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SOME FIXED POINT THEOREMS IN $\mathcal{M}$-FUZZY METRIC SPACE

T. Veerapandi
Associate Professor of Mathematics, P. M. T. College,
Melaneelithanallur – 627953, India
E-mail: vrpnd@yahoo.co.in

G. Uthaya Sankar*
Department of Mathematics,
Mano College (An Institution of M. S. Univerity, Tirunelveli),
Sankarankovil – 627756, India
E-mail: uthayaganapathy@yahoo.com

A. Subramanian
Department of Mathematics, The M. D. T. Hindu College,
Tirunelveli – 627010, India
E-mail:asmani1963@gmail.com

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ABSTRACT

In this paper we prove Banach type fixed point theorem for complete $\mathcal{M}$ – fuzzy metric space. Also we establish some fixed point theorems for generalized contraction mappings in $\mathcal{M}$ – fuzzy metric spaces.

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Keywords: *$\mathcal{M}$ – Fuzzy metric space, $\mathcal{M}$ – Fuzzy contraction mapping, Fixed point theorem.*

1. INTRODUCTION AND PRELIMINARIES:

In 1965, Zadeh introduced the famous theory of fuzzy sets and used it as a tool for dealing with uncertainty arising out of lack of information about certain complex system. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek's study of fuzzy metric spaces paves a way for very soothing machinery to develop fixed point theorems for contractive type maps. Recently Sedghi and Shope [7] introduced D^* - metric space as a probable modification of the definition of D - metric introduced by Dhage, and prove some basic properties in D^* - metric spaces. Using D^* - metric concepts, they [7] define $\mathcal{M}$ – fuzzy metric space and proved a common fixed point theorem in it. In this paper we prove Banach type fixed point theorem for complete $\mathcal{M}$ – fuzzy metric space. Also we prove some fixed point theorems for generalized contraction mappings in $\mathcal{M}$ – fuzzy metric space.

Definition: 1.1 Let X be a nonempty set. A generalized metric (or D^* - metric) on X is a function: $D^*: X^3 \to [0,\infty)$, that satisfies the following conditions for each $x, y, z, a \in X$

(i) $D^*(x, y, z) \geq 0$,
(ii) $D^*(x, y, z) = 0$ iff $x = y = z$,
(iii) $D^*(x, y, z) = D^*(p\{x, y, z\})$, (symmetry) where p is a permutation function,
(iv) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.
The pair (X, D^*), is called a generalized metric (or D^* - metric) space.

Immediate examples of D* - metric are

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$,
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$.

Here, d is the ordinary metric on X.

Corresponding author:** *G. Uthaya Sankar*,E-mail:** *uthayaganapathy@yahoo.com*

Definition: 1.2 A fuzzy set $\mathcal{M}$ in an arbitrary set X is a function with domain X and values in [0, 1].

Definition: 1.3 A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions

(i) $*$ is associative and commutative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$ for all $a \in [0, 1]$,
(iv) $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0, 1]$.

Two typical examples for continuous t-norm are $a*b = ab$ and $a*b = \min\{a, b\}$.

Definition: 1.4 A 3-tuple $(X, \mathcal{M}, *)$ is called $\mathcal{M}$– fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm, and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

$(FM-1)$ $\mathcal{M}(x, y, z, t) > 0$
$(FM-2)$ $\mathcal{M}(x, y, z, t) = 1$ iff $x = y = z$
$(FM-3)$ $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$, where p is a permutation function
$(FM-4)$ $\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$
$(FM-5)$ $\mathcal{M}(x, y, z, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous
$(FM-6)$ $\lim_{t \rightarrow \infty} \mathcal{M}(x, y, z, t) = 1$.

Example: 1.5 Let X be a nonempty set and D^* is the D^* - metric on X. Denote $a*b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$, then $(X, \mathcal{M}, *)$ is a $\mathcal{M}$- fuzzy metric space. We call this $\mathcal{M}$-fuzzy metric induced by D^* - metric space. Thus every D^* - metric induces a $\mathcal{M}$-fuzzy metric.

