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BEST SIMULTANEOUS APPROXIMATION IN FUZZY n-NORMED LINEAR SPACES

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ABSTRACT

The main aim of this paper is to consider the t-best simultaneous approximation in fuzzy n-normed linear spaces. We develop the theory of t-best simultaneous approximation in its quotient spaces. Then we discuss the relationship in t-proximinality and t-Chebyshevity of a given space and its quotient space.

Key Words: *t-best simultaneous approximation, t-proximinality, t-Chebyshevity, Quotient spaces.*

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1. INTRODUCTION:

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Theory of Fuzzy sets was introduced by Zadeh [20] in 1965. The idea of fuzzy norm was initiated by Katsaras in [11]. Felbin [5] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [10]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [12].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [11], Felbin [5], and Bag and Samanta [1]. The concept of 2-norm and n-norm on a linear space has been introduced and developed by Gahler in [6,7]. Following Misiak [14], Malceski [13] and Gunawan [9] developed the theory of n-normed linear space. Narayana and Vijayabalaji [15] introduced the concept of fuzzy n-normed linear space. Vijayabalaji and Thillaigovindan [19] introduced the notion of convergent sequence and Cauchy sequences in fuzzy n-normed linear space and studied the completeness of the fuzzy n-normed linear space. Vaezpour and Karimi [18] introduced the concept of t-best approximation in fuzzy normed linear spaces. Surender Reddy [16] introduced the concept of t-best approximation in fuzzy 2-normed linear spaces. Recently Goudarzi and Vaezpour [8] considered the set of all t-best simultaneous approximation in fuzzy normed linear spaces and used the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces. Surender Reddy [17] considered the set of all t-best simultaneous approximation in fuzzy 2-normed linear spaces and used the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

In this paper, we consider the set of all t-best simultaneous approximation in fuzzy n-normed linear spaces and we use the concept of simultaneous t-proximinality and simultaneous t-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

2. PRELIMINARIES:

Definition 2.1: Let $n \in N$ and let X be a real linear space of dimension $\geq n$. A real valued function $\|\bullet, \bullet, \ldots, \bullet\|$ on $\underbrace{X \times X \times \ldots \times X}_{n} = X^n$ satisfying the following conditions

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

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nN_2 : $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

nN_3: $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, x_2, ..., x_{n-1}, x_n\|$, for every $\alpha \in R$,

nN_4 : $\|x_1, x_2, ..., x_{n-1}, y + z\| \le \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$ for all $y, z, x_1, x_2, ..., x_{n-1} \in X$,

then the function $\|\bullet, \bullet, ..., \bullet\|$ is called an *n*-norm on *X* and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called *n*-normed linear space.

Example 2.2: A trivial example of an *n*-normed linear space is $X = R^n$ equipped with the following Euclidean *n*-norm.

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})| = abs\left(\begin{vmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in R^n$ for each $i = 1, 2, ..., n$.

Definition 2.3: Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times ... \times X}_{n} \times R$ is called a fuzzy *n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2, ..., x_n, y \in X$.

$(n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2, ..., x_n, t) = 0$,

$(n - N_2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

$(n - N_3)$: $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_{n-1}, cx_n, t) = N(x_1, x_2, ..., x_{n-1}, x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$, $N(x_1, x_2, ..., x_{n-1}, x_n + y, s + t) \ge$
$$\min\{N(x_1, x_2, ..., x_{n-1}, x_n, s), N(x_1, x_2, ..., x_{n-1}, y, t)\},$$

$(n - N_6)$: $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then the pair (X, N) is called a fuzzy *n*-normed linear space (briefly F-*n*-NLS).

Remark 2.4: From $(n - N_3)$, it follows that in F-*n*-NLS,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$, $N(x_1, x_2, ..., x_i + x_i', ..., x_n, s + t) \ge$
$$\min\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x_i', ..., x_n, t)\}.$$

Example 2.5: Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2, ..., x_n, t) = \frac{kt^n}{kt^n + m\|x_1, x_2, ..., x_n\|}, \text{ if } t > 0, \ t \in R, \ k, m, n \in R^+, \ x_1, x_2, ..., x_n \in X$$
$$= 0, \text{ if } t \le 0, \ t \in R, \ x_1, x_2, ..., x_n \in X.$$

Then (X, N) is a fuzzy *n*-normed linear space. In particular if $k = m = n = 1$ we have

$$N(x_1, x_2, ..., x_n, t) = \frac{t}{t + \|x_1, x_2, ..., x_n\|}, \text{ if } t > 0, \ t \in R, \ x_1, x_2, ..., x_n \in X$$
$$= 0, \text{ if } t \le 0, \ t \in R, \ x_1, x_2, ..., x_n \in X.$$

which is called the standard fuzzy *n*-norm induced by the *n*-norm $\|\bullet, \bullet, ..., \bullet\|$.

