

ON $g\pi$ US SPACES

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Abstract

In this paper we introduce and study the notion of $g\pi$ -US spaces, $g\pi$ -convergence, sequentially $g\pi$ -compactness, sequentially $g\pi$ -continuity and sequentially $g\pi$ -sub-continuity by utilizing $g\pi$ -open sets.

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1. Introduction and Preliminaries:

In 1967, A. Wilansky [11] introduced and studied the concept of US -spaces. Levine [8] introduced the concept of generalized closed sets of a topological space. Since the advent of these notions, several research papers with interesting results in different respects came to existence (see, [1], [2], [3], [4], [5], [6], [9]). This paper is devoted to deal with the concepts of $g\pi$ -US spaces, $g\pi$ -convergence, sequentially $G\pi O$ -compactness, sequentially $g\pi$ -continuity and sequentially $g\pi$ -sub-continuity.

Throughout the present paper (X, τ) and (Y, σ) (or simply X and Y) denote topological spaces. Let A be a subset of X . We denote the interior and the closure of a set A by $\text{Int}(A)$ and $\text{Cl}(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = \text{int}(\text{cl}(A))$ (resp. $A = \text{cl}(\text{int}(A))$). The finite union of regular open sets is said to be π -open. The complement of a π -open set is said to be π -closed. A $A \subset X$ is called a generalized closed set (briefly g -closed set) of X [8] if $\text{Cl}(A) \subset G$ holds whenever $A \subset G$ and G is open in X . A subset A of X is called a g -open set of X , if its complement A^c is g -closed in X . A $A \subset X$ is called a $g\pi$ -closed set of X [10] if $\pi\text{Cl}(A) \subset G$ holds whenever $A \subset G$ and G is open in X . A subset A of X is called a $g\pi$ -open set of X , if its complement A^c is $g\pi$ -closed in X .

A space X is $G O$ compact if every g -open cover of X has a finite subcover. A subset A of a space X is said to be GO -compact if A is GO -compact as a subspace of X . The product space of two non-empty spaces is GO -compact, if each factor space is GO compact [1]. If A is g -open in X and B is g -open in Y , then $A \times B$ is g -open in $X \times Y$ [8]. A function $f: X \rightarrow Y$ is said to be g -continuous (resp. $g\pi$ continuous) [1] ([10]) if the inverse image of every closed set in Y is g -closed (resp. $g\pi$ -closed) in X .

2. $g\pi$ -US spaces:

Definition: 1 A sequence $\{x_n\}$ in a space X $g\pi$ -converges to a point $x \in X$ if $\{x_n\}$ is eventually in every $g\pi$ -open set containing x .

Definition: 2 A space X is said to be $g\pi$ -US if every sequence in X $g\pi$ -converges to a point of X .

Definition: 3 A space X is said to be:

(1) $g\pi$ - T_1 if for each pair of distinct points x and y in X there exist a $g\pi$ -open set U in X such that $x \in U$ and $y \notin U$ and a $g\pi$ -open set V in X such that $y \in V$ and $x \notin V$.

(2) $g\pi$ - T_2 if for each pair of distinct points x and y in X there exist $g\pi$ -open sets U and V such that $U \cap V = \emptyset$ and $x \in U$, $y \in V$.

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Theorem: 2.1 Every $g\pi$ -US space is $g\pi$ - T_1 .

Proof: Let X be a $g\pi$ -US space and x, y be two distinct points of X . Consider the sequence $\{x_n\}$, where $x_n = x$ for any $n \in \mathbb{N}$. Clearly, $\{x_n\}$ $g\pi$ converges to x . Since $x \neq y$ and X is $g\pi$ -US, $\{x_n\}$ does not $g\pi$ -converge to y , i.e., there exists a $g\pi$ -open set U containing x but not y . Similarly, we obtain a $g\pi$ -open set V containing y but not x . Thus, X is $g\pi$ - T_1 .

Theorem: 2.2 Every $g\pi$ - T_2 space is $g\pi$ -US.

Proof: Let X be a $g\pi$ - T_2 space and $\{x_n\}$ a sequence in X . Assume that $\{x_n\}$ $g\pi$ -converges to two distinct points x and y . Then $\{x_n\}$ is eventually in every $g\pi$ -open set containing x and also in every $g\pi$ -open set containing y . Since X is $g\pi$ - T_2 then $\{x_n\}$ is eventually in two disjoint $g\pi$ -open sets. This is a contradiction. Therefore, X is $g\pi$ -US.

Definition: 4 A subset A of a space X is said to be:

- (1) Sequentially $g\pi$ -closed if every sequence in A $g\pi$ -converges to a point in A ,
- (2) Sequentially $G\pi$ O-compact if every sequence in A has a subsequence which $g\pi$ converges to a point in A .

Theorem: 2.3 A space is $g\pi$ -US if and only if the diagonal set Δ is a sequentially $g\pi$ -closed subset of the product space $X \times X$.