Lemma: 1.6 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. Then for every $t > 0$ and for every $x, y \in X$ we have $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Lemma: 1.7 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$–fuzzy metric space. Then $\mathcal{M}(x, y, z, t)$ is non-decreasing with respect to t, for all x, y, z in X.

Proof: For each $x, y, z, a \in$ X and $t, s > 0$ we have

$$\mathcal{M}(x, y, a, t) * \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$$

If set $a=z$ we get

$$\mathcal{M}(x, y, z, t) * \mathcal{M}(z, z, z, s) \leq \mathcal{M}(x, y, z, t+s)$$

That is $\mathcal{M}(x, y, z, t+s) \geq \mathcal{M}(x, y, z, t)$.

Definition: 1.8 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$-fuzzy metric space. For $t > 0$, the open ball $B_{\mathcal{M}}(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_{\mathcal{M}}(x, r, t) = \{y \in X: \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_{\mathcal{M}}(x, r, t) \subseteq A$.

Definition: 1.9 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$– fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$ is said to be converges to a point $x \in X$ if $\lim_{n \rightarrow \infty} \mathcal{M}(x, x, x_n, t) = 1$ for all $t > 0$
(b) $\{x_n\}$ is called Cauchy sequence if $\lim_{n \rightarrow \infty} \mathcal{M}(x_{n+p}, x_{n+p}, x_n, t) = 1$ for all $t > 0$ and $p > 0$
(c) A $\mathcal{M}$– fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.10 Since * is continuous, it follows from (*FM-4*) that the limit of the sequence is uniquely determined.

Lemma: 1.11 Let $\{x_n\}$ be a sequence in a $\mathcal{M}-$ fuzzy metric space $(X, \mathcal{M}, *)$ with the condition (*FM-6*). If there exists a number $q \in (0,1)$ such that

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_{n-1}, x_n, \frac{t}{q}) \tag{1.11.1}$$

for all $t > 0$ and $n = 1, 2, 3 \ldots$, then $\{x_n\}$ is a Cauchy sequence.

Proof: By the simple induction with the condition (1.11.1), we have for all $t > 0$ and $n = 1, 2 \ldots$

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) \geq \mathcal{M}(x_0, x_0, x_1, \frac{t}{q^n}) \tag{1.11.2}$$

Thus by (*FM-4*) and (1.11.2), for any positive integer p and $t > 0$, we have

$$\mathcal{M}(x_n, x_n, x_{n+p}, t) \geq \mathcal{M}(x_n, x_n, x_{n+1}, \frac{t}{p}) \overset{p \;\; times}{*\ldots\ldots*} \mathcal{M}(x_{n+p-1}, x_{n+p-1}, x_{n+p}, \frac{t}{p})$$

$$\geq \mathcal{M}(x_0, x_0, x_1, \frac{t}{pq^n}) \overset{p \;\; times}{*\ldots\ldots*} \mathcal{M}(x_0, x_0, x_1, \frac{t}{pq^{n+p-1}})$$

Therefore, by (*FM-6*), we have

$$lim_{n \to \infty} \mathcal{M}(x_n, x_n, x_{n+p}, t) \geq 1 \overset{p \;\; times}{*\ldots\ldots*} 1 = 1$$

which implies that $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}-$ fuzzy metric space X.

Lemma 1.12 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}-$ fuzzy metric space with condition (*FM-6*). If for all $x, y, z \in X$, $t > 0$ with positive number $q \in (0,1)$ and $\mathcal{M}(x, y, z, qt) \geq \mathcal{M}(x, y, z, t)$, then $x = y = z$.

Proof: By hypothesis and (*FM-6*), we have

$$\mathcal{M}(x, y, z, t) \geq \mathcal{M}(x, y, z, \frac{t}{q})$$

$$\vdots$$

$$\geq \mathcal{M}(x, y, z, \frac{t}{q^n}) \to 1 \quad \text{as n} \to \infty$$

Hence by (*FM-2*), $x = y = z$.

2. MAIN RESULTS:

Theorem: 2.1 Let X be a complete $\mathcal{M}-$ fuzzy metric space with continuous t-norm * and let $T: X \to X$ be a mapping such that $\mathcal{M}(Tx, Ty, Tz, t) \geq \mathcal{M}(x, y, z, \frac{t}{q})$ for all $x, y, z \in X$ and $t > 0$, where $0 < q < 1$. Then T has a unique fixed point.

