



MODELING THE EFFECT OF HARVESTING OF THE VEGETATION BIOMASS AND
GRAZER POPULATION ON PREDATOR POPULATION
WITH HABITAT COMPLEXITY

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ABSTRACT

In this paper, a nonlinear mathematical model is proposed and analyzed to see the effect of harvesting of vegetation biomass and grazer population on the predator population. The model is formulated as a food chain model by considering the hyperbolic interaction processes. It is assumed that the dynamics of vegetation biomass follows regrowth equation. We also study the effect of the degree of habitat complexity and gestation delay on the stability of a food chain model. It is observed that there are stability switches, and Hopf bifurcation occurs when the delay crosses some critical value. It is observed that the quantitative level of abundance of system populations depends crucially on the delay parameter if the gestation period exceeds some critical value. However, the fluctuations in the population levels can be controlled completely by increasing the degree of habitat complexity.

Keywords: *Vegetation biomass, grazer population, predator population, Harvesting, Habitat complexity, Limit cycle, Stability of Hopf-bifurcation*

Mathematical Subject Classification: 92B05

1. INTRODUCTION:

In aquatic communities trophic interactions regulate the stability and diversity of communities in space and time. Variation in spatial structure influences the outcome of interactions such as predation or resource exploitation (Schindler et al., [13]; Nurminen et al., [12]). Macrophytes, pebbles and detritus are some of the components that add to the structural complexity of habitats. Such complexity adds to the available refuge and thus influences prey–predator interactions (Diehl, [6]; Babbitt and Jordan, [3]). Further, these structures, and Macrophytes in particular, organize the assemblages of different invertebrate species (De Szalay and Resh, [5]) and mediate food web dynamics. Thus heterogeneity may increase the species' diversity and stabilize predator–prey interactions (Gilinsky, [7]), evident from the studies on predation by fishes on chironomid (Diehl, [6]) and mosquito (Hurst et al., [10]) larvae. Structural complexity influences predation by odonate nymphs and hemipteran bugs on tadpoles (Babbitt and Jordan,[3]; Kopp et al.,[11]) as well as spatial orientation of Notonecta and its intraguild prey Buena (Hampton, [8]). The interaction of multiple predators, predation rate and prey selection are also modified by the habitat complexity (Hughes and Grabowski, [9]). For aquatic predatory insects, ricefields, ponds, and temporary pools are habitats with ample heterogeneity with regard to spatial structures and prey species abundance (Bambaradeniya et al., [4]). When ample prey species are available structural complexity is more important in determining predator success.

The theory of harvesting is important in natural resource management and bioeconomics. Most species have a growth rate which more or less maintains a constant population equal to the carrying capacity of the environment K (this of course depend on the population). In this case the growth and death rates are nearly equal. The harvesting of species affects their mortality rates and if the harvesting is not too much the population will adjust to a new equilibrium $N^* < K$. It has been evident that there is need to develop ecologically acceptable strategies for harvesting any renewable resources such as fish, plants, animals etc. It is interesting to note that even if the excess harvest does not threaten extinction, it can cause damage to the resource in the long run. Massive fruit collection from the forest has an adverse effect on regeneration. The problem then is to determine a strategy which ensures steady harvest year after year without a progressive decline in the abundance of the resource. The problem here is how to maximize the sustainable yield (SY) by determining the population growth dynamics so as to obtain a harvesting rate which keeps the population at its maximum growth. We consider a logistic population growth model in which the mortality rate is enhanced by harvesting; by a term that is proportional to the existing population N . We consider a logistic growth model

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$$F(N) = rN\left(1 - \frac{N}{K}\right)$$

Where r is the intrinsic growth rate and K the carrying capacity of the environment. Considering the effort E of harvesting, we assume that the harvest is proportional to the stock level as well as the effort i.e. $h = qEN$ where q is the constant of proportionality called the catchability constant. The effort is measured in man days. If grass is cut with strokes of a sickle, the harvest depends on the number of strokes E on the grass density. If the effort is constant then the harvest as a function of the stock is a straight line passing through the origin. The intersection of the line or the growth curve gives the sustainable yield. The net growth rate after harvest is given by

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - qEN$$

There are numerous studies on the effects of harvesting on population growth. In the context of predator-prey interaction, some studies that treat the populations being harvested as a homogeneous resource include those of Dai and Tang [25], Myerscough et al. [26], Chaudhuri[27] and Leung[28]. In the last few years, researchers have been showing keen interest to investigate the direction and stability of Hopf bifurcation arising from a delay-induced neural network [14-17]. Study of the bifurcation and its stability, however, is relatively new in ecological system. Recently, Song and Wei [18], Yang and Tian [19], Qu and Wei [20], Celik [21,22], have studied the direction and stability of Hopf bifurcation of delay-induced ecological systems. In most of the studies, delay occurs in a first degree term. Sun et al. [23] investigated the direction and stability of a delay-induced eco-epidemiological system with Type I response function [24], where delay occurred in the term of degree two. In this study, we first modify Holling type II response function [24] to incorporate the effect of habitat complexity and then put delay in the second degree term. The objective is to study the interplay between gestation delay and the degree of habitat complexity.

The rest of this paper is organized as follows: In section 2, we introduce our mathematical model. In sections, 3 and 4 our model is analyzed with regard to equilibria and their stabilities, respectively. In section 5, we study the effect of delay on the model. In section 6, we present a numerical example to illustrate the applicability of results obtained and also investigate the occurrence of Hopf bifurcation in absence as well as in the presence of delay for certain set of parameters. We conclude with a short discussion in section 7.