Definition 2.6: A sequence $\{x_k\}$ in a fuzzy *n*-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, ..., x_{n-1}, x_k - x, t) > 1 - r$, $\forall \;\; k \geq n_0$.

Theorem 2.7: In a fuzzy *n*-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$, $\forall \;\; t > 0$.

Definition 2.8: Let (X, N) be a fuzzy *n*-normed linear space. Let $\{x_k\}$ be a sequence in X then $\{x_k\}$ is said to be a Cauchy sequence if $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_{k+p} - x_k, t) = 1$, $\forall \;\; t > 0$ and $p = 1, 2, 3, ...$.

Definition 2.9: A fuzzy *n*-normed linear space (X, N) is said to be complete if every Cauchy sequence in *X* is convergent.

Definition 2.10: A complete fuzzy *n*-normed linear space (X, N) is called a fuzzy *n*-Banach space.

Definition 2.11: Let (X, N) be a fuzzy *n*-normed linear space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) > 1 - r\}$$

$$B[x, r, t] = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) \geq 1 - r\}$$

Definition 2.12: Let (X, N) be a fuzzy *n*-normed linear space. A subset *A* of *X* is said to be open if there exists $r \in (0,1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 2.13: Let (X, N) be a fuzzy *n*-normed linear space. A subset *A* of X is said to be closed if for any sequence $\{x_k\}$ in *A* converges to $x \in A$.

i.e., $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$, for all $t > 0$ implies that $x \in A$.

Corollary 2.14: Let (X, N) be a fuzzy *n*-normed linear space. Then *N* is a continuous function on $\underbrace{X \times X \times ... \times X}_{n} \times R$.

3. *t*-BEST SIMULTANEOUS APPROXIMATION:

Definition 3.1: Let (X, N) be a fuzzy *n*-normed linear space. A subset *A* of X is called *F*-bounded if there exists $t > 0$ and $0 < r < 1$ such that $N(x_1, x_2, ..., x_{n-1}, x, t) > 1 - r$, for all $x \in A$.

Definition 3.2: Let (X, N) be a fuzzy *n*-normed linear space, *W* be a subset of X and *M* be a *F*-bounded subset in *X*. For $t > 0$, we define

$$d(M, W, t) = \sup_{w\in W} \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m - w, t).$$

An element $w_0 \in W$ is called a *t*-best simultaneous approximation to *M* from *W* if for $t > 0$,

$$d(M, W, t) = \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m - w_0, t).$$

The set of all *t*-best simultaneous approximations to *M* from *W* will be denoted by $S_W^t(M)$ and we have

$$S_W^t(M) = \{w \in W : \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M, W, t)\}$$

Definition 3.3: Let W be a subset of a fuzzy *n*-normed linear space (X, N) then *W* is called a simultaneous *t*-proximinal subset of *X* if for each *F*-bounded set *M* in *X*, there exists at least one *t*-best simultaneous approximation from W to M. Also W is called a simultaneous t-Chebyshev subset of X if for each F-bounded set M in X, there exists a unique t-best simultaneous approximation from W to M.

Definition 3.4: Let (X,N) be a fuzzy n-normed linear space. A subset E of X is said to be convex if $(1-\lambda)x+\lambda y\in E$ whenever $x,y\in E$ and $0<\lambda<1$.

Lemma 3.5: Every open ball in a fuzzy n-normed linear space (X,N) is convex.

Theorem 3.6: Suppose that W is a subset of a fuzzy n-normed linear space (X,N) and M is F-bounded in X. Then $S_W^t(M)$ is a F-bounded subset of X and if W is convex and is a closed subset of X then $S_W^t(M)$ is closed and is convex for each F-bounded subspace M of X.

