exist $A_1 \in \tau_x$ and $B_1 \in \tau_y$ such that $\mathcal{C}A_1 \geq P$ or $\mathcal{C}B_1 \geq Q$ and $\mathcal{C}A_1 \times 1 \vee 1 \times \mathcal{C}B_1 = \mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B$.

Lemma 2.16
If A is a fuzzy subset of X and B is a fuzzy subset of Y, then $\mathcal{C}(A \times B) = \mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B$.

Proof: By using Definition 2.14, $\mathcal{C}(A \times B)(x, y) = \max\{\mathcal{C}A(x), \mathcal{C}B(y)\}$
$= \max\{(\mathcal{C}A \times 1)(x, y), (1 \times \mathcal{C}B)(x, y)\} = (\mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B)(x, y)$.

This implies that $\mathcal{C}(A \times B) = \mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B$. This completes the proof.

Lemma 2.17

Let A be a fuzzy $\mathcal{C}$ - closed set of a fuzzy space X and B be a fuzzy $\mathcal{C}$ - closed set of a fuzzy space Y, then $A \times B$ is a fuzzy $\mathcal{C}$ - closed set of the fuzzy product space $X \times Y$ where the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties.

Proof: Let A be a fuzzy $\mathcal{C}$ - closed set of a fuzzy space X. Then by using Definition 1.4, $\mathcal{C}A$ is fuzzy open set in X.

Also if $\mathcal{C}A$ is fuzzy open set in X, then $\mathcal{C}A \times 1$ is fuzzy open in $X \times Y$. Similarly let B be a fuzzy $\mathcal{C}$ - closed set of a fuzzy space X. Then by using Definition 1.4, $\mathcal{C}B$ is fuzzy open set in Y. Also if $\mathcal{C}B$ is fuzzy open set in Y then $\mathcal{C}B \times 1$ is fuzzy open in $X \times Y$. Since the arbitrary union of fuzzy open sets is fuzzy open. So, $\mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B$ is fuzzy open in $X \times Y$. By using Lemma 2.16,

$\mathcal{C}(A \times B) = \mathcal{C}A \times 1 \vee 1 \times \mathcal{C}B$, hence $\mathcal{C}(A \times B)$ is fuzzy open. By using Definition 1.4,

$A \times B$ is fuzzy $\mathcal{C}$ - closed of the fuzzy product space $X \times Y$.

This completes the proof.

Theorem 2.18

Let $\mathcal{C}$ be a complement function that satisfies the monotonic and involutive properties.

If A is a fuzzy subset of a fuzzy space X and B is a fuzzy subset of a fuzzy space Y,

then $Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B \geq Cl_{\mathcal{C}} (A \times B)$.

Proof: By using Definition 2.14, $(Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B)(x, y) = \min\{Cl_{\mathcal{C}} A(x), Cl_{\mathcal{C}} B(y)\}$

$\geq \min\{A(x), B(y)\} = (A \times B)(x, y)$. This shows that $Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B \geq (A \times B)$.

$Cl_{\mathcal{C}} (A \times B) \leq Cl_{\mathcal{C}} (Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B) = Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B$.

This completes the proof.

Theorem 2.19

Let X and Y are $\mathcal{C}$ - product related fuzzy topological spaces. Then for a fuzzy subset A of X and a fuzzy subset B of Y, $Cl_{\mathcal{C}}(A \times B) = Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B$.

Proof: By using Theorem 2.18, it is sufficient to show that $Cl_{\mathcal{C}}(A \times B) \geq Cl_{\mathcal{C}} A \times Cl_{\mathcal{C}} B$. By using Definition 1.5 and by using Definition 2.14, we have

$Cl_{\mathcal{C}}(A \times B) = \inf\{\mathcal{C}(A_\alpha \times B_\beta) : \mathcal{C}(A_\alpha \times B_\beta) \geq A \times B$ where $A_\alpha \in \tau_x$ and $B_\beta \in \tau_y\}$.

$= \inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta \geq A \times B\}$, by using Lemma 2.16.

$= \inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}A_\alpha \geq A$ or $\mathcal{C}B_\beta \geq B\}$.

$= \min(\inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}A_\alpha \geq A\}, \inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}B_\beta \geq B\})$.