2. THE MATHEMATICAL MODEL:

We first consider the instantaneous model given by the following system of autonomous ordinary differential equations as a food chain model with habitat complexity.

$$\begin{aligned} \frac{dV}{dt} &= u_0 \left(1 - \frac{V}{m}\right) - \frac{a_1VN}{b_1 + V} - q_1EV, \\ \frac{dN}{dt} &= \xi N \left(\frac{a_1V}{b_1 + V} - \eta\right) - \frac{a_2(1-c)NP}{1 + a_2(1-c)hN} - q_2EN, \\ \frac{dP}{dt} &= \theta P \left(\frac{a_2(1-c)N}{1 + a_2(1-c)hN} - \mu\right). \end{aligned} \tag{2.1}$$

$$V(0) = V_0 \geq 0, \quad N(0) = N_0 \geq 0, \quad P(0) = P_0 \geq 0,$$

Where, V be the vegetation biomass, N be the density of grazer population and P be the density of predator population. It is assumed that the dynamics of vegetation biomass follows regrowth equation and its depletion due to grazer population is given by a hyperbolic type of interaction involving the density of grazer population N as well as the concentration V (i.e. $VN/(b_1 + V)$). It is further assumed that the rate of depletion of V due to harvesting is proportional to the product of vegetation biomass and applied effort with catchability coefficient q_1 . As grazer population wholly depend upon vegetation biomass, its growth rate is proportional to the interaction term ($a_1VN/(b_1 + V)$). It is considered that the natural depletion rate of density of grazer population is proportional to N . It is further assumed that the depletion rate of grazer population density by its predator is given by the hyperbolic interaction between grazers and its predators(i.e. $a_2(1-c)NP/1 + a_2(1-c)hN$). The depletion rate of grazer population due to harvesting is proportional to the product of concentration of N and applied effort with catchability

coefficient q_2 . The growth rate of predator population is proportional to the interaction term (i.e. $a_2(1-c)NP/1+a_2(1-c)hN$). The natural depletion rate of predator population is considered proportional to P . u_0 is initial regrowth rate of the vegetation biomass, m is the carrying capacity of vegetation biomass, a_1 is the attack rate of grazers, b_1 is the half-saturation constant, ξ is the vegetation-grazer conversion rate, η is the zero population growth grazer intake, a_2 is the saturation killing rate (the maximum killing rate), θ is the prey (grazer)-predator conversion rate, μ is the zero population growth predator intake, h is the handling time and c is a dimensionless parameter that measures the degree or strength of habitat complexity.

The coefficients ξ and θ are positive constants and their magnitudes are less than equal to unity. c is also positive constant but its magnitude is less than unity. It is to be noted, when $c = 0$, i.e. when there is no complexity, we get back the original Holling Type II response function. Therefore, this modified functional response would be suitable for predator-prey interaction with habitat complexity. It may be pointed out that for feasibility of model (2.1), the growth rates of grazer population should be positive. Hence, from the second and third equations of model (2.1), it follows that

$$a_1 > \eta \tag{2.2}$$

and

$$a_2 > \mu \tag{2.3}$$

To analyze the model (2.1), we need the bounds of dependent variables involved. For this we find the region of attraction in the following lemma.

Lemma 2.1: The set

$$\Omega = \left\{ (V, N, P) : 0 \leq V + N + P \leq \frac{u_0}{\alpha} \right\},$$

where $\alpha = \min \left\{ \frac{u_0}{m} + q_1 E, q_2 E + \xi \eta, \theta \mu \right\}$ is the region of attraction for all solutions initiating in the interior of the positive octant.

Proof: Let $(V(t), N(t), P(t))$ be any solution with positive initial conditions (V_0, N_0, P_0) . Define a function

$$W(t) = V(t) + N(t) + P(t).$$

Computing the time derivative of $W(t)$ along solutions of system (2.1), we get

$$\begin{aligned} \frac{dW}{dt} &= u_0 \left(1 - \frac{V}{m} \right) - \frac{a_1 V N}{b_1 + V} - q_1 E V + \xi N \left(\frac{a_1 V}{b_1 + V} - \eta \right) - \frac{a_2 (1-c) NP}{1 + a_2 (1-c) h N} - q_2 E N + \frac{\theta a_2 (1-c) NP}{1 + a_2 (1-c) h N} - \theta P \mu \\ &\leq u_0 - \left(\frac{u_0}{m} + q_1 E \right) V - (q_2 E + \xi \eta) N - \theta P \mu \\ &\leq u_0 - \alpha W, \end{aligned}$$

where $\alpha = \min \left\{ \frac{u_0}{m} + q_1 E, q_2 E + \xi \eta, \theta \mu \right\}$.

Thus $W'(t) + \alpha W(t) \leq u_0$.

Applying a theorem in differential inequalities [2], we obtain

$$0 \leq W(V, N, P) \leq \frac{u_0}{\alpha} + \frac{W(V_0, N_0, P_0)}{\exp(\alpha t)},$$

and for any $t \rightarrow \infty$, $0 \leq W \leq \frac{u_0}{\alpha}$. Therefore all solutions of system (2.1) enter into the region

$$\Omega = \left\{ (V, N, P) : 0 \leq W \leq \frac{u_0}{\alpha} \right\}.$$

This completes the proof of lemma.

3. EQUILIBRIUM ANALYSIS:

There exist following three equilibria of the system (2.1)

- (i) $E_1(V_1, 0, 0)$, where $V_1 = \frac{mu_0}{u_0 + mq_1E} > 0$,
- (ii) $E_2(V_2, N_2, 0)$, and
- (iii) $E_3(V^*, N^*, P^*)$.

The equilibrium E_2 exists provided that

$$\xi a_1 - (\xi \eta + q_2 E) \left(1 + \frac{b_1}{m} + \frac{b_1 q_1 E}{u_0} \right) > 0,$$

where

$$V_2 = \frac{b_1 (\xi \eta + q_2 E)}{\xi a_1 - (\xi \eta + q_2 E)}, \quad N_2 = \frac{u_0 \xi \left[\xi a_1 - (\xi \eta + q_2 E) \left(1 + \frac{b_1}{m} + \frac{b_1 q_1 E}{u_0} \right) \right]}{[\xi a_1 - (\xi \eta + q_2 E)] (\xi \eta + q_2 E)}.$$