Proof: Since M is F-bounded, there exists $t>0$ and $0<r<1$ such that $N(x_1,x_2,...,x_{n-1},x,t)>1-r$, for all $x\in M$. If $w\in S_W^t(M)$, then

$$\inf_{m\in M} N(x_1,x_2,...,x_{n-1},m-w,t)=d(M,W,t).$$

Now, for all $m\in M$ and $w\in S_W^t(M)$,

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},w,2t) &= N(x_1,x_2,...,x_{n-1},w-m+m,2t)\\ &\ge \min\{N(x_1,x_2,...,x_{n-1},w-m,t),N(x_1,x_2,...,x_{n-1},m,t)\}\\ &\ge \inf_{m\in M}\min\{N(x_1,x_2,...,x_{n-1},w-m,t),N(x_1,x_2,...,x_{n-1},m,t)\}\\ &\ge \min\{\inf_{m\in M}N(x_1,x_2,...,x_{n-1},w-m,t),\inf_{m\in M}N(x_1,x_2,...,x_{n-1},m,t)\}\\ &\ge \min\{d(M,W,t),(1-r)\}\ge(1-r_0),\text{ for some } 0<r_0<1.\end{aligned}$$

Then $S_W^t(M)$ is F-bounded. Suppose that W is convex and is a closed subset of X. We show that $S_W^t(M)$ is convex and closed. Let $x,y\in S_W^t(M)$ and $0<\lambda<1$. Since W is convex, there exists $z_\lambda\in W$ such that $z_\lambda=\lambda x+(1-\lambda)y$, for each $0<\lambda<1$. Now for $t>0$ we have,

$$\begin{aligned}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},(\lambda x+(1-\lambda)y)-m,t) &= \inf_{m\in M}N(x_1,x_2,...,x_{n-1},z_\lambda-m,t)\\ &\le d(M,W,t).\end{aligned}$$

On the other hand, for a given $t>0$, take the natural number n such that $t>\frac{1}{n}$, we have

$$\begin{aligned}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},(\lambda x+(1-\lambda)y)-m,t) &= \inf_{m\in M}N(x_1,x_2,...,x_{n-1},\lambda(x-y)+y-m,t)\\ &\ge \inf_{m\in M}\min\{N(x_1,x_2,...,x_{n-1},x-y,\frac{1}{\lambda n}),N(x_1,x_2,...,x_{n-1},y-m,t-\frac{1}{n})\}\\ &= \min\{N(x_1,x_2,...,x_{n-1},x-y,\frac{1}{\lambda n}),\inf_{m\in M}N(x_1,x_2,...,x_{n-1},y-m,t-\frac{1}{n})\}\\ &\ge \lim_{n\to\infty}\left(\inf_{m\in M}N(x_1,x_2,...,x_{n-1},y-m,t-\frac{1}{n})\right)=d(M,W,t).\end{aligned}$$

So $S_W^t(M)$ is convex. Finally let $\{w_n\}\subset S_W^t(M)$ and suppose $\{w_n\}$ converges to some w in X. Since $\{w_n\}\subset W$ and W is closed so $w\in W$. Therefore by Corollary 2.14, for $t>0$ we have

$$\begin{aligned}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},m-w,t) &= \inf_{m\in M}N(x_1,x_2,...,x_{n-1},\lim_{n\to\infty}w_n-m,t)\\ &= \lim_{n\to\infty}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},w_n-m,t)=d(M,W,t).\end{aligned}$$

Theorem 3.7: The following assertions are hold for $t > 0$,

(i) $d(M + x, W + x, t) = d(M, W, t)$, $\forall \ x \in X$,

(ii) $d(\lambda M, \lambda W, t) = d(M, W, \frac{t}{|\lambda|})$, $\forall \ \lambda \in C$,

(iii) $S_{W+x}^{t}(M + x) = S_{W}^{t}(M) + x$, $\forall \ x \in X$,

(iv) $S_{\lambda W}^{|\lambda| t}(\lambda M) = \lambda S_{W}^{t}(M) + x$, $\forall \ \lambda \in C$,

Proof: (i) $d(M + x, W + x, t) = \sup_{w \in W} \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, (m + x) - (w + x), t)$

$$= \sup_{w \in W} \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M, W, t)$$