Proof: Let $x_0 \in X$ be any arbitrary fixed element in X.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots$

For $n \geq 1$, we have

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) = \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)$$

$$\geq \mathcal{M}(x_{n-1}, x_{n-1}, x_n, \frac{t}{q})$$

Therefore, by lemma 1.11, $\{x_n\}$ is a Cauchy sequence in $\mathcal{M}$ – fuzzy metric space X.

Since X is $\mathcal{M}$ – fuzzy complete, sequence $\{x_n\}$ converges to a point $x \in X$.

Thus we have

$$\begin{aligned}
\mathcal{M}(Tx, Tx, x, t) &\geq \mathcal{M}(Tx, Tx, x_{n+1}, \frac{t}{2}) * \mathcal{M}(x_{n+1}, x, x, \frac{t}{2}) \\
&= \mathcal{M}(Tx, Tx, Tx_n, \frac{t}{2}) * \mathcal{M}(x_{n+1}, x, x, \frac{t}{2}) \\
&\geq \mathcal{M}(x, x, x_n, \frac{t}{2q}) * \mathcal{M}(x_{n+1}, x, x, \frac{t}{2}) \\
&\to 1 \quad \text{as } n \to \infty
\end{aligned}$$

By (FM-2) we get $Tx = x$, a fixed point.

Uniqueness: Suppose there exists $y \in X$ such that $Ty = y$ and $y \neq x$

Now $\mathcal{M}(x, x, y, t) = \mathcal{M}(Tx, Tx, Ty, t)$

$$\begin{aligned}
&\geq \mathcal{M}(x, x, y, \frac{t}{q}) \\
&\quad\vdots \\
&\geq \mathcal{M}(x, x, y, \frac{t}{q^n}) \to 1 \quad \text{as } n \to \infty
\end{aligned}$$

By (FM-2) we get $x = y$.

Theorem: 2.2 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $T: X \to X$ be a mapping such that for all x, $y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, x, Tx, t), \mathcal{M}(y, y, Ty, t), \mathcal{M}(z, z, Tz, t)\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is a unique fixed point of T.

Proof: Let $x_0 \in X$ be any arbitrary fixed element in X.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots$

Suppose $x_n = x_{n+1}$ for some n. Then $x_n = Tx_n$. Thus $x_n = u$ is a fixed point of T.

Let us assume that $x_n \neq x_{n+1}$ for all n.

For $n \geq 1$, we have

$$\begin{aligned}
\mathcal{M}(x_n, x_n, x_{n+1}, t) &= \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t) \\
&> \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t), \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t), \mathcal{M}(x_n, x_n, Tx_n, t)\} \\
&= \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_n, x_n, x_{n+1}, t)\} \\
&= \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t).
\end{aligned}$$

Hence $\mathcal{M}(x_n, x_n, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$ for all $n \geq 1$. Thus $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ is monotonically increasing sequence of positive real number bounded above by 1, it is convergent to a positive real number, say L.

Therefore $lim_{n \to \infty}\ \mathcal{M}(x_n, x_n, x_{n+1}, t) = L$.

Also the sequence $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ has a subsequence $\{\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\}$ converges to L

That is $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$.

Now we prove that $L = 1$.

Suppose $L < 1$

Since $x_n = T^n x_0$ has a subsequence x_{n_k} converges to u

We have $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$ (2.2.1)

Now $1 > L = \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$

$$\geq \lim_{k\to\infty}\{\mathcal{M}(x_{n_k}, x_{n_k}, u, \frac{t}{2}) * (u, x_{n_k+1}, x_{n_k+1}, \frac{t}{2})\}$$

$$= 1*1 \quad \text{using (2.2.1)}$$

which is contradiction.

Therefore $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$

Now we prove u is fixed point of T.

