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R-CLOSED SETS IN TOPOLOGICAL SPACES

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ABSTRACT

New classes of sets called R-closed sets and R_s-closed sets; R-open and R_s-open sets are introduced and study some of their properties. Moreover the notions of R-Continuity and R_s-continuity are introduced and study some of their properties.

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1. INTRODUCTION:

The study of generalized closed sets in a topological space was initiated by Levine [5] and the concept of $T_{1/2}$ space was introduced. In 1996, H. Maki, J. Umehara and T. Noiri[8] introduced the class of pregeneralized closed sets and used them to obtain properties of pre- $T_{1/2}$ spaces. The modified forms of generalized closed sets and generalized continuity were studied by K. Balachandran, P. Sundaram and H.Maki[2]. M.Sheik john introduced ω-closed sets and ω-open sets [12]. In this paper we introduced a new classes of sets called R-closed sets for topological spaces.

2. PRELIMINARIES:

Throughout this paper(X, τ),(Y, σ) and (Z, η) will always denote topological spaces on which no separation axioms are assumed, unless otherwise mentioned. When A is a subset of (X, τ), cl(A), Int(A) denote the closure, the interior of A. We recall some known definitions needed in this paper.

Definition: 2.1 Let (X, τ) be a topological space. A subset A of the space X is said to be
(1) Pre open [9] if A ⊆ Int (cl(A)) and preclosed if cl(Int(A)) ⊆ A.
(2) Semi open [6] if A ⊆ cl(Int(A)) and semiclosed if Int (cl(A)) ⊆ A.
(3) α -open [10] if A ⊆ Int (cl(Int(A))) and α-closed if cl(Int(cl(A)) ⊆ A.
(4) Semi preopen [1] if A ⊆ cl((Int(cl(A))) and semi preclosed if Int(cl(Int(A))) ⊆ A.
(5) Regular open [4] if A =Int (cl(A)) and regular closed if A = cl(Int(A)).

Lemma: 2.2 [1] For any subset A of X, the following relations hold.
1. Scl (A) = AU (Int(cl(A)))
2. αcl(A) = AUcl(Int(cl(A)))
3. Pcl (A) = AUcl(Int(A))
4. Spcl (A) = AU Int (cl(Int(A)))

Definition: 2.3 Let (X, τ) be a topological space.A subset A⊆X is said to be

(i) a generalized closed set [5] (briefly g-closed) if cl(A) ⊆U whenever A⊆U and U is open in (X,τ);the complement of a g- closed set is called a g-open set.

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(ii) an α-generalized closed set [7] (briefly αg-closed) if αcl(A) ⊆U whenever A⊆U and U is open in (X, τ); the complement of a αg- closed set is called a αg-open set.

(iii) a generalized semi preclosed set [3] (briefly gsp-closed) if spcl(A) ⊆U whenever A⊆U and U is open in (X,τ); the complement of a gsp- closed set is called a gsp-open set.

(iv) an ω-closed set [12] if cl(A) ⊆U whenever A⊆U and U is semi open in (X, τ);the complement of a ω-closed set is called an ω-open set.

(v) a generalized preclosed set [11] (briefly gp-closed) if pcl(A) ⊆U whenever A⊆U and U is open in (X, τ); the complement of a gp- closed set is called a gp-open set.

(vi) a $g\alpha^*$ closed set [13] if αcl(A) ⊆IntU whenever A⊆U and U is α-open in (X,τ);the complement of a $g\alpha^*$ closed set is called a $g\alpha^*$ open set.

(vii) a generalized pre regular closed set [4] (briefly gpR-closed) if pcl(A) ⊆U whenever A⊆U and U is regular open in (X, τ);the complement of a gpr- closed set is called a gpr-open set.

Definition: 2.4 A function f :(X, τ)→(Y,σ) is said to be g-continuous[2] if $f^{-1}(V)$ is g-closed in (X, τ) for every closed set V of (Y, σ).

3. BASIC PROPERTIES OF R-CLOSED SETS:

We introduce the following definition.