(ii) Clearly equality holds for $\lambda = 0$, so suppose that $\lambda \neq 0$. Then,

$$d(\lambda M, \lambda W, t) = \sup_{w \in W} \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, \lambda(m - w), t)$$

$$= \sup_{w \in W} \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, \frac{t}{|\lambda|}) = d(M, W, \frac{t}{|\lambda|})$$

(iii) $x + W \in S_{W+x}^{t}(M + x)$ if and only if,

$$\inf_{m+x \in M+x} N(x_1, x_2, ..., x_{n-1}, m + x - w - x, t) = d(M + x, W + x, t)$$

and by (i), the above equality holds if and only if,

$$\inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M, W, t)$$

for all $w \in W$ and this shows that $w \in S_{W}^{t}(M)$. So $x + w \in S_{W}^{t}(M) + x$.

(iv) $y_0 \in S_{\lambda W}^{|\lambda| t}(\lambda M)$ if and only if $y_0 \in \lambda W$ and,

$$d(\lambda W, \lambda M, |\lambda| t) = \inf_{\lambda m \in \lambda M} N(x_1, x_2, ..., x_{n-1}, y_0 - \lambda m, |\lambda| t)$$

$$= \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, \frac{y_0}{\lambda} - m, t)$$

But by (ii), we have $d(\lambda M, \lambda W, |\lambda| t) = d(W, M, t)$. So we have $\frac{y_0}{\lambda} \in W$ and $d(M, W, t) = \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, \frac{y_0}{\lambda} - m, t)$ or equivalently $\frac{y_0}{\lambda} \in S_{W}^{t}(M)$ and the proof is completed.

Corollary 3.8: Let A be a nonempty subset of a fuzzy *n*-normed linear space (X, N) then the following statements are hold.

(i) A is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if $A+y$ is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev), for each $y \in X$,

(ii) A is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if αA is simultaneous $|\alpha| t$-proximinal (respectively simultaneous $|\alpha| t$-Chebyshev), for each $\alpha \in C$.

Corollary 3.9: Let A be a nonempty subspace of a fuzzy *n*-normed linear space X and M be a *F*-bounded subset of X. Then for $t > 0$,

(i) $d(A, M + y, t) = d(A, M, t)$, $\forall \ y \in A$,

(ii) $S_{A}^{t}(M + y) = S_{A}^{t}(M) + y$, $\forall \ y \in A$,

(iii) $d(A, \alpha M, |\alpha| t) = d(A, M, t)$, for $0 \neq \alpha \in C$,

(iv) $S_A^{|\alpha| t}(\alpha M) = \alpha S_A^t(M)$, for $0 \neq \alpha \in C$.

4. SIMULTANEOUS t-PROXIMINALITY AND SIMULTANEOUS t-CHEBYSHEVITY IN QUOTIENT SPACES:

In this section we give characterization of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity in quotient spaces.

Definition 4.1: Let (X, N) be a fuzzy *n*-normed linear space, M be a linear manifold in *X* and let $Q : X \to X/M$ be the natural map $Qx = x + M$. We define
$$N(x_1, x_2, ..., x_{n-1}, x + M, t) = \sup\{N(x_1, x_2, ..., x_{n-1}, x + y, t) : y \in M\}, \quad t > 0$$

Theorem 4.2: If M is a closed subspace of a fuzzy *n*-normed linear space (X, N) and $N(x_1, x_2, ..., x_{n-1}, x + M, t)$ is defined as above then

(a) N is a fuzzy n-norm on X/M.

(b) $N(x_1, x_2, ..., x_{n-1}, Qx, t) \geq N(x_1, x_2, ..., x_{n-1}, x, t)$.

(c) If (X, N) is a fuzzy *n*-Banach space then so is $(X/M, N)$.

Proof: (a) It is clear that $N(x_1, x_2, ..., x_{n-1}, x + M, t) = 0$ for $t \leq 0$.
Let $N(x_1, x_2, ..., x_{n-1}, x + M, t) = 1$ for $t > 0$. By definition there is a sequence $\{x_k\}$ in M such that $N(x_1, x_2, ..., x_{n-1}, x + x_k, t) \to 1$. So $x + x_k \to 0$ or equivalently $x_k \to (-x)$ and since *M* is closed so $x \in M$ and $x + M = M$, the zero element of X/M. On the other hand we have,
$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, (x+M) + (y+M), t) &= N(x_1, x_2, ..., x_{n-1}, (x+y) + M, t) \\ &\geq N(x_1, x_2, ..., x_{n-1}, (x+m) + (y+n), t) \\ &\geq \min\{N(x_1, x_2, ..., x_{n-1}, x+m, t_1), N(x_1, x_2, ..., x_{n-1}, y+n, t_2)\} \end{aligned}$$