The equilibrium E_3^* exists provided that

$$c < 1, \mu h < 1, \quad \frac{\xi a_1 V^*}{b_1 + V^*} > \xi \eta + q_2 E,$$

where

$$N^* = \frac{\mu}{a_2(1-c)(1-\mu h)} > 0, \quad P^* = \frac{1}{a_2(1-c)(1-\mu h)} \left[\xi \left(\frac{a_1 V^*}{b_1 + V^*} - \eta \right) - q_2 E \right],$$

and V^* is the unique positive root of the following equation

$$a_2(1-c)(1-\mu h)(u_0 + mq_1E)V^2 + [(u_0 b_1 + mq_1 b_1 E - u_0 m)(a_2(1-c)(1-\mu h)) + a_1 m \mu]V - u_0 m b_1 a_2(1-c)(1-\mu h) = 0.$$

4. STABILITY ANALYSIS:

To discuss the local stability of system (2.1), we compute the variational matrix of system (2.1). The sign of the real parts of the eigenvalues of the variational matrix evaluated at a given equilibria determine its stability. The entries of general variational matrix are given by differentiating the right side of system (2.1) with respect to V , N and P i.e.

$$M(V, N, P) = \begin{bmatrix} \frac{u_0}{m} - q_1 E - \frac{a_1 b_1 N}{(b_1 + V)^2} & -\frac{a_1 V}{b_1 + V} & 0 \\ \frac{\xi a_1 b_1 N}{(b_1 + V)^2} & \xi \left(\frac{a_1 V}{b_1 + V} - \eta \right) - \frac{a_2(1-c)P}{(1+a_2(1-c)hN)^2} - q_2 E & -\frac{a_2(1-c)N}{1+a_2(1-c)hN} \\ 0 & \frac{a_2(1-c)P}{(1+a_2(1-c)hN)^2} & \theta \left(\frac{a_2(1-c)N}{1+a_2(1-c)hN} - \mu \right) \end{bmatrix}$$

Using analogous notations to the equilibria (i.e. $M(E_1)$ is the variational matrix corresponding to E_1 , $M(E_2)$ is the variational matrix corresponding to E_2 and $M(E_3)$ is the variational matrix corresponding to E_3 , we get

$$M(E_1) = \begin{bmatrix} -\frac{u_0}{m} - q_1 E & -\frac{a_1 V_1}{b_1 + V_1} & 0 \\ 0 & \xi \left(\frac{a_1 V_1}{b_1 + V_1} - q_2 E \right) & 0 \\ 0 & 0 & -\theta \mu \end{bmatrix}.$$

The three eigenvalues of $M(E_1)$ are $-\left(\frac{u_0}{m} + q_1 E\right)$, $\xi \left(\frac{a_1 V_1}{b_1 + V_1} - \eta \right) - q_2 E$ and $-\theta \mu$. This implies E_1 is locally stable in the $V - P$ plane. As to the N - direction, it is locally unstable if $\xi \left(\frac{a_1 V_1}{b_1 + V_1} - \eta \right) - q_2 E$ is positive (which is the necessary condition for the existence of E_2) whenever E_2 exists and stable if $\xi \left(\frac{a_1 V_1}{b_1 + V_1} - \eta \right) - q_2 E$ is negative. Hence E_1 is unstable saddle point whenever E_2 exists otherwise it is locally asymptotically stable.

$$M(E_2) = \begin{bmatrix} -\frac{u_0}{m} - q_1 E - \frac{a_1 b_1 N_2}{(b_1 + V_2)^2} & -\frac{a_1 V_2}{(b_1 + V_2)} & 0 \\ \frac{\xi a_1 b_1 N_2}{(b_1 + V_2)^2} & 0 & -\frac{a_2 (1-c) N_2}{1 + a_2 (1-c) h N_2} \\ 0 & 0 & \theta \left(\frac{a_2 (1-c) N_2}{1 + a_2 (1-c) h N_2} - \mu \right) \end{bmatrix}.$$

The eigenvalues of $M(E_2)$ are $\theta \left(\frac{a_2 (1-c) N_2}{1 + a_2 (1-c) h N_2} - \mu \right)$ and

$$\lambda_{2\pm} = \frac{1}{2} \left[-\left\{ \frac{u_0}{m} + q_1 E + \frac{a_1 b_1 N_2}{(b_1 + V_2)^2} \right\} \pm \sqrt{\left\{ \frac{u_0}{m} + q_1 E + \frac{a_1 b_1 N_2}{(b_1 + V_2)^2} \right\}^2 - \frac{4 a_1^2 b_1 \xi N_2 V_2}{(b_1 + V_2)^3}} \right].$$

The signs of the real parts of λ_{2+} and λ_{2-} are negative. This implies that E_2 is locally asymptotically stable in the $V - N$ plane. It can be easily shown that E_2 is globally asymptotically stable in $V - N$ plane whenever it is locally asymptotically stable. E_2 is asymptotically stable or unstable in P - direction according to whether $\theta \left(\frac{a_2 (1-c) N_2}{1 + a_2 (1-c) h N_2} - \mu \right)$ is negative or positive.

Now we discuss the stability of the interior equilibrium point

$$M(E_3) = \begin{bmatrix} -\frac{u_0}{m} - q_1 E - \frac{a_1 b_1 N^*}{(b_1 + V^*)^2} & -\frac{a_1 V^*}{(b_1 + V^*)} & 0 \\ \frac{\xi a_1 b_1 N^*}{(b_1 + V^*)^2} & \frac{a_2^2 (1-c)^2 h P^* N^*}{(1 + a_2 (1-c) h N^*)^2} & -\frac{a_2 (1-c) N^*}{(1 + a_2 (1-c) h N^*)} \\ 0 & \frac{\theta a_2 (1-c) P^*}{(1 + a_2 (1-c) h N^*)^2} & 0 \end{bmatrix}.$$

The characteristic equation corresponding to this variational matrix is given by

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

where

$$A_1 = \frac{u_0}{m} + q_1E + \frac{a_1b_1N^*}{(b_1 + V^*)^2} - \frac{a_2^2(1-c)^2hP^*N^*}{(1 + a_2(1-c)hN^*)^2},$$

$$A_2 = -\left(\frac{u_0}{m} + \frac{a_1b_1N^*}{(b_1 + V^*)^2} + q_1E\right)\left(\frac{a_2^2(1-c)^2hP^*N^*}{(1 + a_2(1-c)hN^*)^2}\right) + \frac{\theta a_2^2(1-c)^2P^*N^*}{(1 + a_2(1-c)hN^*)^3} + \frac{\xi N^*V^*a_1^2b_1}{(b_1 + V^*)^3}$$

$$A_3 = \left(\frac{\theta a_2^2(1-c)^2P^*N^*}{(1 + a_2(1-c)hN^*)^3}\right)\left(\frac{u_0}{m} + q_1E + \frac{a_1b_1N^*}{(b_1 + V^*)^2}\right).$$

Then by Routh-Hurwitz criteria equilibrium E_3^* is locally asymptotically stable if $A_1 > 0$, $A_3 > 0$, and $A_1A_2 > A_3$ and unstable if either of these conditions is not satisfied.