Definition: 3.1 A subset A of a topological space (X, τ) is said to be R-closed in (X, τ) if αcl(A)⊆Int(U) whenever A⊆U and U is ω-open in (X,τ).

Theorem: 3.2 Every open and α closed subset of (X, τ) is R-closed but not conversely.

Proof: Let A be open and α-closed subset of (X, τ).

Let A ⊆ U and U be ω-open in x.

Since A is α-closed, αcl(A)=A.

∴ αcl(A) = A= Int(A) (∵ A is open)
⊆ Int (U) (∵ A ⊆ U)

∴ αcl(A) ⊆ Int(U)

∴ A is R-closed.

Conversely,

R-closed sets need not be open and α-closed.

(i) Let X= {a, b, c, d}, τ={X, ø, {a, b}}

Here A = {a} is R- closed and α-closed but A is not open.

ii)Let X={a, b, c} τ={X, ø,{c}}.Here A={a, c} is not open and not α-closed But A={a, c} is R-closed.

Theorem: 3.3 Every R-closed set is αg closed but not conversely.

Proof: Let A is R-closed.

Let A ⊆ U, U is open ⇒ U is ω-open.

(ie) A ⊆ U,U is ω-open. ∴ A is R-closed αcl(A) ⊆ Int(U)

∴ αcl(A) ⊆ U. ∴ A is αg-closed.

The converse is not true.

(ie) every αg-closed set need not be R-closed.

Let X= {a, b, c} τ={X, ø,{a},{b, c}}

Here A= {b} is αg-closed.but it is not R-closed.

Theorem: 3.4 Every R-closed set is gα* closed but not conversely.

Proof: Let A ⊆ U, U is α-open ⇒ U is ω-open

∴ αcl(A) ⊆ Int(U)

∴ A is gα* closed.

Conversly, every gα* closed set need not be R-closed.

Let X= {a, b, c},τ={X, ø,{c},{a, b}}

Let A= {a, c}.A is gα* closed but not R-closed

Theorem: 3.5 Every R-closed set is gsp closed but not conversely.

Proof: Every R-closed set is αg-closed and every αg-closed set is gsp closed.

⇒ Every R-closed set is gsp closed.

Since every αg-closed set is not R-closed, we conclude the converse part is not true.

For eg Let X= {a, b, c} τ={X, ø,{a, c}}

Let A= {a}. Spcl{a}={a} ⊆ {a, c}

∵ A ⊆ U ⇒ spcl(A) ⊆ U

∴ A is gsp closed but not R-closed.

Theorem: 3.6 Every R-closed set is gp closed but not conversely.

Proof:

Let A be R-closed.

Let A ⊆ U, U is open ⇒ U is ω-open.

∵ αcl(A) ⊆ Int(U) ⊆ U

∵ pcl(A) ⊆ U ⇒ A is gp closed.

Conversely every gp closed set need not be R-closed

Let X= {a, b, c}. τ={X, ø, {a, b}}

Let A = {a} Here A is gp closed but not R-closed.

Theorem: 3.7 Every R-closed set is gpr closed.

Proof: Let A be R-closed

Let A ⊆ U, U is regular open ⇒ U is open
⇒ U is ω-open.

∴αcl(A) ⊆ Int(U) ⊆ U

∴Pcl(A) ⊆ U

⇒ A is gpr closed.

Conversely every gpr closed sets need not be R-closed.

Let X= {a, b, c}, τ={X, ø,{a, b}} clearly {a, b}is regular open. Let A= {a} Here A is gpr closed but A is not R-closed.

Theorem: 3.8 Every R-closed set is gp closed but not conversely

Proof: Let A ⊆ U, U is open ⇒ U is ω-open

∴αcl(A) ⊆ Int(U) ⊆ U

∴Pcl(A) ⊆ U.

Converse need not be true.

Let X= {a, b, c} τ={X, ø,{a,}}. Let A= {a}.

Here {a} is gp closed but not R-closed.