Proof: Suppose that X is a $g\pi$ -US space and $\{(x_n, x_n)\}$ is a sequence in the diagonal Δ . It follows that $\{x_n\}$ is a sequence in X . Since X is $g\pi$ -US, the sequence $\{x_n\}$ $g\pi$ -converges to a unique point, say $x \in X$. This implies that the sequence $\{(x_n, x_n)\}$ $g\pi$ -converges to (x, x) which clearly belongs to Δ . Therefore, Δ is a sequentially $g\pi$ -closed subset of $X \times X$.

Conversely, suppose that the diagonal Δ is a sequentially $g\pi$ -closed subset of $X \times X$. Assume that a sequence $\{x_n\}$ is $g\pi$ -converging to x and y . Then it follows that $\{(x_n, x_n)\}$ $g\pi$ -converges to (x, y) . By hypothesis, since Δ is sequentially $g\pi$ -closed, we have $(x, y) \in \Delta$. Thus, $x = y$. Therefore, X is $g\pi$ -US.

Theorem: 2.4 If a space X is $g\pi$ -US and a subset M of X is sequentially $G\pi$ O compact, then M is sequentially $g\pi$ -closed.

Proof: Assume that $\{x_n\}$ is any sequence in M which $g\pi$ -converges to a point $x \in X$. Since M is sequentially $G\pi$ O-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ $g\pi$ -converges to $m \in M$. Since X is $g\pi$ -US, we have $x = m$. This shows that M is sequentially $g\pi$ -closed.

Theorem: 2.5 The product space of an arbitrary family of $g\pi$ -US topological spaces is a $g\pi$ -US topological space.

Proof: Let $\{X_\lambda: \lambda \in \Delta\}$ be a family of $g\pi$ -US topological spaces with the index set Δ . The product space of $\{X_\lambda: \lambda \in \Delta\}$ is denoted by ΠX_λ . Let $\{x_n(\lambda)\}$ be a sequence in ΠX_λ . Suppose that $\{x_n(\lambda)\}$ $g\pi$ -converges to two distinct points x and y in ΠX_λ . Then there exists a $\lambda_0 \in \Delta$ such that $x(\lambda_0) \neq y(\lambda_0)$. Then $\{x_n(\lambda_0)\}$ is a sequence in X_{λ_0} . Let V_{λ_0} be any $g\pi$ -open set in X_{λ_0} containing $x(\lambda_0)$. Then $V = V_{\lambda_0} \times \prod_{\lambda \neq \lambda_0} X_\lambda$ is a $g\pi$ -open set of ΠX_λ containing x . Therefore, $\{x_n(\lambda)\}$ is eventually in V . Thus, $\{x_n(\lambda_0)\}$ is eventually in V_{λ_0} and it $g\pi$ -converges to $x(\lambda_0)$. Similarly, the sequence $\{x_n(\lambda_0)\}$ $g\pi$ -converges to $y(\lambda_0)$. This is a contradiction as X_{λ_0} is a $g\pi$ -US space. Therefore, the product space ΠX_λ is $g\pi$ -US.

3. Sequentially $G\pi$ O-compact preserving functions:

Definition: 5 A function $f: X \rightarrow Y$ is said to be:

- (1) Sequentially $g\pi$ -continuous at $x \in X$ if the sequence $\{f(x_n)\}$ $g\pi$ -converges to $f(x)$ whenever a sequence $\{x_n\}$ $g\pi$ -converges to x . If f is sequentially $g\pi$ -continuous at each $x \in X$, then it is said to be sequentially $g\pi$ -continuous.
- (2) Sequentially nearly $g\pi$ -continuous, if for each sequence $\{x_n\}$ in X that $g\pi$ -converges to $x \in X$, there exists subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that the sequence $\{f(x_{n_k})\}$ $g\pi$ -converges to $\{f(x_n)\}$.
- (3) Sequentially sub $g\pi$ -continuous if for each point $x \in X$ and each sequence $\{x_n\}$ in X $g\pi$ -converging to x , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $y \in Y$ such that the sequence $\{f(x_{n_k})\}$ $g\pi$ -converge to y .
- (4) Sequentially $G\pi$ O-compact preserving if the image $f(M)$ of every sequentially $G\pi$ O-compact set M of X is a sequentially $G\pi$ O-compact subset of Y .

Theorem: 3.1 Let $f_1: X \rightarrow Y$ and $f_2: X \rightarrow Y$ be two sequentially $g\pi$ - continuous functions. If Y is $g\pi$ -US, then the set $E = \{x \in X: f_1(x) = f_2(x)\}$ is sequentially $g\pi$ -closed.

Proof: Suppose that Y is $g\pi$ -US and $\{x_n\}$ is any sequence in E that f_1 -converges to $x \in X$. Since f_1 and f_2 are sequentially $g\pi$ -continuous functions, the sequence $\{f_1(x_n)\}$ (respectively, $\{f_2(x_n)\}$) converges to $f_1(x)$ respectively, $f_2(x)$. Since $x_n \in E$ for each $n \in \mathbb{N}$ and Y is $g\pi$ -US, $f_1(x) = f_2(x)$ and hence $x \in E$. This shows that E is sequentially $g\pi$ -closed.