for $m, n \in M$, $x_1, x_2, ..., x_{n-1}, x, y \in X$ and $t_1 + t_2 = t$. Now if we take supremum on both sides, we have

$$\begin{aligned} &N(x_1, x_2, ..., x_{n-1}, (x+M) + (y+M), t) \\ &\qquad \geq \min\{N(x_1, x_2, ..., x_{n-1}, x+M, t_1), N(x_1, x_2, ..., x_{n-1}, y+M, t_2)\}. \end{aligned}$$

Also we have,
$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, \alpha(x+M), t) &= N(x_1, x_2, ..., x_{n-1}, \alpha x + M, t) \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, \alpha x + \alpha y, t) : y \in M\} \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, x + y, \frac{t}{|\alpha|}) : y \in M\} \\ &= N(x_1, x_2, ..., x_{n-1}, x + M, \frac{t}{|\alpha|}) \end{aligned}$$
and the remaining properties are obviously true. Therefore *N* is a fuzzy *n*-norm on X/M.

(b) We have,
$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, Qx, t) &= N(x_1, x_2, ..., x_{n-1}, x + M, t) \\ &= \sup\{N(x_1, x_2, ..., x_{n-1}, x + y, t) : y \in M\} \\ &\geq N(x_1, x_2, ..., x_{n-1}, x, t) \end{aligned}$$

(c) Let $\{y_k + M\}$ be a Cauchy sequence in X/M. Then there exists $\varepsilon_k > 0$ such that $\varepsilon_k \to 0$ and $N(x_1, x_2, ..., x_{n-1}, (y_k + M) - (y_{k+1} + M), t) \geq 1 - \varepsilon_k$. Let $z_1 = 0$. We choose $z_2 \in M$ such that,

$$N(x_1, x_2, ..., x_{n-1}, y_1 - (y_2 - z_2), t) \geq \min\{N(x_1, x_2, ..., x_{n-1}, (y_1 - y_2) + M, t), (1-\varepsilon_1)\}.$$

But $N(x_1, x_2, ..., x_{n-1}, (y_1 - y_2) + M, t) \geq (1-\varepsilon_1)$. Therefore,

$$N(x_1, x_2, ..., x_{n-1}, y_1 - (y_2 - z_2), t) \geq \min\{(1-\varepsilon_1), (1-\varepsilon_1)\} = (1-\varepsilon_1).$$

Now suppose z_{k-1} has been chosen, $z_k \in M$ can be chosen such that

$$N(x_1, x_2, ..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \geq \min\{N(x_1, x_2, ..., x_{n-1}, (y_{k-1} - y_k) + M, t), (1-\varepsilon_{k-1})\}$$

and therefore,

$$N(x_1, x_2, ..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \geq \min\{(1-\varepsilon_{k-1}), (1-\varepsilon_{k-1})\} = (1-\varepsilon_{k-1}).$$

Thus, $\{y_k + z_k\}$ is Cauchy sequence in *X*. Since *X* is complete, there is an y_0 in *X* such that $y_k + z_k \to y_0$ in *X*. On the other hand $y_k + M = Q(y_k + z_k) \to Q(y_0) = y_0 + M$. Therefore every Cauchy sequence $\{y_k + M\}$ is convergent in X/M and so X/M is complete and $(X/M, N)$ is a fuzzy *n*-Banach space.

Definition 4.3: Let *A* be a nonempty set in a fuzzy *n*-normed linear space (X, N). For $x \in X$ and $t > 0$, we shall denote the set of all elements of *t*-best approximation to *x* from *A* by $P_A^t(x)$;

i.e., $$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, y - x, t)\}.$$

where, $d(A, x, t) = \sup\{N(x_1, x_2, ..., x_{n-1}, y - x, t) : y \in A\} = \sup_{y \in A} N(x_1, x_2, ..., x_{n-1}, y - x, t)$.

If each $x \in X$ has at least (respectively exactly) one *t*-best approximation in *A* then *A* is called a *t*-proximinal (respectively *t*-Chebyshev) set.

