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APPLICATION OF HOMOTOPY PERTURBATION METHOD TO SOLVE MULTIDIMENSIONAL SCHRODINGER'S EQUATIONS

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ABSTRACT

In this paper, an application of homotopy perturbation method (HPM) is applied to solve a multidimensional Schrodionger's equation. Comparisons are made between the Adomians decomposition method and HPM. The results reveal that HPM is very effective and simple and gives the exact solution.

Keywords and phrases: *Multidimensional partial differential equations; Homotopy perturbation method (HPM); Adomian decomposition method (ADM); Schrodinger's equation.*

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1. INTRODUCTION:

The homotopy perturbation method (HPM) [3], first proposed by He is 1998, was further developed and improved by him ([4]-[6]). This method had been successfully applied to solve many types of linear and nonlinear problems. The method yields a very rapid convergence of the solution series in the most Cases. Usually, one iteration leads to high accuracy of the solution [8]. The goal of He's HPM was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems. This method was successfully applied to various engineering problems. In this method, the solution is considered as an infinite series which usually converges rapidly to the exact solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a small parameter. Considerable research works have been conducted recently in applying this method to a class of linear and non – linear equations [2].

In this paper, we apply He's HPM to multidimensional Schrodinger's equation. The results reveal that the proposed method is very effective and simple.

The solution of the two – dimensional Schrodinger equation has been a subject of considerable interest. This equation is the fundamental equation of physics for describing quantum mechanical behavior [1]. It is also often called the Schrodinger wave equation, and is a partial differential equation that describes how the wave function of a physical system evolves over time. Also this equation appears in electromagnetic wave propagations [11], in underwater acoustics (paraxial approximation of the wave equations [12] or also in optic (Fresnel equation [10]) and design of certain optoelectronic devices [7] as it nodes an electromagnetic wave equation in two – dimensional weakly guiding structure. Consider the following two – dimensional Schrodinger equation

$$-i\frac{\partial v}{\partial t}(x,y,t)=\frac{\partial^2 v}{\partial x^2}(x,y,t)+\frac{\partial^2 v}{\partial y^2}(x,y,t)-V(x,y)v(x,y,t)+v^2\bar{v}, \tag{1}$$

with the following initial condition

$$v(x,y,0)=\phi(x,y),(x,y,t)\in[a,b]\times[a,b]\times[0,T],$$

where $v(x,y,t)$ is the wave function in continuous and *V*(*x, y*) is an arbitrary potential function and $i^2=-1$.

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In this latter, we will use homotopy perturbation method to study two – dimensional Schrodinger equations.
In section 2, we recall basic idea of HPM. In section 3, the application of HPM in multidimensional Schrodinger's equation is proposed. Finally, two multidimensional Schrodinger's equations are solved by using HPM in section 4.

2. BASIC IDEA OF HPM:

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation.

$$A(u)-f(r)=0, r\in\Omega. \tag{2}$$

With the following boundary conditions.

$$B(u,\frac{\partial u}{\partial n}), r\in\Gamma, \tag{3}$$

Where *A* is general differential operator, *B* a boundary operator, *f*(*r*) is a known analytical function and *G* is the boundary of the domain Ω . The operator *A* can be decomposed into two operators, *L* and *N*, where *L* is a linear, and *N* a nonlinear operator. Equation (2) can be, therefore, written as follows:

$$L(u)+N(u)-f(r)=0. \tag{4}$$

Using the homotopy technique, we construct a homotopy $v(r,p):\Omega\times[0,1]\to R$, which satisfies:

$$H(v,p)=(1-p)[L(v)-L(u_0)]+p[A(v)-f(v)-f(r)]=0, p\in[0,1], r\in\Omega. \tag{5}$$

or

$$H(v,p)=L(v)-L(u_0)+pL(u_0)+p[N(v)-f(r)]=0. \tag{6}$$

Where $p\in[0,1]$ is an embedding parameter, u_0 is an initial approximation for the solution of Eq. (2), which satisfies the boundary conditions. Obviously, from Eqs. (5) and (6) we will have