Theorem: 3.9 A subset A of a topological space(X, τ)is R-closed then αcl(A) contains no non ø ω-closed set.

Proof: Let F be a non ø ω-closed set such that F ⊆ αcl(A)-A

Then F⊆ αcl(A) and A ⊆ x-F, is ω-open.

Since A is R closed, αcl(A) ⇒ Int(X-F) = X-cl(F)

⇒ cl(F) ⊆ X- αcl(A).

(ie) F ⊆ αcl(A) and cl(F) ⊆ X- αcl(A).

(ie) F ⊆ αcl(A) and F ⊆ X- αcl(A).

(ie) F ⊆ αcl(A) ∩(X- αcl(A))= ø

⇒ αcl(A) –A contains no non ø ω-closed set.

Conversely if αcl(A) –A contains no non ø ω-closed set, then A need not be R-closed.

Let X= {a, b ,c} τ={X, ø,{a, b}}. Let A= {a}

Here A= {a} is not R-closed. But αcl(A) –A =X-{a} = {b, c}

Theorem: 3.10 If A and B are R-closed then AUB is R-closed

Proof: Let A and B are R-closed sets.

Let AUB ⊆ U, U be ω-open.

∴αcl(A) ⊆ Int(U), αcl(B) ⊆ Int(U)(∵A and B are α closed)

∴αcl(AUB)= αcl(A) U αcl(B) ⊆ Int(U).

⇒ AUB is R-closed.

Remark: 3.11 The intersection of two R- closed sets need not be R closed.

Let X= {a, b, c} τ ={X, ø,{a,}}

Here {a, b} and {a, c} are R-closed, but {a, b} ∩ {a, c}={a} is not R-closed.

Theorem: 3.12 If A is R-closed and A ⊆ B ⊆ αcl(A) then B is R-closed.

Proof: Let U be ω-open set of X s.t B ⊆ U. Let A ⊆ B ⊆ αcl(A)

∴A ⊆ U and U is ω-open.

∴αcl(A) ⊆ Int(U).

Also B ⊆ αcl(A) ⇒ αcl(B)⊆ αcl(αcl(A))= αcl(A)⊆ Int(U)

∴αcl(B) ⊆ Int(U). ∴ B is R-closed.

Theorem: 3.13 If a subset A of (X, τ) is ω-open and R-closed then A is α-closed in (X, τ)

Proof: Let A be ω-open and R-closed.

Then αcl(A)⊆ Int(A)⊆ A

∴αcl(A) ⊆ A

∴A is α-closed.

Theorem: 3.14 Let A be R-closed in (X, τ) then A is α-closed in (X, τ) iff αcl(A)-A is ω-closed.

Proof: Given A is R-closed.

(⇒)

Let A be α-closed. ∴αcl(A)=A

(ie) αcl(A)-A= ø which is ω-closed.

(⇐)

If αcl(A)-A is ω-closed,since A is R-closed, αcl(A)-A does not contain any non ø ω-closed set.

∴αcl(A)-A= ø

∴αcl(A) ⊆ A

(ie) A is α-closed.

Theorem: 3.15 An open set A of (X, τ) is αg-closed iff A is R-closed.

Proof:

(⇒)

Let A be an open and αg closed set.

Let A ⊆ U,U is ω-open

Since A ⊆ U, Int(A) ⊆ Int(U)

∴A ⊆ Int(U) which is open.

∴ αcl(A) ⊆ Int (U)(∵ A is αg cld)

⇒ A is R-closed.
(⇐)

Let A be R-closed.

Let A ⊆ U, U is open ⇒ U is ω-open.

∴ αcl(A) ⊆ Int(U) ⊆ U

∴ αcl(A) ⊆ U

⇒ A is αg –closed.

Theorem: 3.16 In a topological space X, for each x∈X, {x} is ω-closed or its complement X-{x} is R-closed in (X, τ)

Proof: Let (X, τ) be a topology.

To prove{x} is ω-closed or X-{x} is R-closed in (x, τ).