5. ANALYSIS OF THE MODEL WITH DELAY

In this section, we consider the same model (2.1) with delay. Introducing the gestation delay $\tau(> 0)$ in the model system (2.1), we get the desired delay-induced food-chain model with habitat complexity as follows:

$$\begin{aligned} \frac{dV}{dt} &= u_0\left(1 - \frac{V}{m}\right) - \frac{a_1VN}{b_1 + V} - q_1EV, \\ \frac{dN}{dt} &= \xi N\left(\frac{a_1V}{b_1 + V} - \eta\right) - \frac{a_2(1-c)NP}{1 + a_2(1-c)hN} - q_2EN, \\ \frac{dP}{dt} &= \frac{\theta a_2(1-c)N(t-\tau)P(t-\tau)}{1 + a_2(1-c)hN(t-\tau)} - \theta P\mu. \end{aligned} \tag{5.1}$$

with initial condition given by

$$V(\phi) = \psi_1(\phi) \geq 0, \quad N(\phi) = \psi_2(\phi) \geq 0, \quad P(\phi) = \psi_3(\phi) \geq 0, \quad \phi \in (-\tau, 0]$$

Now we linearize system (5.1) about E_3 and obtain

$$\begin{aligned} \frac{dv}{dt} &= -\frac{u_0v}{m} - \frac{a_1V^*n}{b_1 + V^*} - \frac{a_1b_1N^*v}{(b_1 + V^*)^2} - q_1Ev, \\ \frac{dn}{dt} &= \frac{\xi a_1b_1N^*v}{(b_1 + V^*)^2} - \xi\eta n - q_2En - \frac{a_2(1-c)N^*p}{1 + a_2(1-c)hN^*} - \frac{a_2(1-c)P^*n}{\{1 + a_2(1-c)hN^*\}^2} + \frac{\xi a_1V^*n}{b_1 + V^*} \\ \frac{dp}{dt} &= \frac{\theta a_2(1-c)P^*n(t-\tau)}{\{1 + a_2(1-c)hN^*\}^2} + \frac{\theta a_2(1-c)p(t-\tau)N^*}{\{1 + a_2(1-c)hN^*\}} - \theta\mu p(t-\tau) \end{aligned} \tag{5.2}$$

where v , n and p are small perturbations given to V , N and P , respectively such that $V = V^* + v$, $N = N^* + n$ and $P = P^* + p$.

$$M = \begin{bmatrix} -\frac{u_0}{m} - q_1 E - \frac{a_1 b_1 N^*}{(b_1 + V^*)^2} & -\frac{a_1 V^*}{(b_1 + V^*)} & 0 \\ \frac{\xi a_1 b_1 N^*}{(b_1 + V^*)^2} & -\frac{a_2(1-c)P^*}{(1+a_2(1-c)hN^*)^2} - q_2 E - \xi \eta + \frac{\xi a_1 V^*}{b_1 + V^*} & -\frac{a_2(1-c)N^*}{(1+a_2(1-c)hN^*)} \\ 0 & \frac{\theta a_2(1-c)P^* e^{-\lambda \tau}}{(1+a_2(1-c)hN^*)^2} & \frac{\theta a_2(1-c)N^* e^{-\lambda \tau}}{\{1+a_2(1-c)hN^*\}} - \theta \mu e^{-\lambda \mu} \end{bmatrix}$$

The characteristic equation associated with system (5.2) is given by

$$\lambda^3 - (R_1 + R_2)\lambda^2 + (R_1 R_2 + R_3)\lambda - e^{-\lambda \tau} [R_3 \lambda^2 - (R_1 R_3 + R_2 R_3 + R_4)\lambda + R_1 R_2 R_3 + R_1 R_4 + R_3 R_5] = 0, \quad (5.3)$$

where

$$\begin{aligned} R_1 &= -\frac{u_0}{m} - \frac{a_1 b_1 N^*}{(b_1 + V^*)^2} - q_1 E, \\ R_2 &= -\xi \eta - q_2 E - \frac{a_2(1-c)P^*}{\{1+a_2(1-c)hN^*\}^2} + \frac{\xi a_1 V^*}{b_1 + V^*}, \\ R_3 &= \frac{\theta a_2(1-c)N^*}{1+a_2(1-c)hN^*} - \theta \mu, \\ R_4 &= \frac{\theta a_2^2(1-c)^2 N^* P^*}{\{1+a_2(1-c)hN^*\}^3} \\ R_5 &= \frac{\xi a_1^2 b_1 V^* N^*}{(b_1 + V^*)^3} \end{aligned}$$

We have already shown that the interior equilibrium point E_3 is locally asymptotically stable in the absence of delay.