Lemma: 3.2 Every function $f: X \rightarrow Y$ is sequentially sub- $g\pi$ -continuous if Y is sequentially $G\pi O$ -compact.

Proof: Let $\{x_n\}$ be a sequence in X that $g\pi$ converges to $x \in X$. It follows that $\{f(x_n)\}$ is a sequence in Y . Since Y is sequentially $G\pi O$ compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that $g\pi$ -converges to a point $y \in Y$.

Therefore, $f: X \rightarrow Y$ is sequentially sub- $g\pi$ -continuous.

Theorem 3.3 Every sequentially nearly $g\pi$ -continuous function is sequentially $G\pi O$ -compact preserving.

Proof: Let $f: X \rightarrow Y$ be a sequentially nearly $g\pi$ -continuous function and M be any sequentially $G\pi O$ -compact subset of X . We will show that $f(M)$ is a sequentially $G\pi O$ -compact subset of Y . So, assume that $\{y_n\}$ is any sequence in $f(M)$. Then for each $n \in \mathbb{N}$, there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Now M is sequentially $G\pi O$ -compact, so there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that $g\pi$ -converges to a point $x \in M$. Since f is sequentially nearly $g\pi$ -continuous, there exists a subsequence $\{x_{n_k(i)}\}$ of $\{x_{n_k}\}$ such that $\{f(x_{n_k(i)})\}$ $g\pi$ -converges to $f(x)$. Therefore, there exists a subsequence $\{y_{n_k(i)}\}$ of $\{y_n\}$ that $g\pi$ -converges to $f(x)$. This implies that $f(M)$ is a sequentially $G\pi O$ -compact set of Y .

Theorem: 3.4 Every sequentially $G\pi O$ -compact preserving function is sequentially sub- $g\pi$ -continuous.

Proof: Suppose that $f: X \rightarrow Y$ is a sequentially $G\pi O$ -compact preserving function. Let x be any point of X and $\{x_n\}$ a sequence that $g\pi$ converges to x . We denote the set $\{x_n : n \in \mathbb{N}\}$ by A and put $M = A \cup \{x\}$. Since $\{x_n\}$ $g\pi$ -converges to x , M is sequentially $G\pi O$ -compact. By hypothesis, f is sequentially $G\pi O$ -compact preserving and hence $f(M)$ is a sequentially $G\pi O$ -compact subset of Y . Now in $f(M)$ there exists a subsequence $\{f(x_{n_k})\}$ of $\{f(x_n)\}$ that $g\pi$ -converges to a point $y \in f(M)$. This implies that f sequentially sub- $g\pi$ -continuous.

Theorem 3.5 A function $f: X \rightarrow Y$ is sequentially $G\pi O$ -compact preserving if and only if $f|_M: M \rightarrow f(M)$ is sequentially sub $g\pi$ -continuous for each sequentially $G\pi O$ -compact set M of X .

Proof: Necessity: suppose that $f: X \rightarrow Y$ is a sequentially $G\pi O$ -compact preserving function. Then $f(M)$ is sequentially $G\pi O$ -compact in Y for each sequentially $G\pi O$ -compact subset M of X . Therefore, by Theorem 3.4 $f|_M: M \rightarrow f(M)$ is sequentially sub- $g\pi$ -continuous.

Sufficiency: Let M be any sequentially $G\pi O$ -compact set of X . We will show that $f(M)$ is sequentially $G\pi O$ -compact subset of Y . Let $\{y_n\}$ be any sequence in $f(M)$. Then for each $n \in \mathbb{N}$, there exists a point $x_n \in M$ such that $f(x_n) = y_n$.

Since $\{x_n\}$ is a sequence in the sequentially $G\pi O$ -compact set M there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that $g\pi$ -converges to a point in M . By hypothesis $f|_M: M \rightarrow f(M)$ is sequentially sub- $g\pi$ -continuous hence there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ that $g\pi$ -converges to $y \in f(M)$. This implies that $f(M)$ is sequentially $G\pi O$ -compact in Y .

Corollary: 3.6 If a function $f: X \rightarrow Y$ is sequentially sub- $g\pi$ -continuous and $f(M)$ is sequentially $g\pi$ -closed in Y for each sequentially $G\pi O$ -compact set M of X , then f is sequentially $G\pi O$ -compact preserving.

Proof: It will suffice to show that f is sequentially sub- $g\pi$ -continuous for each sequentially $G\pi O$ -compact set M of X , and by Lemma 3.2 we are done. So, let $\{x_n\}$ be any sequence in M that $g\pi$ -converges to a point $x \in M$. Then, since f is sequentially sub- $g\pi$ -continuous there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a point $y \in Y$ such that $\{f(x_{n_k})\}$ $g\pi$ converges to y .

Since $\{f(x_{n_k})\}$ is a sequence in the sequentially $g\pi$ -closed set $f(M)$ of Y , we obtain $y \in f(M)$. This implies that $f|_M: M \rightarrow f(M)$ is sequentially sub- $g\pi$ -continuous.

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