If {x} is not ω-closed in (X, τ) then X-{x} is not ω-open and the only ω-open set containing X-{x} is X.

∴ αcl{X-{x}} ⊆ X=Int (X)

∴ αcl{X-{x}} ⊆ Int (X) ⇒ X-{x} is R-closed.

Remark: 3.17 Closedness and R-closedness are independent.

Let X= {a, b, c}, τ={X, ø, {b}, {a, b}}

A= {b, c} is R-closed but not closed.

Let X= {a, b, c}, τ={X, ø, {b}, {c}, {b, c}}

A= {a} is closed but not R-closed.

Remark: 3.18 R-closedness and g-closedness are independent.

Let X= {a, b, c} τ={X, ø,{a},{a, b}}

A= {b} is R-closed but not g-closed.

Let X= {a, b, c} τ ={X, ø, {a},{a, c}}

A= {b} is g-closed but not R-closed.

Remark: 3.19 R-closedness and preclosedness are independent

Let X= {a, b, c} τ={X, ø, {a}}

Here A = {a, b} is R-closed but not preclosed.

Let X= {a, b, c} τ={X, ø, {a, b}}

Here A = {a} is preclosed but not R-closed.

∴ R-closedness and preclosedness are independent.

Remark: 3.20 R-closedness and α closedness are independent.

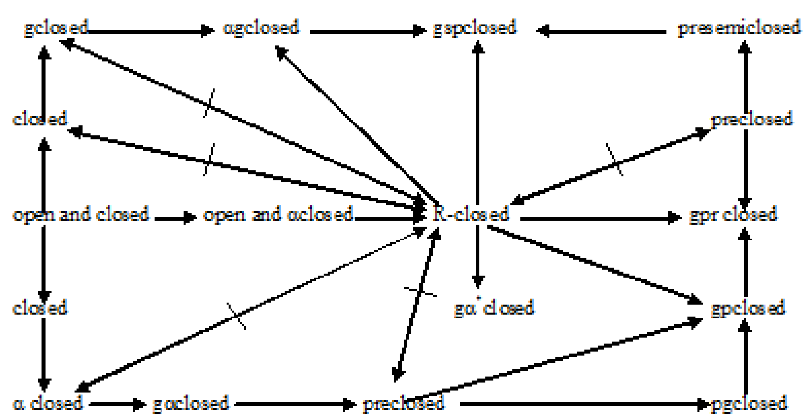
Let X= {a, b, c}, τ={X, ø, {a}}

Here A= {a, b} is R-closed but not α-closed.

Let X= {a, b, c}, τ={X, ø, {b, c}}

Here A= {a} is α-closed but not R-closed.

Remark: 3.21 From the above discussions and known results we have the following implications. A⟶B (A↮B) reperesents A implies B but not conversely (A and B are independent of each other).



4. PROPERTIES OF R-CLOSED SETS:

Definition: 4.1 The Intersection of all ω-open Subsets of (X, τ) containing A is called the ω-Kernal of A and denoted by ω-Ker (A)

Theorem: 4.2 If a subset A of (X, τ) is R-closed then αcl(A) ⊆ ω-Ker(A).

Proof: Let A be R-closed. Let A ⊆ U and U is ω-open.

∴αcl(A) ⊆ Int(U) Whenever A ⊆ U ,U is ω-open.

Let x ∈ αcl(A)

If x∉ ω-Ker(A), then there exists a ω-open set containing A subset x∉U.

∴x∉A ⇒ x∉ αcl(A)

⇒ ⇐ to x∈ αcl(A)

∴αcl(A) ⊆ ω-Ker(A)

Definition: 4.3 A subset A of (X, τ) is said to be R_s-closed in (X, τ) if αcl (A) ⊆ Int (cl(U)) Whenever A ⊆ U and U is ω-open in (X, τ)

Theorem: 4.4 Every R-closed set is R_s-closed.

Proof: Let A be any R-closed set.

Let A ⊆ U and U be ω-open in X.