Lemma 4.4: Let (X, N) be a fuzzy *n*-normed linear space and *M* be a *t*-proximinal subspace of *X*. For each nonempty *F*-bounded set *S* in X and $t > 0$,

$$d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t)$$

Proof: Since *M* is *t*-proximinal it follows that for each $s \in S$ there exists $m_s \in P_M^t(S)$ such that for $t > 0$,

$$N(x_1, x_2, ..., x_{n-1}, s - m_s, t) = \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t).$$

So,
$$\begin{aligned} d(S, M, t) &= \sup_{m \in M} \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m, t) \\ &\geq \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, t) \\ &= \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t) \\ &\geq \sup_{m \in M} \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m, t) = d(S, M, t) \end{aligned}$$

This implies that, $d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t)$.

Example 4.5: Let $(X = R^n, \|\bullet, \bullet, ..., \bullet\|)$ be *n*-normed linear space and consider (X, N) as its standard induced fuzzy *n*-normed linear space (Example 2.5). A nonempty subset *S* of X is *F*-bounded if and only if *S* is bounded in $(X, \|\bullet, \bullet, ..., \bullet\|)$. If we take $M = R$ we can easily prove that *M* is proximinal in $(X, \|\bullet, \bullet, ..., \bullet\|)$.

Lemma 4.6: Let (X, N) be a fuzzy *n*-normed linear space, *M* be a *t*-proximinal subspace of X and S be an arbitrary subset of X then the following assertions are equivalent:

(i) S is a *F*-bounded subset of *X*.

(ii) S/M is a *F*-bounded subset of X/M.

Proof: Suppose that S be a F-bounded subset of X. Then there exist $t > 0$, $0 < r < 1$ such that, $N(x_1, x_2, ..., x_{n-1}, x, t) > 1 - r$, for all $x \in S$. But,
$N(x_1, x_2, ..., x_{n-1}, x + M, t) = \sup_{y \in M} N(x_1, x_2, ..., x_{n-1}, x + y, t) \geq N(x_1, x_2, ..., x_{n-1}, x, t) \geq 1 - r$.

So, $(i) \Rightarrow (ii)$ is proved. Now to prove that $(ii) \Rightarrow (i)$. Let S/M be a F-bounded subset of X/M. Since M is t-proximinal, then for each $s \in S$ there exists $m_s \in M$ such that $m_s \in P_M^t(S)$. So for each $s \in S$,

$$N(x_1, x_2, ..., x_{n-1}, s - m_s, t) = \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t) \tag{1}$$

Now from Lemma 4.4, we conclude that for $t > 0$,

$$\begin{aligned}\inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, t) &= \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t) \\ &= \sup_{m \in M} \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m, t).\end{aligned}$$

Then for $0 < r < 1$ such that $\inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, t) \geq r$ and $t > 0$ there exists $m_r \in M$ such that

$$\inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_r, t) \geq \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, t) - r \geq 0.$$

So by (1), for all $s \in S$ we have

$$\begin{aligned}N(x_1, x_2, ..., x_{n-1}, s, t) &= N(x_1, x_2, ..., x_{n-1}, s - m_r + m_r, t) \\ &\geq \min\{N(x_1, x_2, ..., x_{n-1}, s - m_r, \tfrac{t}{2}), N(x_1, x_2, ..., x_{n-1}, m_r, \tfrac{t}{2})\} \\ &\geq \inf_{s \in S} \min\{N(x_1, x_2, ..., x_{n-1}, s - m_r, \tfrac{t}{2}), N(x_1, x_2, ..., x_{n-1}, m_r, \tfrac{t}{2})\} \\ &\geq \min\{(\inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, \tfrac{t}{2}) - r), N(x_1, x_2, ..., x_{n-1}, m_r, \tfrac{t}{2})\} \\ &= \min\{(\inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, \tfrac{t}{2}) - r), N(x_1, x_2, ..., x_{n-1}, m_r, \tfrac{t}{2})\} \\ &\geq \min\{(\inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s + M, \tfrac{t}{2}) - r), N(x_1, x_2, ..., x_{n-1}, m_r, \tfrac{t}{2})\}. \end{aligned} \tag{2}$$

Since S/M is F-bounded, by its definition we can find $0 < r_0 < 1$ such that in the right hand side of (2) be greater than or equal to $(1 - r_0)$ and this completes the proof.