Suppose $u \neq T(u)$ we have

$$\mathcal{M}(u, u, Tu, t) = \lim_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)$$
$$= \lim_{k\to\infty} \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)$$
$$> \lim_{k\to\infty} \min\{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k}, Tx_{n_k}, t), \mathcal{M}(x_{n_k+1}, x_{n_k+1}, Tx_{n_k+1}, t), \mathcal{M}(u, u, Tu, t)\}$$
$$= \lim_{k\to\infty} \min\{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t), \mathcal{M}(x_{n_k+1}, x_{n_k+1}, x_{n_k+2}, t), \mathcal{M}(u, u, Tu, t)\}$$
$$= \min\{\mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, Tu, t)\}$$
$$= \mathcal{M}(u, u, Tu, t)$$

which is contradiction.

Therefore $u = Tu$.

Uniqueness: Suppose there exists $v \in X$ such that $Tv = v$ and $v \neq u$.

Now consider $\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$

$$> \min\{\mathcal{M}(u, u, v, t), \mathcal{M}(u, u, Tu, t), \mathcal{M}(u, u, Tu, t), \mathcal{M}(v, v, Tv, t)\}$$
$$= \min\{\mathcal{M}(u, u, v, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(v, v, v, t)\}$$
$$= \mathcal{M}(u, u, v, t)$$

which is contradiction.

Therefore u is a unique fixed point of T.

Theorem: 2.3 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$– fuzzy metric space and $T: X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, Tx, Ty, t), \mathcal{M}(z, Ty, Tz, t), \mathcal{M}(x, y, Tx, t), \mathcal{M}(y, z, Ty, t)\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is a unique fixed point of T.

Proof: Let $x_0 \in X$ be any arbitrary fixed element in X.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots$

Suppose $x_n = x_{n+1}$ for some n. Then $x_n = Tx_n$. Thus $x_n = u$ is a fixed point of T.

Let us assume that $x_n \neq x_{n+1}$ for all n.

For $n \geq 1$, we have

$$\begin{aligned}
\mathcal{M}(x_n, x_n, x_{n+1}, t) &= \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)\\
&> \min \{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, Tx_{n-1}, Tx_{n-1}, t), \mathcal{M}(x_n, Tx_{n-1}, Tx_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t), \mathcal{M}(x_{n-1}, x_n, Tx_{n-1}, t)\}\\
&= \min \{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_n, x_n, t), \mathcal{M}(x_n, x_n, x_{n+1}, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_n, x_n, t)\}\\
&= \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t).
\end{aligned}$$

Hence $\mathcal{M}(x_n, x_n, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$ for all $n \geq 1$. Thus $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ is monotonically increasing sequence of positive real number bounded above by 1, it is convergent to a positive real number, say L.

Therefore $\lim_{n\to\infty} \mathcal{M}(x_n, x_n, x_{n+1}, t) = L$.

Also the sequence $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ has a subsequence $\{\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\}$ converges to L

That is $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$.

Now we prove that $L = 1$.

Suppose $L < 1$

Since $x_n = T^n x_0$ has a subsequence x_{n_k} converges to u

We have $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$ (2.3.1)

$$\begin{aligned}
\text{Now } 1>L &= \lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\\
&\geq \lim_{k\to\infty} \{\mathcal{M}(x_{n_k}, x_{n_k}, u, \tfrac{t}{2}) * (u, x_{n_k+1}, x_{n_k+1}, \tfrac{t}{2})\}\\
&= 1*1 \quad \text{using (2.3.1)}
\end{aligned}$$

which is contradiction.

Therefore $\lim_{k\to\infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$

Now we prove u is fixed point of T.

Suppose $u \neq T(u)$ we have

$$\begin{aligned}
\mathcal{M}(u, u, Tu, t) &= \lim_{k\to\infty} \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)\\
&= \lim_{k\to\infty} \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)\\
&> \lim_{k\to\infty} \min \{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, Tx_{n_k}, Tx_{n_k+1}, t), \mathcal{M}(u, Tx_{n_k+1}, Tu, t),\\
&\qquad \mathcal{M}(x_{n_k}, x_{n_k+1}, Tx_{n_k}, t), \mathcal{M}(x_{n_k+1}, u, Tx_{n_k+1}, t)\}\\
&= \lim_{k\to\infty} \min \{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k+1}, x_{n_k+2}, t), \mathcal{M}(u, x_{n_k+2}, Tu, t),\\
&\qquad \mathcal{M}(x_{n_k}, x_{n_k+1}, x_{n_k+1}, t), \mathcal{M}(x_{n_k+1}, u, x_{n_k+2}, t)\}\\
&= \min \{\mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, Tu, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t)\}\\
&= \mathcal{M}(u, u, Tu, t)
\end{aligned}$$

which is contradiction.