⇒ αcl(A) ⊆ Int U ⊆ Int (cl(U))

⇒ A is R_s-closed.

5. R-OPEN AND R_S OPEN SETS:

Definition: 5.1 A subset A of (X, τ) is said to be R-open in (X, τ) if its complement X-A is R-closed in(X, τ).

Definition 5.2: A subset A of (X, τ) is said to be R_s-open in (X, τ) if its complment X-A is R_s-closed in(X, τ)

Theorem: 5.3 Let (X, τ) be a topological space and A $\subseteq$ X.

(i) A is a R-open set iff cl (U)$\subseteq$ αint(A) whenever U $\subseteq$ A and U is ω-closed.
(ii) A is R_s-open set iff cl(Int(U)) $\subseteq$ αint (A) whenever U $\subseteq$ A and U is ω-closed.
(iii) If A is R-open then A is R_s-open.

Proof: ($\Rightarrow$) Let A be an R-open set in (X, τ)

Let U $\subseteq$ A and U is ω-closed.

Then X-A is R-closed and X-A $\subseteq$ X-U and X-U is ω-open.

∴αcl(X-A) $\subseteq$ Int (X-U)

x-αint(A) $\subseteq$ X-cl(U)

∴cl(U) $\subseteq$ αint(A)

($\Leftarrow$) whenever U $\subseteq$ A and U is ω-closed then cl(Int(U)) $\subseteq$ αint(A)

(i) Let A $\subseteq$V and V is ω-closed

A $\subseteq$ V $\Rightarrow$ X-A$\supseteq$X-V which is ω-open.

∴cl(X-V) $\subseteq$ αint(X-A)

X-int(V) $\subseteq$ X-αcl(A)

∴αcl(A) $\subseteq$ Int(V)

∴A is R-closed.

(ii) Let A be an R_s open set.

Let F $\subseteq$ A and F is ω-closed

∴X-A is R_s-closed and X-F is ω-open subset such that X-A $\subseteq$ X-F

∴αcl(X-A) $\subseteq$ Int (cl(X-F))

(ie)X-αint(A) $\subseteq$ Int (X-int(F))

$\Rightarrow$ X-αint(A) $\subseteq$ X-cl(Int(F))

(ie) cl(Int(F)) $\subseteq$ αint(A)

(iii)Let A be R open.

To prove A is R_s open.

Let K $\subseteq$ A and K is ω-closed.

$\Rightarrow$ cl(K) $\subseteq$ αint(A)

(ie) cl(int(K)) $\subseteq$ cl(K) $\subseteq$ αint(A)

$\Rightarrow$ A is R_s-open.

6. R-CONTINUITY AND R_S-CONTINUITY:

Let f: (X, τ) $\rightarrow$ (Y, σ)be a function from a topological space (X, τ) into a topological space(Y, σ).

Definition: 6.1 A function f: (X, τ) $\rightarrow$ (Y, σ) is said to be R-Continuous (respectively R_s-Continuous) if f^{-1}(V) is R-closed (respectively R_s-closed) in (X, τ) for every closed set V of (Y, σ).

Definition: 6.2 A function f: (X, τ) $\rightarrow$ (Y, σ) is said to be R-irresolute (respectively R_s irresolute) if f^{-1}(v) is R-closed (respectively R_s-closed)set V of (Y, σ)

Example for R-continuous mapping.

Let x= {a, b, c}, τ={X, ø,{a},{a, b}}, σ={X, ø,{a},{a, c}}

Let f :(X, τ) $\rightarrow$ (X, σ) is defined by f (a) =a, f (b) =c, f(c) = b.

Then 'f' is R-continuous.

Example for R-irresolute mapping.

Let X= {a, b, c} τ={X, ø, {a}, {a, b}{a, c}}, σ={X, ø,{b},{a, b}{b, c}}

Let f :(X, τ) $\rightarrow$ (X, σ) defined by, f (a) =b, f (b) =c, f(c) =a

Then 'f' is R-irresolute.