$$H(v,0)=L(v)-L(u_0)=0, \tag{7}$$

$$H(v,1)=A(v)-f(r)=0. \tag{8}$$

The changing process of p from zero to unity is just that of $U(p,r)$ from $u_0(r)$ to $u(r)$. In topology, this is called homotopy. According to the (HPM), we can first use the embedding parameter p as a small parameter, and assume that solution of Equations (5) and (6) can be written as a power series in p :

$$v=v_0+pv_1+p^2v_2+\ldots \tag{9}$$

Setting $p=1$ results in the approximate solution of Equation (2)

$$u=\lim_{p\to 1}v=v_0+v_1+v_2+\ldots \tag{10}$$

The combination of the perturbation method and the homotopy method is called the HPM, which had eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the

traditional perturbation techniques; The series (9) is convergent for most cases. Some criteria are suggested for convergence of the series (9) in [9].

3. APPLICATION OF HPM IN SCHRODINGER'S EQUATION

For solving Eq.(1), by homotopy perturbation method, we construct the following homotopy:

$$(1-p)(\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t})+p(\frac{\partial v}{\partial t}-i(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-V(x,y)v-v^2\bar{v}))=0,$$

$$or \tag{11}$$

$$\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}+p(-i(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}-V(x,y)v-v^2\bar{v})+\frac{\partial u_0}{\partial t})=0$$

Suppose that the solution of Eq. (11) to be in the following form

$$v = v_0 + pv_1 + p^2 v_2 + \ldots \tag{12}$$

Substituting Eq. (12) into Eq. (11), and equation the coefficients of the terms with the identical powers of p,

$$p^0 \quad : \quad \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0,$$

$$p^1 : \quad \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} - i\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} - V(x,y)v_0 - v_0^2 \overline{v}_0\right) = 0, v_1(x,y,0) = 0,$$

$$p^2 : \quad \frac{\partial v_2}{\partial t} - i\left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} - V(x,y)v_1 - \sum_{z=0}^{1}\sum_{k=0}^{1-z} v_z v_k \overline{v_{j-k-z-1}}\right) = 0, v_2(x,y,0) = 0,$$

$$p^3 \quad : \quad \frac{\partial v_3}{\partial t} - i\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} - V(x,y)v_2 - \sum_{z=0}^{2}\sum_{k=0}^{2-z} v_z v_k \overline{v_{j-k-z-1}}\right) = 0, v_3(x,y,0) = 0,$$

$$\vdots$$

$$p^j \quad : \quad \frac{\partial v_i}{\partial t} - i\left(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2} - V(x,y)v_{j-1} - \sum_{z=0}^{j-1}\sum_{k=0}^{j-z-1} v_z v_k \overline{v_{j-k-z-1}}\right) = 0, v_j(x,y,0) = 0,$$

$$\vdots$$

For simplicity we take

$$v_0(x,y,t) = u_0(x,y,t) = u(x,y,0) = \varphi(x,y) \tag{13}$$

Having this assumption we get the following iterative equation

$$v_j = i\int_0^t \left(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2} - V(x,y)v_{j-1} - \sum_{z=0}^{j-1}\sum_{k=0}^{j-z-1} v_z v_k \overline{v_{j-k-z-1}}\right)dt, i = 1,2,3\ldots \tag{14}$$

Therefore, the approximated solutions of Eq. (1) can be obtained, by setting $p = 1$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \tag{15}$$

Similar to the above discussion can be written for three – dimensional. Schrodinger's equation.

4. EXAMPLES:

To illustrate the method and to show the ability of the method two examples are presented.