Therefore $u = Tu$.

Uniqueness: Suppose there exists $v \in X$ such that $Tv = v$ and $v \neq u$.

Now consider $\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$

$$> \min\{\mathcal{M}(u, u, v, t), \mathcal{M}(u, Tu, Tu, t), \mathcal{M}(v, Tu, Tv, t), \mathcal{M}(u, u, Tu, t), \mathcal{M}(u, v, Tu, t)\}$$
$$= \min\{\mathcal{M}(u, u, v, t), \mathcal{M}(u, u, u, t), \mathcal{M}(v, u, v, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, v, u, t)\}$$
$$= \mathcal{M}(u, u, v, t)$$

which is contradiction.

Therefore u is a unique fixed point of T.

Corollary: 2.4 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $T: X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, Tx, Ty, t), \mathcal{M}(z, Ty, Tz, t)\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is a unique fixed point of T.

Corollary: 2.5 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$ – fuzzy metric space and $T: X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, y, Tx, t), \mathcal{M}(y, z, Ty, t)\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is a unique fixed point of T.

Theorem: 2.6 Let $(X, \mathcal{M},*)$ be a $\mathcal{M}$ –fuzzy metric space with $t*t \geq t$ and $T: X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min\{\mathcal{M}(x, y, z, t), \mathcal{M}(x, y, Tx, t)* \mathcal{M}(y, z, Ty, t), \mathcal{M}(x, x, Tx, t) * \mathcal{M}(y, y, Ty, t)\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is a unique fixed point of T.

Proof: Let $x_0 \in X$ be any arbitrary fixed element in X.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2 \ldots$

Suppose $x_n = x_{n+1}$ for some n. Then $x_n = Tx_n$. Thus $x_n = u$ is a fixed point of T.

Let us assume that $x_n \neq x_{n+1}$ for all n.

For $n \geq 1$, we have

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) = \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)$$
$$> \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t) * \mathcal{M}(x_{n-1}, x_n, Tx_{n-1}, t), \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t)* \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t)\}$$
$$= \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t) * \mathcal{M}(x_{n-1}, x_n, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)* \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)\}$$
$$= \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t) * \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)* \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)\}$$
$$= \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t).$$

Hence $\mathcal{M}(x_n, x_n, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$ for all $n \geq 1$. Thus $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ is monotonically increasing sequence of positive real number bounded above by 1, it is convergent to a positive real number, say L.

Therefore $\lim_{n \to \infty} \mathcal{M}(x_n, x_n, x_{n+1}, t) = L$.

Also the sequence $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ has a subsequence $\{\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\}$ converges to L

That is $\lim_{k \to \infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$.

Now we prove that $L = 1$.

Suppose $L < 1$

Since $x_n = T^n x_0$ has a subsequence x_{n_k} converges to u

We have $lim_{k \to \infty}\ \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$ (2.6.1)

Now $1 > L = lim_{k \to \infty}\ \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)$

$$\geq lim_{k \to \infty} \{\mathcal{M}(x_{n_k}, x_{n_k}, u, \frac{t}{2}) * (u, x_{n_k+1}, x_{n_k+1}, \frac{t}{2})\}$$

$= 1*1$ using (2.6.1)

which is contradiction.

Therefore $lim_{k \to \infty}\ \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$

Now we prove u is fixed point of T.