Lemma 4.7: Let M be a t-proximinal subspace of a fuzzy n-normed linear space (X, N) and $W \supseteq M$ a subspace of X. Let K be F-bounded in X. If $w_0 \in S_W^t(K)$, then
$w_0 + M \in S_{W/M}^t(K/M)$.

Proof: Since K is F-bounded by Lemma 4.6, K/M is F-bounded in X/M. Assume that $w_0 \in S_W^t(K)$ and $w_0 + M \notin S_{W/M}^t(K/M)$. Thus there exists $w' \in M$ such that for $t > 0$,

$$\begin{aligned}\inf_{k \in K} N(x_1, x_2, ..., x_{n-1}, k - (w' + M), t) &> \inf_{k \in K} N(x_1, x_2, ..., x_{n-1}, k - (w_0 + M), t) \\ &\geq \inf_{k \in K} N(x_1, x_2, ..., x_{n-1}, k - w_0, t) = d(K, W, t)\end{aligned} \tag{3}$$

On the other hand for each $k \in K$ and for $t > 0$,

$$N(x_1, x_2, ..., x_{n-1}, k - (w' + M), t) = \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, k - (w' + m), t)$$

Then for each $0<\varepsilon<1$ and $k\in K$ there exists $m_k \in M$ such that for $t>0$,

$$N(x_1, x_2, ..., x_{n-1}, k-(w'+m_k), t) \geq N(x_1, x_2, ..., x_{n-1}, k-(w'+M), t) - \varepsilon .$$

Since $w'+m_k \in M$ we conclude that

$$\begin{aligned} d(K,W,t) &\geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w'+m_k), t) \\ &\geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w'+M), t) - \varepsilon \end{aligned}$$

Thus, $$d(K,W,t) \geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w'+M), t) \qquad (4)$$

By (3) and (4) we get,

$$d(K,W,t) \geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w'+M), t) > d(K,W,t),$$

and this is a contradiction. Therefore $w_0 + M \in S^t_{W/M}(K/M)$ and the proof is completed.

Corollary 4.8: Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space (X, N) and $W \supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then W/M is a simultaneous *t*-proximinal subspace of X/M .

Corollary 4.9: Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space (X, N) and $W \supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then for each *F*-bounded set *K* in *X*,

$$Q(S^t_W(K)) \subseteq S^t_{W/M}(K/M).$$

Theorem 4.10: Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space (X, N) and $W \supseteq M$ subspace of *X*. If *K* is *F*-bounded set in *X* such that $w_0 + M \in S^t_{W/M}(K/M)$ and $m_0 \in S^t_M(K-w_0)$, then $w_0 + m_0 \in S^t_W(K)$.

Proof: In view of Lemma 4.4, for $t>0$ we have,

$$\begin{aligned} \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, (k-w_0)-m_0, t) &= \sup_{m\in M} \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, (k-w_0)-m, t) \\ &= \inf_{k\in K} \sup_{m\in M} N(x_1, x_2, ..., x_{n-1}, k-(w_0+m), t) \\ &= \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w_0+M), t) \\ &\geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w+M), t) \quad \forall\ w\in W \\ &\geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-w, t) \quad \forall\ w\in W . \end{aligned}$$

Hence, $\inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-(w_0+m_0), t) \geq \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k-w, t)\ \ \forall\ w\in W$.

But $w_0 + m_0 \in W$. Then $w_0 + m_0 \in S^t_W(K)$ and so the proof is completed.

Theorem 4.11: Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space (X, N) and $W \supseteq M$ a simultaneous *t*-proximinal subspace of *X*. Then for each *F*-bounded set *K* in *X*,

$$Q(S^t_W(K)) = S^t_{W/M}(K/M)$$

Proof: By Corollary 4.9, we obtain

$$Q(S^t_W(K)) \subseteq S^t_{W/M}(K/M).$$

Also by Lemma 4.6, W/M is simultaneous *t*-proximinal in X/M. Now let, $w_0 + M \in S^t_{W/M}(K/M)$, where $w_0 \in W$. By simultaneous *t*-proximinality of *M* there exists $m_0 \in M$ such that $m_0 \in S^t_M(K - w_0)$. Then in view of Theorem 4.10, we conclude that $w_0 + m_0 \in S^t_W(K)$.