Now $\inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}A_\alpha \geq A\} \geq \inf\{\mathcal{C}A_\alpha \times 1 : \mathcal{C}A_\alpha \geq A\}$.

$= \inf\{\mathcal{C}A_\alpha : \mathcal{C}A_\alpha \geq A\} \times 1$.

$= (Cl_{\mathcal{C}} A) \times 1$.

Also $\inf\{\mathcal{C}A_\alpha \times 1 \vee 1 \times \mathcal{C}B_\beta : \mathcal{C}B_\beta \geq B\} \geq \inf\{1 \times \mathcal{C}B_\beta : \mathcal{C}B_\beta \geq B\}$.

$= 1 \times \inf\{\mathcal{C}B_\beta : \mathcal{C}B_\beta \geq B\}$.

$= 1 \times Cl_{\mathcal{C}}B$.

The above discussions *imply* that $Cl_{\mathcal{C}}(A \times B) \geq \min(Cl_{\mathcal{C}}A \times 1, 1 \times Cl_{\mathcal{C}}B)$.

$= Cl_{\mathcal{C}}A \times Cl_{\mathcal{C}}B$.

This completes the proof.

Lemma 2.20 [Athar, Ahmad [1]]

Let (X, τ) be a fuzzy topological space. If A, B, C, D are the fuzzy subsets of X then $(A \wedge B) \times (C \wedge D) = (A \times D) \wedge (B \times C)$.

Theorem 2.21

Let X_i, i = 1, 2 ….n, be a family of $\mathcal{C}$ - product related fuzzy topological spaces. Each A_i is a fuzzy subset of X_i, and the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties, then $Bd_{\mathcal{C}} \prod_{i=1}^{n} A_i$

$= [Bd_{\mathcal{C}}A_1 \times Cl_{\mathcal{C}} A_2 \times \ldots \times Cl_{\mathcal{C}} A_n] \vee [Cl_{\mathcal{C}}A_1 \times Bd_{\mathcal{C}}A_2 \times Cl_{\mathcal{C}}A_3 \times \ldots \times Cl_{\mathcal{C}}A_n] \vee \ldots \vee [Cl_{\mathcal{C}}A_1 \times Cl_{\mathcal{C}} A_2 \times \ldots \times Bd_{\mathcal{C}}A_n]$.

Proof: It suffices to prove this for n = 2. By using Proposition 2.8, we have

$Bd_{\mathcal{C}}(A_1 \times A_2) = Cl_{\mathcal{C}}(A_1 \times A_2) \wedge \mathcal{C}(Int(A_1 \times A_2))$.

$= (Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge \mathcal{C}(IntA_1 \times IntA_2)$, by using Theorem 2.19.

$= (Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge \mathcal{C}[(Int A_1 \wedge Cl_{\mathcal{C}} A_1) \times (Int A_2 \wedge Cl_{\mathcal{C}} A_2)]$.

$=(Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge [\mathcal{C}(IntA_1 \wedge Cl_{\mathcal{C}}A_1) \times 1 \vee 1 \times \mathcal{C}(IntA_2 \wedge Cl_{\mathcal{C}} A_1)]$,by using Lemma 2.16.

Since $\mathcal{C}$ satisfies the monotonic and involutive properties, by using Lemma 1.6 (i) and (ii), and also by using Lemma 1.3 (ii),

$Bd_{\mathcal{C}}(A_1 \times A_2)$

$= (Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge [(Cl_{\mathcal{C}}\mathcal{C}A_1 \vee Int\mathcal{C}A_1) \times 1 \vee 1 \times (Cl_{\mathcal{C}} \mathcal{C}A_2 \vee Int\mathcal{C}A_2)]$.

$= (Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge [Cl_{\mathcal{C}} \mathcal{C}A_1 \times 1 \vee 1 \times Cl_{\mathcal{C}} \mathcal{C}A_2]$.

$= [(Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge (Cl_{\mathcal{C}} \mathcal{C}A_1 \times 1)] \vee [(Cl_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2) \wedge (1 \times Cl_{\mathcal{C}} \mathcal{C}A_2)]$.