Now when $\tau \neq 0$, stability of the system (5.1) can change only if there exists at least one root of equation (5.3) such that $\text{Re}(\lambda) = 0$. Let $\lambda = i\omega$ be one such root. Substituting this in equation (5.3) and equating real and imaginary parts, we get

$$\omega S_3 \cos \omega \tau + (S_4 - \omega^2 R_3) \sin \omega \tau = \omega^3 - \omega S_2 \quad (5.3a)$$

$$\omega S_3 \sin \omega \tau - (S_4 - \omega^2 R_3) \cos \omega \tau = -\omega^2 S_1 \quad (5.3b)$$

Where,

$$S_1 = R_1 + R_2, S_2 = R_1 R_2 + R_3, S_3 = R_1 R_3 + R_2 R_3 + R_4, S_4 = R_1 R_2 R_3 + R_1 R_4 + R_3 R_5$$

Squaring and adding equations (5.3a) and (5.3b), we get

$$\omega^6 + \omega^4 \{-2S_2 + S_1^2 - R_3^2\} + \omega^2 \{S_2^2 - S_3^2 + 2S_4 R_3\} - S_4^2 = 0 \quad (5.4)$$

Substituting $\omega^2 = \sigma$ equation (5.4) becomes

$$\sigma^3 + \sigma^2 \{S_1^2 - R_3^2 - 2S_2\} + \sigma \{S_2^2 - S_3^2 + 2S_4 R_3\} - S_4^2 = 0 \quad (5.5)$$

Now, let assume,

$$(H_1) \quad S_1^2 - R_3^2 - 2S_2 > 0 \quad \text{and}$$

$$(H_2) \quad S_2^2 - S_3^2 + 2S_4 R_3 < 0$$

So by Descartes' rule of sign, equation (5.5) has at least one positive real root. This implies that equation (5.4) has a real solution. Hence E_3 does not remain stable for all $\tau > 0$. So stability change can occur.

Again solving (5.3a) and (5.3b), we get a critical value of delay that is given as follows

$$\tau_c = \frac{1}{\omega} \cos^{-1} \left[\frac{\omega^4 (S_3 - S_1 R_3) + \omega^2 (S_1 S_4 - S_2 S_3)}{\omega^2 S_3^2 + (S_4 - \omega^2 R_3)^2} \right].$$

This is the least positive value of delay for which stability change can occur. To establish Hopf bifurcation, we can verify the following transversality condition:

$$\frac{d \operatorname{Re}(\lambda)}{d\tau} > 0 \tag{5.6}$$

at $\tau = \tau_c$, where τ_c is the critical value of τ .

Differentiating equation (5.3) with respect to τ , we obtain

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = - \frac{[3\lambda^2 - 2\lambda(R_1 + R_2) + R_1 R_2 + R_5]}{\lambda[\lambda^3 - \lambda^2(R_1 + R_2) + \lambda(R_1 R_2 + R_5)]} - \frac{[2\lambda R_3 - (R_1 R_3 + R_2 R_3 + R_4)]}{\lambda \left[\lambda^2 R_3 - \lambda(R_1 R_3 + R_2 R_3 + R_4) + \frac{\tau}{\lambda} [R_1 R_2 R_3 + R_1 R_4 + R_3 R_5] \right]} - \frac{\tau}{\lambda}$$

$$\left(\frac{d(\operatorname{Re} \lambda)}{d\tau} \right)^{-1} = \frac{R_1^2 R_2^2 \omega^2 + 2R_1 R_2 R_3 \omega^2 + R_5^2 \omega^2 + 2R_1^2 \omega^4 + 2R_2^2 \omega^4 + 2R_3^2 \omega^4 - 4R_5 \omega^4 + 3\omega^6}{\{-\omega^4 + \omega^2 (R_1 R_2 + R_5)\}^2 + \omega^6 (R_1 + R_2)^2} +$$

$$\frac{2R_3^2 R_5 \omega^2 - R_1^2 R_3^2 \omega^2 - R_2^2 R_3^2 \omega^2 - 2R_2 R_3 R_4 \omega^2 - R_4^2 \omega^2 - 2R_3^2 \omega^4}{\omega^4 (R_1 R_3 + R_2 R_3 + R_4)^2 + [(R_1 R_2 R_3 + R_1 R_4 + R_3 R_5)\omega - \omega^3 R_3]^2} \tag{5.7}$$

We verify the condition (5.6) numerically. Shows that transversality condition holds and hence Hopf bifurcation occurs at $\tau = \tau_c$.

6. NUMERICAL SIMULATION FOR BIFURCATION:

In this section, we present numerical simulation to illustrate results obtained in previous sections. The system (2.1) is integrated using fourth order Runge – Kutta Method with the help of MATLAB software package.

Hopf bifurcation analysis of the instantaneous model-

We shall vary c in system (2.1) so as to obtain a Hopf bifurcation. Now, we write autonomous system (2.1) in the form:

$$\dot{x} = F(x, k), \text{ where } x = (V, N, P), k = (u_0, m, a_1, b_1, q_1, \xi, E, \eta, a_2, c, h, q_2, \theta, \mu)$$

we say that an ordered pair (x_0, k_0) is a Hopf bifurcation point if,

- (i) $F(x_0, k_0) = 0$
- (ii) $J(x, k)$ has two complex conjugate eigenvalues $\lambda_{1,2}$ around (x_0, k_0) , $\lambda_{1,2} = a(x, k) \pm ib(x, k)$
- (iii) $a(x_0, k_0) = 0, b(x_0, k_0) \neq 0$
- (iv) The third eigenvalue $\lambda_3(x_0, k_0) \neq 0$

Extensive numerical simulations are carried out for various values of parameters and for different sets of initial conditions. We take the parameters of the system (2.1) as,

$$u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07$$

$$E = 0.5, \theta = 1.0, \mu = 0.6 \text{ and } h = 0.04$$

We consider the system,

$$\begin{aligned} \frac{dV}{dt} &= 3\left(1 - \frac{V}{50}\right) - \frac{1 \times VN}{0.5 + V} - 0.5 \times 0.5 \times V, \\ \frac{dN}{dt} &= 1 \times N \left(\frac{1 \times V}{0.5 + V} - 0.5 \right) - \frac{1 \times (1 - c)NP}{1 + 1 \times (1 - c) \times 0.04 \times N} - 0.07 \times 0.5 \times N, \\ \frac{dP}{dt} &= 1 \times P \left(\frac{1 \times (1 - c)N}{1 + 1 \times (1 - c) \times 0.04 \times N} - 0.6 \right). \end{aligned} \quad (6.1)$$

The system (6.1) always has non-negative equilibrium $E_1(1.17188, 0, 0)$. The system (6.1) has positive equilibria

$E_2(V_2, N_2, 0)$ and $E_3(V^*, N^*, P^*)$ if and if $c \in [0, 1)$.