Therefore $w_0 + M \in Q(S^t_W(K))$ and the proof is completed.

Corollary 4.12: Let *W* and *M* be subspaces of a fuzzy *n*-normed linear space (X, N). If M is simultaneous *t*-proximinal then the following assertions are equivalent:

(i) W/M is simultaneous *t*-proximinal in X/M.

(ii) $W + M$ is simultaneous *t*-proximinal in *X*.

Proof: $(i) \Rightarrow (ii)$. Let *K* be an arbitrary *F*-bounded set in *X*. Then by Lemma 4.6, K/M is a *F*-bounded set in X/M. Since $(W+M)/M = W/M$ and *M* are simultaneous *t*-proximinal it follows that there exists $w_0 + M \in (W+M)/M$ and $m_0 \in M$ such that $w_0 + M \in S^t_{(W+M)/M}(K/M)$ and $m_0 \in S^t_M(K - w_0)$. By Theorem 4.10, we have $w_0 + m_0 \in S^t_{W+M}(K)$. This shows that $W + M$ is simultaneous *t*-proximinal in *X*.

$(ii) \Rightarrow (i)$. Since $W + M$ is simultaneous *t*-proximinal and $W + M \supseteq M$, by Corollary 4.8, we have $(W+M)/M = W/M$ is simultaneous *t*-proximinal.

Theorem 4.13: Let *W* and *M* be subspaces of a fuzzy *n*-normed linear space (X, N). If M is simultaneous *t*-Chebyshev then the following assertions are equivalent:

(i) W/M is simultaneous *t*-Chebyshev in X/M.

(ii) $W + M$ is simultaneous *t*-Chebyshev in *X*.

Proof: $(i) \Rightarrow (ii)$, By hypothesis $(W+M)/M = W/M$ is simultaneous *t*-Chebyshev. Assume that (ii) is false. Then some *F*-bounded subset *K* of *X* has two distinct simultaneous *t*-best approximations such as l_0 and l_1 in $W + M$. Thus we have,

$$l_0,\ l_1 \in S^t_{W+M}(K). \tag{5}$$

Since $W + M \supseteq M$ by lemma 4.6, $l_0 + M$, $l_1 + M \in S^t_{(W+M)/M}(K/M) = S^t_{W/M}(K/M)$.

Since W/M is simultaneous *t*-Chebyshev, $l_0 + M = l_1 + M$. So there exists $0 \neq m_0 \in M$ such that $l_1 = l_0 + m_0$.

By (5) for all $t > 0$,

$$\begin{aligned}
\inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, (k - l_0) - m_0, t) &= \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_1, t) \\
&= \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_0, t) \\
&= d(K, W + M, t) \\
&= d(K - l_0, W + M, t) \geq d(K - l_0, M, t)
\end{aligned}$$

This shows that both *m* and zero are simultaneous *t*-best approximations to $S - l_0$ from *M* and this is a contradiction.

$(ii) \Rightarrow (i)$. Assume that (i) does not hold. Then for some *F*-bounded subset *K* of *X*, K/M has two distinct simultaneous *t*-best approximations such as $w + M$ and $w' + M$ in W/M. Thus $w - w' \notin M$. Since *M* is simultaneous *t*-proximinal there exists simultaneous *t*-best approximations *m* and m' to $K - w$ and $K - w'$ from *M* respectively. Therefore $m \in S^t_M(K - w)$ and $m' \in S^t_M(K - w')$. Since $W + M \supseteq M$, $w + M$ and $w' + M$ are in $S^t_{W/M}(K/M) = S^t_{(K+M)/M}(K/M)$, by Theorem 4.10, $w + m$ and $w' + m' \in S^t_{W+M}(K)$. But $W + M$ is simultaneous *t*-Chebyshev. Thus $w + m = w' + m'$ and so $w - w' \in M$, which is a contradiction.

Corollary 4.14: Let *M* be simultaneous *t*-Chebyshev subspace of a fuzzy *n*-normed linear space (X,N). If $W \supseteq M$ is a simultaneous *t*-Chebyshev subspace in *X*, then the following assertions are equivalent:
(i) *W* is simultaneous *t*-Chebyshev in *X*.
(ii) W/M is simultaneous *t*-Chebyshev in X/M.

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