Remark: 6.3 The composition of two R-continuous functions need not be R-continuous.

Let X= {a, b, c}   τ ={X, ø,{a},{a, b}{a, c}}, σ ={X, ø,{a},{a, b}},η ={x, ø,{a, b}}

Let f: (X, τ) $\rightarrow$ (X, σ) defined by f (a) =b, f (b) =c, f(c) = a.

Let g: (x, σ) $\rightarrow$ (x, η) defined by g(a)=b, g(b) = c, g(c) = a.

clearly f and g are R-continuous.

But $(fog)^{-1}$(c) =f^{-1}(b) = a which is not R-closed.

$\therefore$fog is not R-continuous.

Theorem: 6.4 Let f:(X, τ) $\rightarrow$ (Y, σ) and g: (Y, σ) $\rightarrow$ (Z, η) be two functions. Then
(i) gof is R-continuous if g is continuous and f is R-continuous
(ii) gof is R-irresolute if g is R-irresolute and f is R-irresolute
(iii) gof is R-continuous if g is R-continuous and f is R-irresolute

Proof: (i) Let V be closed in(Z, η)

$\Rightarrow$ g^{-1}(v) is closed in(Y, σ)

$\Rightarrow$ f^{-1}(g^{-1}(v)=$(gof)^{-1}$(v) is R-closed in(X, τ)

(ii)Let v be R-closed in (Z, η)

$\Rightarrow$ g^{-1}(v) is R-closed in (Y, σ)

$\Rightarrow$ $f^{-1}(g^{-1}(v))=(gof)^{-1}(v)$ is R-closed in (X, τ)

(iii)Let V be closed in (Z, η)

Since g is R-continuous ,$g^{-1}(v)$ is R-closed in (Y, σ)

$\Rightarrow$ $f^{-1}(g^{-1}(v))=(gof)^{-1}(v)$ is R-closed in (X, τ)

∴gof is R-continuous.

Definition: 6.5 A space x is called an αω space if the intersection of α closed set with a ω-closed set is ω-closed.

Theorem: 6.6 For a subset A of an αω-space (X, τ) the following are equivalent.

(1) A is R-closed
(2) cl{x}∩A≠ ø for each x ∉ αcl(A)
(3) αcl(A)-A contains no non ø ω-closed set.

Proof: (1) Let A be R-closed.

Let x∈αcl(A).If cl{x}∩A= ø then A ⊆ X-cl{x}, X-cl{x} is open and hence X-cl{x} is ω-open

Let U=X-cl{x}

(ie) A ⊆ U, U is ω open $\Rightarrow$ αcl(A) ⊆ Int(U)

(ie) αcl(A) ⊆ Int(X-cl{x}) =X-cl(cl{x}) =X-cl{x}

(ie) αcl(A) ⊆ X-cl{x}

Since x∈ αcl(A),x∈ X-cl{x} which is not possible

∴cl{x}∩A≠ ø

(2) If cl{x}∩A≠ ø for x∈αcl(A),to prove αcl(A)-A contains no non ø ω-closed set.

Let us assume αcl(A)-A contains no non ø ω-closed set.

Let K ⊆ αcl(A)-A is a non ø ω-closed set

Then K ⊆ αcl(A) and A ⊆ X-K

Let x∈K then x∈αcl(A) then by (ii),cl{x}∩A≠ ø

cl{x}∩A ⊆ K∩A ⊆ (αcl(A) –A) ∩ A

Which is a ⇒⇐

Hence αcl(A)-Acontains no non ø ω-closed sets.

(3) If αcl(A)-A contains no non ø ω-closed set.

Le A ⊆ U,Uis ω-open.

If αcl(A) ⊈Int (U) then αcl(A) ∩ $(int(U))^c$ = ø

Since the space is a αω space,

αcl(A) ∩ $(int (U))^c$ is a non ø ω-closed subset of αcl(A)-Awhich is a $\Rightarrow$ $\Leftarrow$

∴A is R-closed

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