Now take $c = 0.4$

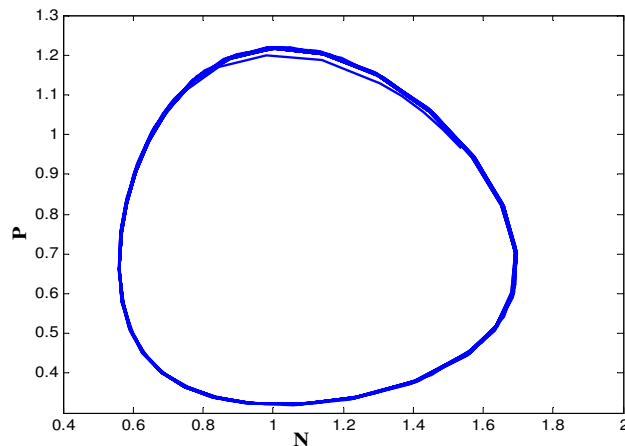
$$k = (3, 50, 1, 0.5, 0.5, 1, 0.5, 0.6, 1, 0.4, 0.04, 0.07, 1, 0.6)$$

The coordinates of E_3 and the corresponding eigenvalues are:

$$x_1 = (6.60489, 1.02459, 0.673883)$$

$$\lambda_1 = 0 + 0.488218i, \lambda_2 = 0 - 0.488218i, \lambda_3 = -0.258037$$

In this way ordered pair (x_0, k_0) is satisfied above all conditions (i-iv). So ordered pair (x_0, k_0) is Hopf point.



Fig(1) Here $V(0) = 5, N(0) = 1, P(0) = 0.96$ and $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.4, h = 0.04$

For $c = 0.3 < 0.4$

$$k = (3, 50, 1, 0.5, 0.5, 1, 0.5, 0.6, 1, 0.3, 0.04, 0.07, 1, 0.6)$$

The coordinates of E_3 and the corresponding eigenvalues are :

$$x_1 = (7.0325, 0.87822, 0.583462)$$

$\lambda_1 = 0.00244973 + 0.488876i, \lambda_2 = 0.00244973 - 0.488876i, \lambda_3 = -0.257589$. All eigenvalues have not negative real parts, only λ_3 has negative real part, so E_3 is always saddle point at $c = 0.3$.

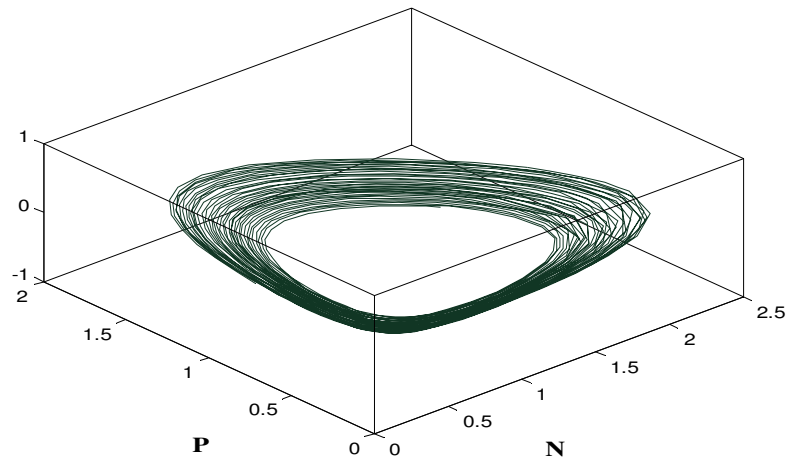


Fig (2) Here $V(0) = 5, N(0) = 1, P(0) = 0.96$ and $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5,$
 $q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.3, h = 0.04$

But if we take $c = 0.8 > 0.4$

$$k = (3, 50, 1, 0.5, 0.5, 1, 0.5, 0.6, 1, 0.8, 0.04, 0.07, 1, 0.6)$$

The coordinates of E_3 and the corresponding eigenvalues are:

$$x_1 = (1.86145, 3.07377, 1.29747)$$

$\lambda_1 = -0.130436 + 0.52648i, \lambda_2 = -0.130436 - 0.52648i, \lambda_3 = -0.267997$. All eigenvalues have negative real parts, so equilibrium point is locally asymptotically stable at $c = 0.8$.

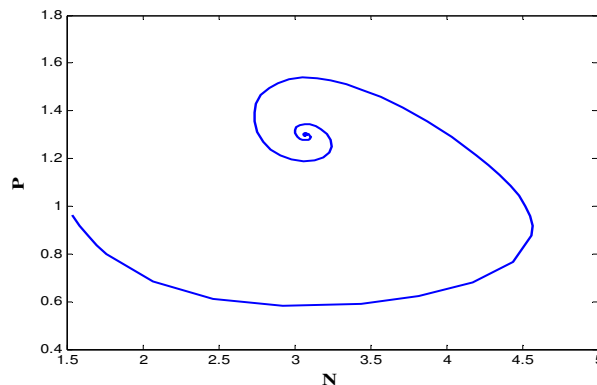


Fig (3) Here $V(0) = 5, N(0) = 1, P(0) = 0.96$ and $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5,$
 $q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.8, h = 0.04$

It has been showed numerically that the Hopf point is found when $c = 0.4$. E_3 is unstable when $c < 0.4$ and stable when $c > 0.4$. The numerical study presented here shows that, using the parameter c as control, it is possible to break the stable behavior of the system (6.1) and drive it to an unstable state. Also, it is possible to keep the population levels at a required state using the above control.