Again by using Lemma 2.16, we get

$Bd_{\mathcal{C}}(A_1 \times A_2)$

$= [(Cl_{\mathcal{C}} A_1 \wedge Cl_{\mathcal{C}} \mathcal{C}A_1) \times (1 \wedge Cl_{\mathcal{C}} A_2)] \vee [(Cl_{\mathcal{C}} A_2 \wedge Cl_{\mathcal{C}} \mathcal{C}A_2) \times (1 \wedge Cl_{\mathcal{C}} A_1)]$.

$= [(Cl_{\mathcal{C}} A_1 \wedge Cl_{\mathcal{C}} \mathcal{C}A_1) \times Cl_{\mathcal{C}} A_2] \vee [(Cl_{\mathcal{C}} A_2 \wedge Cl_{\mathcal{C}} \mathcal{C}A_2) \times Cl_{\mathcal{C}} A_1]$.

$Bd_{\mathcal{C}}(A_1 \times A_2) = [Bd_{\mathcal{C}} A_1 \times Cl_{\mathcal{C}} A_2] \vee [Cl_{\mathcal{C}} A_1 \times Bd_{\mathcal{C}} A_2]$.

This completes the proof.

Theorem 2.22

Let f: X→Y be a fuzzy continuous function. Suppose the complement function $\mathcal{C}$ satisfies the monotonic and involutive properties. Then $Bd_{\mathcal{C}} f^{-1}(B) \le f^{-1}(Bd_{\mathcal{C}} B)$, for any fuzzy subset B in Y.

Proof: Let f be a fuzzy continuous function and B be a fuzzy subset in Y. By using Definition 2.1, we have

$Bd_{\mathcal{C}} f^{-1}(B) = Cl_{\mathcal{C}} f^{-1}(B) \wedge Cl_{\mathcal{C}} (\mathcal{C}(f^{-1}(B)))$.
$= Cl_{\mathcal{C}} f^{-1}(B) \wedge Cl_{\mathcal{C}} (f^{-1}(\mathcal{C}B))$.

Since f is fuzzy continuous and since $f^{-1}(B) \le f^{-1}(Cl_{\mathcal{C}} B)$, it follows

$Cl_{\mathcal{C}} f^{-1}(B) \le f^{-1}(Cl_{\mathcal{C}} B)$. This together with the above imply that $Bd_{\mathcal{C}} f^{-1}(B) \le f^{-1}(Cl_{\mathcal{C}} B) \wedge f^{-1}(Cl_{\mathcal{C}} (\mathcal{C}B)) = f^{-1}(Cl_{\mathcal{C}} B \wedge Cl_{\mathcal{C}} (\mathcal{C}B))$. That is $Bd_{\mathcal{C}} f^{-1}(B) \le f^{-1}(Bd_{\mathcal{C}} B)$.

This completes the proof.

3. Examples

Athar and Ahmad[1] studied the properties of fuzzy boundary with respect to usual complement function $\mathcal{C}(x) = 1-x$, $0 \le x \le 1$.

In section 2, we extended some of these properties to fuzzy $\mathcal{C}$ - boundary when $\mathcal{C}$ satisfies the monotonic and involutive properties. Such properties are not true when $\mathcal{C}$ does not satisfy monotonic and / or involutive properties as shown in the following examples.

Example 3.1

Let X = {a, b} be associated with fuzzy topology
$\tau = \{ 0, \{a_{.4}, b_{.8}\}, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.8}\}, \{a_{.5}, b_{.2}\}, \{a_{.8}, b_{.7}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.7}\}, \{a_{.4}, b_{.2}\}, \{a_{.6}, b_{.8}\}, \{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, 1 \}$.

Let $\mathcal{C}(x) = \dfrac{1-x}{1+x^2}$, $0 \le x \le 1$ be the complement function. Then the complement function $\mathcal{C}$ does not satisfy the involutive property.

The family of all fuzzy $\mathcal{C}$- closed sets is given by
$\tau^1 = \{ 0, \{a_{.5}, b_{.1}\}, \{a_{.3}, b_{.1}\}, \{a_{.4}, b_{.3}\}, \{a_{.4}, b_{.8}\}, \{a_{.1}, b_{.2}\}, \{a_{.6}, b_{.8}\}, \{a_{.5}, b_{.3}\}, \{a_{.5}, b_{.8}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.1}\}, \{a_{.1}, b_{.1}\}, 1 \}$.