Example: 4.1 Consider the following two – dimensional Schrodinger's equation [14]

$$i\frac{\partial v(x,y,t)}{\partial t} = -\frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + V(x,y)v + (v)^2\overline{v}; i^2 = -1 \tag{16}$$

With the initial conditions,

$$\begin{aligned} &(x,y) \in [0,2\pi]\times[0,2\pi] \\ &\upsilon(x,y,0) = \sin x \sin y \\ &V(x,y) = 1 - \sin^2 x \sin^2 y \end{aligned} \tag{17}$$

He's HPM consists of the following scheme.

$$(1-p)[\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}]+p[\frac{\partial v}{\partial t}-i(\frac{1}{2}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2})-V(x,y)v-v^2\bar{v})]=0 \quad (18)$$

Starting with $v_0=u_0=\sin x\ \sin y$, by suing (14), we obtain the recurrence relation

$$v_1=i\int_0^t [\frac{1}{2}(\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2})-(1-\sin^2 x\ \sin^2 y)v_0-v_0^2\overline{v_0}]dt \quad (19)$$

$$v_j=i\int_0^t [\frac{1}{2}(\frac{\partial^2 v_j}{\partial x^2}+\frac{\partial^2 v_{j-1}}{\partial y^2})-V(x,y)v_{j-1}-\sum_{z=0}^{j-1}\sum_{k=0}^{j-z-1}v_z v_k\overline{v_{j-k-z-1}})dt\,;i=2,3,4... \quad (20)$$

The solution reads

$$v_1(x,y,t)=\frac{-2it}{1!}\sin x\ \sin y$$
$$v_2(x,y,t)=\frac{(-2it)^2}{2!}\sin x\ \sin y$$
$$v_3(x,y,t)=\frac{(-2it)^3}{3!}\sin x\ \sin y$$
$$\vdots$$

The exact solution will be as

$$u(x,y,t)=\sum_{n=0}^{\infty}u_n(x,y,t)=e^{-2it}\sin x\ \sin y$$

Which is exactly the same as obtained by Adomain decomposition method.

Example: 4.2 Let us solve the following three – dimensional Schrodinger's equation [14]

$$i\frac{\partial v(x,y,z,t)}{\partial t}=-\frac{1}{2}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})+V(x,y,z)v+(v)^2\bar{v}\,;i^2=-1 \quad (21)$$

With initial conditions

$$(x,y,z)\in[0,2\pi]\times[0,2\pi]\times[0,2\pi]$$
$$v(x,y,z,0)=\sin x\ \sin y\ \sin z \quad (22)$$
$$V(x,y,z)=1-\sin^2 x\ \sin^2 y\ \sin^2 z$$

We construct a homotopy $\Omega\times[0,1]\rightarrow R$ which satisfies.

$$(1-p)[\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}]+p[\frac{\partial v}{\partial t}-i(\frac{1}{2}(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})-V(x,y,z)v-(v^2)\bar{v})]=0$$

(23)

From Equation (1), we have the following scheme

$$u_0(x,y,z)=\sin x\ \sin y\ \sin z$$

$$v_1 = i\int_0^t [\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial z^2} - (1-\sin^2 x \sin^2 y \sin^2 z)v_0 - v_0^2\overline{v}_0]dt$$

$$v_i = i\int_0^t [\frac{1}{2}(\frac{\partial^2 v_{j-1}}{\partial x^2} + \frac{\partial^2 v_{j-1}}{\partial y^2} + \frac{\partial^2 v_{j-1}}{\partial z^2}) - V(x,y)v_{j-1} - \sum_{z=0}^{j-1}\sum_{k=0}^{j-z-1} v_z v_k \overline{v_{j-k-z-1}}]dt$$

For the first few j, we derive

$$v_1(x,y,z) = \frac{1}{1!}(\frac{-5it}{2})\sin x \sin y \sin z$$

$$v_2(x,y,z) = \frac{1}{2!}(\frac{-5it}{2})^2\sin x \sin y \sin z$$

$$v_3(x,y,z) = \frac{1}{3!}(\frac{-5it}{2})^3\sin x \sin y \sin z$$

$$\vdots$$

$$v(x,y,z,t) = \sum_{n=0}^{\infty} u_n(x,y,z,t) = e^{-\frac{5it}{2}}\sin x \sin y \sin z$$

Which is the exact solution.

5. CONCLUSIONS:

The main goal of this paper has been considered to drive an analytical solution for the general from of a nonlinear Schrodinger's equation in multidimensional case by applying HPM. The examples illustrated the HPM in two and three dimensional cases is simpler, faster and more efficient than the ADM.

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