Hopf bifurcation analysis of the time-delay model:

To check the feasibility of our analysis regarding stability conditions, we have conducted some numerical computation using MATLAB 7.9 by choosing the following set of parameters values in model system (2.1)

$$u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, c = 0.6$$

$$E = 0.5, \theta = 1.0, \mu = 0.6, h = 0.04$$

For the above set of parameter values, the equilibrium point E_3^* exists and is given by

$$V^* = 5.15785, N^* = 1.53689, P^* = 0.964721.$$

Variational matrix $M(E_3)$ corresponding to the equilibrium E_3 is given by

$$M(E_3) = \begin{bmatrix} -0.334005 & -0.911627 & 0 \\ 0.0240055 & 0.00903908 & -0.600002 \\ 0 & 0.367588 & 0 \end{bmatrix}.$$

The characteristic equation resulting from this is given by

$$\lambda^3 + 0.324966\lambda^2 + 0.239418\lambda + 0.073661 = 0.$$

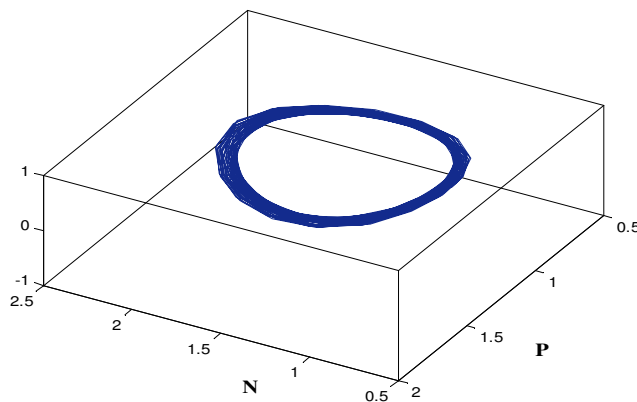
From this characteristic equation, we note that all conditions of Routh – Hurwitz criteria are satisfied and eigenvalues of $M(E_3)$ are given by $-0.00613417 \pm 0.485329i$ and -0.312698 . Hence, E_3 is locally asymptotically stable equilibrium point. Further, for the above set of parameters, in the presence of delay, stability change occurs and its critical value is given by

$$\tau_c = 0.0545544$$

Here we show that the transversality condition (5.6) is satisfied as:

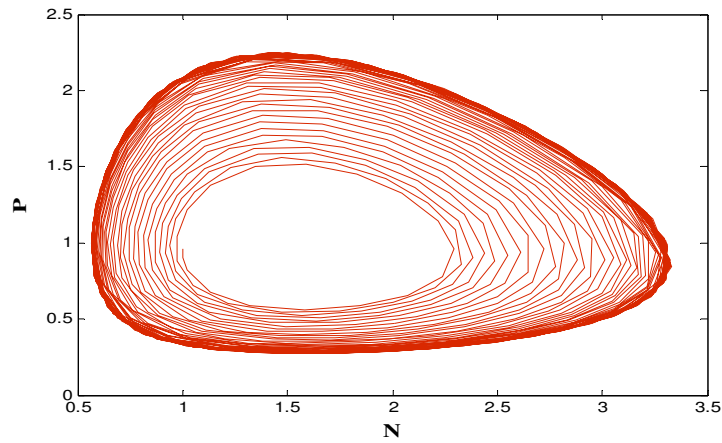
$$\left[\frac{d \operatorname{Re}(\lambda)}{d\tau} \right]_{\tau=\tau_c} = 14.6426 > 0$$

Therefore (V^*, N^*, P^*) is asymptotically stable for $\tau < \tau_c = 0.0545544$ (see Fig (4c)) and unstable for $\tau > \tau_c = 0.0545544$ (see Fig (4b)).

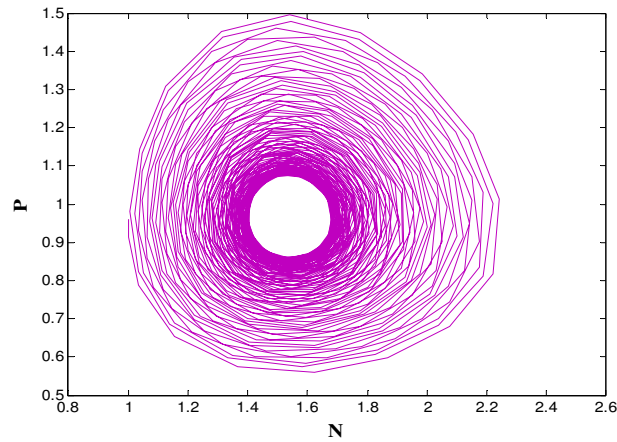


Fig(4a) when $\tau = 0.05$ and

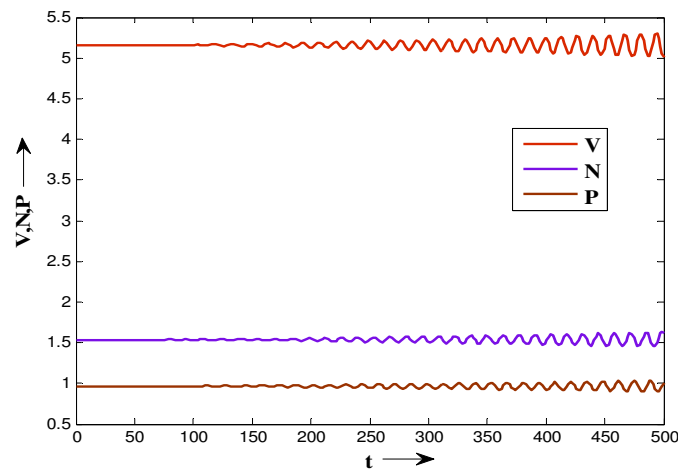
$$V(0) = 5, N(0) = 1, P(0) = 0.96, u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$$



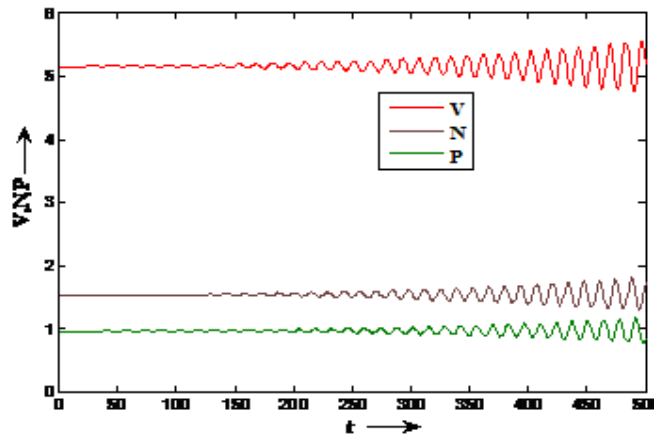
Fig(4b) when $\tau = 0.1$ and $V(0) = 5, N(0) = 1, P(0) = 0.96, u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