Let $A = \{a_{.2}, b_{.8}\}$. Then $Cl_{\mathcal{C}} A = \{a_{.4}, b_{.8}\}$. Now $\mathcal{C}A = \{a_{.8}, b_{.1}\}$ and $Cl_{\mathcal{C}} \mathcal{C}A = 1$.

$Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A) = \{a_{.4}, b_{.8}\} \wedge 1 = \{a_{.4}, b_{.8}\}$.

Also $\mathcal{C}A = \{a_{.8}, b_{.1}\}$, then $Cl_{\mathcal{C}} \mathcal{C}A = 1$ and $\mathcal{C}(\mathcal{C}A) = \{a_{.1}, b_{.9}\}$, $Cl_{\mathcal{C}} \mathcal{C}(\mathcal{C}A) = 1$.

$Bd_{\mathcal{C}} \mathcal{C}A = Cl_{\mathcal{C}} \mathcal{C}A \wedge Cl_{\mathcal{C}} \mathcal{C}(\mathcal{C}A) = 1$. This implies that $Bd_{\mathcal{C}} A \ne Bd_{\mathcal{C}} \mathcal{C}A$.

The above example shows that, the word "involutive" can not be dropped from the hypothesis of the Proposition 2.2.

Example 3.2

Let X = {a, b} be associated with fuzzy topology
$\tau = \{ 0, \{a_{.4}, b_{.8}\}, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.8}\}, \{a_{.5}, b_{.2}\}, \{a_{.8}, b_{.7}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.7}\}, \{a_{.4}, b_{.2}\},$

$\{a_{.6}, b_{.8}\}, \{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, 1 \}$.

Let $\mathcal{C}(x) = \dfrac{1-x}{1+x^2}$, $0 \le x \le 1$ be the complement function. Then the complement function $\mathcal{C}$ satisfies the monotonic property but does not satisfy the involutive property.

The family of all fuzzy $\mathcal{C}$ - closed sets
$\tau^1 = \{ 0, \{a_{.5}, b_{.1}\}, \{a_{.3}, b_{.1}\}, \{a_{.4}, b_{.2}\}, \{a_{.4}, b_{.8}\}, \{a_{.1}, b_{.2}\}, \{a_{.6}, b_{.8}\}, \{a_{.5}, b_{.2}\}, \{a_{5}, b_{.8}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.1}\}, \{a_{.1}, b_{.1}\}, 1 \}$.

Let $A = \{a_{.5}, b_{.4}\}$ and $B = \{a_{.5}, b_{.2}\}$. Then $Cl_{\mathcal{C}} A = \{a_{.5}, b_{.8}\}$ and $Cl_{\mathcal{C}} \mathcal{C}A = \{a_{.4}, b_{.8}\}$.

Now $Bd_{\mathcal{C}} A = Cl_{\mathcal{C}} A \wedge Cl_{\mathcal{C}}(\mathcal{C}A) = \{a_{.4}, b_{.8}\}$. This implies that $Bd_{\mathcal{C}} A \not\le B$.

Example 3.3

Suppose we define the complement function $\mathcal{C}$ as $\mathcal{C}(x) = \dfrac{2x}{1+x}$, $0 \le x \le 1$. Then $\mathcal{C}$ does not satisfy the monotonic and involutive properties.

Let X = {a, b} be associated with fuzzy topology as given in Example 3.1.

The family of all fuzzy $\mathcal{C}$- closed sets is given by
$\tau^1 = \{ 0, \{a_{.6}, b_{.9}\}, \{a_{.8}, b_{.9}\}, \{a_{.7}, b_{.8}\}, \{a_{.7}, b_{.3}\}, \{a_{.9}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.5}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.8}, b_{.8}\}, \{a_{.7}, b_{.9}\}, \{a_{.9}, b_{.9}\}, 1 \}$.

Let $A = \{a_{.4}, b_{.6}\}$ and $B = \{a_{.5}, b_{.3}\}$. Then $Cl_{\mathcal{C}} A = \{a_{.6}, b_{.8}\}$ and $Cl_{\mathcal{C}} \mathcal{C}A = \{a_{.6}, b_{.8}\}$.