Suppose $u \neq T(u)$ we have

$\mathcal{M}(u, u, Tu, t) = lim_{k \to \infty}\ \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)$

$$= lim_{k \to \infty}\ \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)$$

$$> lim_{k \to \infty} \min \{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k+1}, Tx_{n_k}, t) * \mathcal{M}(x_{n_k+1}, u, Tx_{n_k+1}, t),$$
$$\mathcal{M}(x_{n_k}, x_{n_k}, Tx_{n_k}, t) * \mathcal{M}(x_{n_k+1}, x_{n_k+1}, Tx_{n_k+1}, t)\}$$

$$= lim_{k \to \infty} \min \{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), \mathcal{M}(x_{n_k}, x_{n_k+1}, x_{n_k+1}, t) * \mathcal{M}(x_{n_k+1}, u, x_{n_k+2}, t),$$
$$\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) * \mathcal{M}(x_{n_k+1}, x_{n_k+1}, x_{n_k+2}, t)\}$$

$$= \min \{\mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t) * \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t) * \mathcal{M}(u, u, u, t)\}$$
$$= \mathcal{M}(u, u, u, t) = 1$$

Therefore $\mathcal{M}(u, u, Tu, t) = 1$, for each $t > 0$.

Hence $u = Tu$.

Uniqueness: Suppose there exists $v \in X$ such that $Tv = v$ and $v \neq u$.

Now consider $\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$

$$> \min \{\mathcal{M}(u, u, v, t), \mathcal{M}(u, u, Tu, t)*\mathcal{M}(u, v, Tu, t), \mathcal{M}(u, u, Tu, t)*\mathcal{M}(u, u, Tu, t)\}$$
$$= \min \{\mathcal{M}(u, u, v, t), \mathcal{M}(u, u, u, t) * \mathcal{M}(u, v, u, t), \mathcal{M}(u, u, u, t) * \mathcal{M}(u, u, u, t)\}$$
$$= \min \{\mathcal{M}(u, u, v, t), 1* \mathcal{M}(u, v, u, t), 1*1\}$$
$$= \min \{\mathcal{M}(u, u, v, t), \mathcal{M}(u, v, u, t), 1\}$$
$$= \mathcal{M}(u, u, v, t)$$

which is contraction. Therefore u is a unique fixed point of T.

Theorem: 2.7 Let $(X, \mathcal{M}, *)$ be a $\mathcal{M}$– fuzzy metric space and $T: X \to X$ be a mapping such that for all $x, y, z \in X$ and $t > 0$, with $x \neq y$ or $y \neq z$ or $z \neq x$

$$\mathcal{M}(Tx, Ty, Tz, t) > \min \{\mathcal{M}(x, y, z, t), ½[\mathcal{M}(x, y, Tx, t) + \mathcal{M}(y, z, Ty, t)], ½[\mathcal{M}(x, x, Tx, t) + \mathcal{M}(y, y, Ty, t)]\}.$$

For any point $x_0 \in X$ the sequence $\{T^n(x_0)\}$ has a subsequence converges to u. Then u is unique fixed point of T.

Proof: Let $x_0 \in X$ be any arbitrary fixed element in X.

Define a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \ldots$

Suppose $x_n = x_{n+1}$ for some n. Then $x_n = Tx_n$. Thus $x_n = u$ is a fixed point of T.

Let us assume that $x_n \neq x_{n+1}$ for all n.

For $n \geq 1$, we have

$$\mathcal{M}(x_n, x_n, x_{n+1}, t) = \mathcal{M}(Tx_{n-1}, Tx_{n-1}, Tx_n, t)$$

$$\begin{aligned}
&> \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), ½\,[\mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t) + \mathcal{M}(x_{n-1}, x_n, Tx_{n-1}, t)],\\
&\qquad ½\,[\mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t) + \mathcal{M}(x_{n-1}, x_{n-1}, Tx_{n-1}, t)]\}\\
&= \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), ½\,[\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t) + \mathcal{M}(x_{n-1}, x_n, x_n, t)],\\
&\qquad ½\,[\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t) + \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)]\}\\
&= \min\{\mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t), \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)\}\\
&= \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t).
\end{aligned}$$

Hence $\mathcal{M}(x_n, x_n, x_{n+1}, t) > \mathcal{M}(x_{n-1}, x_{n-1}, x_n, t)$ for all $n \geq 1$. Thus $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ is monotonically increasing sequence of positive real number bounded above by 1, it is convergent to a positive real number, say L.

Therefore $\lim_{n \to \infty} \mathcal{M}(x_n, x_n, x_{n+1}, t) = L$.

Also the sequence $\{\mathcal{M}(x_n, x_n, x_{n+1}, t)\}$ has a subsequence $\{\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\}$ converges to L.