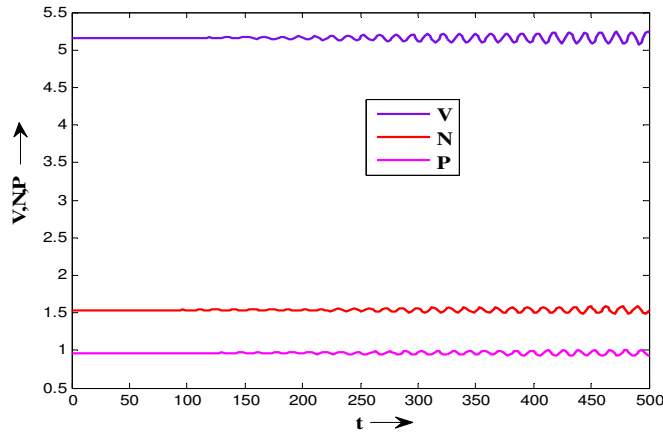
Fig(4c) when $\tau = 0.03$ and $V(0) = 5, N(0) = 1, P(0) = 0.96, u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



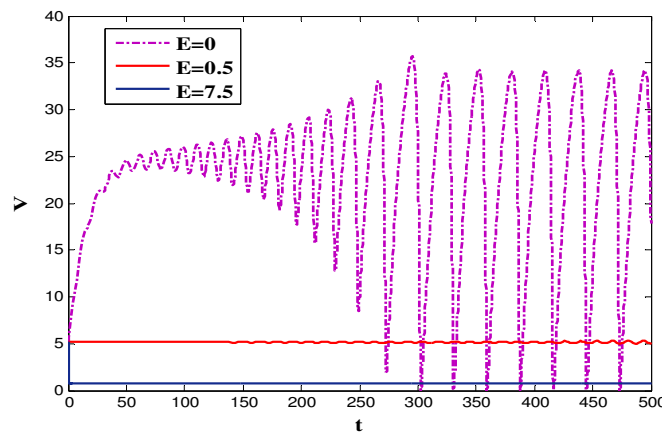
Fig(5a) when $\tau = 0.0545544$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721, u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07, E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



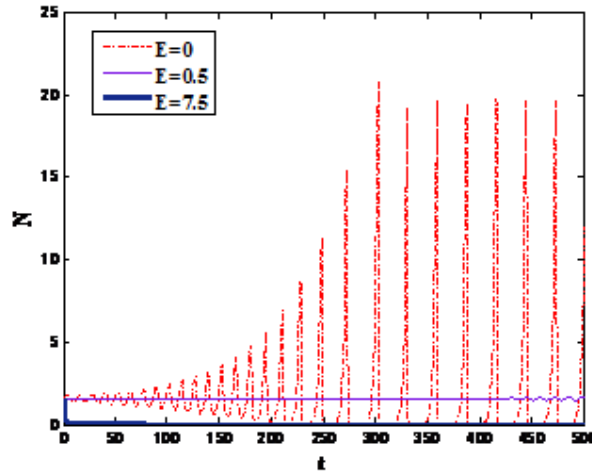
Fig(5b) when $\tau = 0.1$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721,$
 $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07,$
 $E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



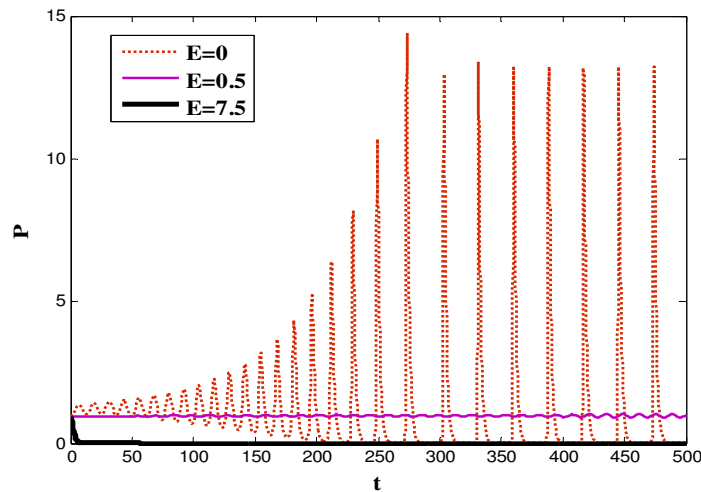
Fig(5c) when $\tau = 0.03$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721,$
 $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07,$
 $E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



Fig(6) when $\tau = 0.0545544$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721,$
 $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07,$
 $E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



Fig(7) when $\tau = 0.0545544$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721,$
 $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07,$
 $E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$



Fig(8) when $\tau = 0.0545544$ and $V(0) = 5.15785, N(0) = 1.53689, P(0) = 0.964721,$
 $u_0 = 3, m = 50, a_1 = 1.0, b_1 = 0.5, q_1 = 0.5, \xi = 1.0, \eta = 0.5, a_2 = 1.0, q_2 = 0.07,$
 $E = 0.5, \theta = 1.0, \mu = 0.6, c = 0.6, h = 0.04$

Figure (6) is the plot of vegetation biomass against t for different values of E . From figure (6), we note that the density of vegetation biomass decreases as E increases. Figures(7) are the plot of grazer population against t for different values of E . From these figures, it can be inferred that grazer population first remains constant as E increases, then starts decreasing as E tends towards the value 7.5. Figures (8) are the plots of predator population against t for different value E . These figure show that predator population decreases and becomes extinct if $E \geq 7.5$

7. CONCLUSION:

In this paper, we have studied a delay-induced food-chain model in presence of habitat complexity. Using stability theory of differential equations, we have obtained conditions for the existence of different equilibria and discussed their stabilities. Our numerical study shows that habitat complexity behaves as a control, it is possible that it breaks the stable behavior of the system (2.1) and drives it to an unstable state. Also, it is possible to keep the population levels at a required state using the above control. Further, we see the effect of harvesting of vegetation biomass and grazer population on the predator population. In the presence of delay, critical value of delay for which stability change occurs is obtained. It is obtained from the analysis that as the catchability coefficient E increases without any limit then predator population goes to extinction.

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