So, $Bd_{\mathcal{C}} A = \{a_{.6}, b_{.8}\} \not\le B$.

The Example 3.2 and Example 3.3 show that, if the complement function $\mathcal{C}$ does not satisfy the monotonic and involutive properties, the conclusion of the Proposition 2.5 is false.

Example 3.4

Let X = {a, b} be associated with fuzzy topology
$\tau = \{ 0, \{a_{.4}, b_{.8}\}, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.7}\}, \{a_{.5}, b_{.2}\}, \{a_{.8}, b_{.7}\}, \{a_{.3}, b_{.2}\}, \{a_{.4}, b_{.7}\}, \{a_{.4}, b_{.2}\}, \{a_{.6}, b_{.7}\}, \{a_{.5}, b_{.8}\}, \{a_{.8}, b_{.8}\}, \{a_{.6}, b_{.8}\}, \{a_{.8}, b_{.9}\}, 1 \}$.

and the complement function defined in Example 3.2 satisfies the monotonic property but does not satisfy the involutive property.

The family of all fuzzy $\mathcal{C}$- closed sets is given by

τ^1 = { 0, {a.5, b.1}, {a.3, b.1}, {a.4, b.2}, {a.4, b.8}, {a.1, b.2}, {a.6, b.8}, {a.5, b.2}, {a5, b.8}, {a.3, b.2}, {a.4, b.1}, {a.1, b.1}, 1}.

Let A = {a.4, b.3} and B = {a.5, b.8}. Then $\mathcal{C}$A = {a.5, b.6}, $\mathcal{C}$B = {a.4, b.1}.

Now $Cl_{\mathcal{C}}$ A= {a.4, b.8}, $Cl_{\mathcal{C}}$ $\mathcal{C}$A = {a.5, b.8}. So, $Bd_{\mathcal{C}}$A = $Cl_{\mathcal{C}}$ A $\wedge$ $Cl_{\mathcal{C}}$ ($\mathcal{C}$A) = {a.4, b.8}.
This implies $Bd_{\mathcal{C}}A \not\leq \mathcal{C}B$.

Example 3.5

Let X = {a, b} be associated with fuzzy topology τ = { 0, {a.5, b.3}, {a.6, b.5}, {a.3, b.2}, {a.7, b.8}, 1 }.

Let $\mathcal{C}(x) = \dfrac{1-x^2}{(1+x)^2}$, 0 ≤ x ≤ 1 be the complement function.

Then the complement function $\mathcal{C}$ satisfies the monotonic property but does not satisfy the involutive property. The family of all fuzzy $\mathcal{C}$- closed sets is given by

τ^1 = { 0, {a.3, b.5}, {a.3, b.3}, {a.5, b.7}, {a.2, b.1}, 1 }. Let A = {a.5, b.4} and
B = {a.6, b.5}. Then $Cl_{\mathcal{C}}$ A ={a.5, b.7} and $Cl_{\mathcal{C}}\mathcal{C}$ A ={a.3, b.5}. This implies that
$Bd_{\mathcal{C}}$A ={a.3, b.5} $\not\leq \mathcal{C}$B.

The above examples show that the condition "monotonic and involutive" can not be dropped from the hypothesis of the Proposition 2.6.

Example 3.6

Let X = {a, b} be associated with fuzzy topology as given in Example 3.2 and the complement function also defined in Example 3.2 satisfies the monotonic property but does not satisfy the involutive property.

The family of all fuzzy $\mathcal{C}$- closed sets is

τ^1 = { 0, {a.5, b.1}, {a.3, b.1}, {a.4, b.3}, {a.4, b.8}, {a.1, b.2}, {a.6, b.8}, {a.5, b.3}, {a5, b.8}, {a.3, b.2}, {a.4, b.1}, {a.1, b.1}, 1}.

Let A = {a.4, b.3} then $Cl_{\mathcal{C}}$A = {a.4, b.8} and
$Cl_{\mathcal{C}}\mathcal{C}$A = {a.5, b.8}. This implies that
$Bd_{\mathcal{C}}$ A = {a.4, b.8}. Now $\mathcal{C}$($Bd_{\mathcal{C}}$ A) = {a.5, b.1} and *Int* A = {a.4, b.2}, *Int* $\mathcal{C}$A = {a.5, b.2}.
So, *Int* A $\vee$ *Int* $\mathcal{C}$A = {a.5, b.2} that implies $\mathcal{C}$($Bd_{\mathcal{C}}$ A) ≠ *Int* A $\vee$ *Int* $\mathcal{C}$A.