That is $\lim_{k \to \infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = L$.

Now we prove that $L = 1$.

Suppose $L < 1$

Since $x_n = T^n x_0$ has a subsequence x_{n_k} converges to u.

We have $\lim_{k \to \infty} \mathcal{M}(x_{n_k}, x_{n_k}, u, t) = 1$ (2.7.1)

$$\begin{aligned}
\text{Now } 1 > L &= \lim_{k \to \infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t)\\
&\geq \lim_{k \to \infty} \{\mathcal{M}(x_{n_k}, x_{n_k}, u, \tfrac{t}{2}) * (u, x_{n_k+1}, x_{n_k+1}, \tfrac{t}{2})\}\\
&= 1*1 \quad \text{using (2.7.1)}
\end{aligned}$$

which is contradiction.

Therefore $\lim_{k \to \infty} \mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) = 1$

Now we prove u is fixed point of T.

Suppose $u \neq T(u)$ we have

$$\begin{aligned}
\mathcal{M}(u, u, Tu, t) &= \lim_{k \to \infty} \mathcal{M}(x_{n_k+1}, x_{n_k+2}, Tu, t)\\
&= \lim_{k \to \infty} \mathcal{M}(Tx_{n_k}, Tx_{n_k+1}, Tu, t)\\
&> \lim_{k \to \infty} \min\{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), ½[\mathcal{M}(x_{n_k}, x_{n_k+1}, Tx_{n_k}, t) + \mathcal{M}(x_{n_k+1}, u, Tx_{n_k+1}, t)],\\
&\qquad ½[\mathcal{M}(x_{n_k}, x_{n_k}, Tx_{n_k}, t) + \mathcal{M}(x_{n_k+1}, x_{n_k+1}, Tx_{n_k+1}, t)]\}\\
&= \lim_{k \to \infty} \min\{\mathcal{M}(x_{n_k}, x_{n_k+1}, u, t), ½[\mathcal{M}(x_{n_k}, x_{n_k+1}, x_{n_k+1}, t) + \mathcal{M}(x_{n_k+1}, u, x_{n_k+2}, t)],\\
&\qquad ½[\mathcal{M}(x_{n_k}, x_{n_k}, x_{n_k+1}, t) + \mathcal{M}(x_{n_k+1}, x_{n_k+1}, x_{n_k+2}, t)]\}\\
&= \min\{\mathcal{M}(u, u, u, t), ½[\mathcal{M}(u, u, u, t) + \mathcal{M}(u, u, u, t)], ½[\mathcal{M}(u, u, u, t) + \mathcal{M}(u, u, u, t)]\}\\
&= \min\{\mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t), \mathcal{M}(u, u, u, t)\}\\
&= \mathcal{M}(u, u, u, t) = 1
\end{aligned}$$

Therefore $\mathcal{M}(u, u, Tu, t) = 1$, for each $t > 0$.

Hence $u = Tu$.

Uniqueness: Suppose there exists $v \in X$ such that $Tv = v$ and $v \neq u$.

Now consider $\mathcal{M}(u, u, v, t) = \mathcal{M}(Tu, Tu, Tv, t)$

$$
\begin{aligned}
&> \min \{\mathcal{M}(u, u, v, t), ½ [\mathcal{M}(u, u, Tu, t) + \mathcal{M}(u, v, Tu, t)], ½ [\mathcal{M}(u, u, Tu, t) + \mathcal{M}(u, u, Tu, t)]\} \\
&= \min \{\mathcal{M}(u, u, v, t), ½[\mathcal{M}(u, u, u, t) + \mathcal{M}(u, v, u, t)], ½[\mathcal{M}(u, u, u, t) + \mathcal{M}(u, u, u, t)]\} \\
&= \min \{\mathcal{M}(u, u, v, t), ½ [1 + \mathcal{M}(u, u, v, t)], 1\} \\
&= \mathcal{M}(u, u, v, t), \text{ for all } t > 0
\end{aligned}
$$

Because, if $½ [1+\mathcal{M}(u, u, v, t)] < \mathcal{M}(u, u, v, t)$, then $\mathcal{M}(u, u, v, t) > 1$, which is contradiction.

Hence u is a unique fixed point of T.

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