Example 3.7

Let the complement function $\mathcal{C}$ as $\mathcal{C}(x) = \sqrt{x}$, 0 ≤ x ≤ 1. Then the complement function $\mathcal{C}$ does not satisfy the monotonic and involutive properties.

Let X = {a, b} be associated with fuzzy topology.
τ = { 0, {a.3, b.8}, {a.2, b.5}, {a.7, b.1}, {a.3, b.5}, {a.3, b.1}, {a.2, b.1}, {a.7, b.8},{a.7, b.5}, 1 }.

The family of all fuzzy $\mathcal{C}$- closed sets is

τ^1 = { 0, {a.5, b.9}, {a.4, b.7}, {a.8, b.3}, {a.5, b.7}, {a.5, b.3}, {a.4, b.3},{a.8, b.9}, {a.8, b.7}, 1 }.

Let A = {a.5, b.4}. Then $Cl_{\mathcal{C}}$A = {a.5, b.7}
and $Cl_{\mathcal{C}}\mathcal{C}$A = {a.8, b.7}

Hence $Bd_{\mathcal{C}}$ A = {a.5, b.7}. Now $\mathcal{C}$($Bd_{\mathcal{C}}$ A) = {a.7, b.8}
and *Int* A = {a.3, b.1},
Int $\mathcal{C}$A = {a.7, b.5}. So, *Int* A $\vee$ *Int* $\mathcal{C}$A = {a.7, b.5}.

This implies that
$\mathcal{C}$($Bd_{\mathcal{C}}$ A) ≠ *Int* A $\vee$ *Int* $\mathcal{C}$A.

If the complement function $\mathcal{C}$ does not satisfy the monotonic and involutive properties, then the conclusion of the Proposition 2.7 is false.

Example 3.8

Let X = {a, b} be associated with fuzzy topology.
τ = { 0, {a.8, b.2}, {a.3, b.6}, {a.1, b.5}, {a.4, b.7}, {a.3, b.2}, {a.8, b.6},{a.1, b.2},{a.8, b.5},{a.4, b.2}{a.8, b.7},{a.3, b.5}, {a.4, b.6}, 1}.

Let $\mathcal{C}(x) = \dfrac{1-x^3}{(1+x)^3}$, 0 ≤ x ≤ 1 be the complement function.

Then the complement function $\mathcal{C}$ satisfies the monotonic property but does not satisfy the involutive property.

The family of all fuzzy $\mathcal{C}$- closed sets is

τ^1 = { 0, {a.1, b.6}, {a.4, b.2}, {a.7, b.5}, {a.3, b.1}, {a.4, b.6}, {a.1, b.2}, {a.7, b.6}, {a.1, b.3}, {a.3, b.6}, {a.1, b.1}, {a.4, b.3}, {a.3, b.2}, 1 }.

Let A = {a.7, b.2}. Then *Int* A = {a.4, b.2}.
Now $Bd_{\mathcal{C}}$ {a.4, b.2} = $Cl_{\mathcal{C}}$ {a.4, b.2} $\wedge$ $Cl_{\mathcal{C}}$ {a.3, b.6} = {a.4, b.2} $\wedge$ {a.3, b.6} = {a.3, b.2}.
So, $Bd_{\mathcal{C}}$ *Int* A = {a.3, b.2}. Also, $Cl_{\mathcal{C}}$A ={a.8, b.3} and $Cl_{\mathcal{C}}\mathcal{C}$A = $Cl_{\mathcal{C}}$ {a.1, b.6} = {a.1, b.6}.

This implies that $Bd_{\mathcal{C}}$ A = {a.8, b.3} $\wedge$ {a.1, b.6} = {a.1, b.3}. So, $Bd_{\mathcal{C}}$ *Int* A ≥ $Bd_{\mathcal{C}}$ A.

This example shows that, the condition "monotonic and involutive" can not be dropped from the hypothesis of the Proposition 2.9.

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