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# BEST SIMULTANEOUS APPROXIMATION IN FUZZY *n*-NORMED LINEAR SPACES

**B. Surender Reddy***

***Department of Mathematics, PGCS, Saifabad, Osmania University, Hyderabad-500004, AP, INDIA***
***E-mail: bsrmathou@yahoo.com***

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***ABSTRACT***
*The main aim of this paper is to consider the t-best simultaneous approximation in fuzzy n-normed linear spaces. We develop the theory of t-best simultaneous approximation in its quotient spaces. Then we discuss the relationship in t-proximinality and t-Chebyshevity of a given space and its quotient space.*

***Key Words:*** *t-best simultaneous approximation, t-proximinality, t-Chebyshevity, Quotient spaces.*

***2000 AMS Subject Classification No:*** *46A30, 46S40, 46A70, 54A40.*

## 1. INTRODUCTION:

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. Theory of Fuzzy sets was introduced by Zadeh [20] in 1965. The idea of fuzzy norm was initiated by Katsaras in [11]. Felbin [5] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [10]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [12].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [12]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [11], Felbin [5], and Bag and Samanta [1]. The concept of 2-norm and *n*-norm on a linear space has been introduced and developed by Gahler in [6,7]. Following Misiak [14], Malceski [13] and Gunawan [9] developed the theory of *n*-normed linear space. Narayana and Vijayabalaji [15] introduced the concept of fuzzy *n*-normed linear space. Vijayabalaji and Thillaigovindan [19] introduced the notion of convergent sequence and Cauchy sequences in fuzzy *n*-normed linear space and studied the completeness of the fuzzy *n*-normed linear space. Vaezpour and Karimi [18] introduced the concept of *t*-best approximation in fuzzy normed linear spaces. Surender Reddy [16] introduced the concept of *t*-best approximation in fuzzy 2-normed linear spaces. Recently Goudarzi and Vaezpour [8] considered the set of all *t*-best simultaneous approximation in fuzzy normed linear spaces and used the concept of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity to introduce the theory of *t*-best simultaneous approximation in quotient spaces. Surender Reddy [17] considered the set of all *t*-best simultaneous approximation in fuzzy 2-normed linear spaces and used the concept of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity to introduce the theory of t-best simultaneous approximation in quotient spaces.

In this paper, we consider the set of all *t*-best simultaneous approximation in fuzzy *n*-normed linear spaces and we use the concept of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity to introduce the theory of *t*-best simultaneous approximation in quotient spaces.

## 2. PRELIMINARIES:

**Definition 2.1:** Let $n \in N$ and let X be a real linear space of dimension $\geq n$. A real valued function $\|\bullet,\bullet,\ldots,\bullet\|$ on $\underbrace{X \times X \times \ldots \times X}_{n} = X^n$ satisfying the following conditions

$nN_1$: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

**Corresponding author: B. Surender Reddy*, *E-mail: bsrmathou@yahoo.com*

$nN_2$ : $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$nN_3$ : $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha| \, \|x_1, x_2, ..., x_{n-1}, x_n\|$, for every $\alpha \in R$,

$nN_4$ : $\|x_1, x_2, ..., x_{n-1}, y + z\| \le \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$ for all $y, z, x_1, x_2, ..., x_{n-1} \in X$,

then the function $\|\bullet, \bullet, ..., \bullet\|$ is called an *n*-norm on *X* and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called *n*-normed linear space.

**Example 2.2:** A trivial example of an *n*-normed linear space is $X = R^n$ equipped with the following Euclidean *n*-norm.

$$\|x_1, x_2, ..., x_n\|_E = |\det(x_{ij})| = abs\left(\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}\right),$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in R^n$ for each $i = 1, 2, ..., n$.

**Definition 2.3:** Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times ... \times X}_{n} \times R$ is called a fuzzy *n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2, ..., x_n, y \in X$.

$(n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2, ..., x_n, t) = 0$,

$(n - N_2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

$(n - N_3)$: $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_{n-1}, cx_n, t) = N(x_1, x_2, ..., x_{n-1}, x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$, $N(x_1, x_2, ..., x_{n-1}, x_n + y, s + t) \ge$
$$\min\{N(x_1, x_2, ..., x_{n-1}, x_n, s), N(x_1, x_2, ..., x_{n-1}, y, t)\},$$

$(n - N_6)$: $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t \to \infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then the pair $(X, N)$ is called a fuzzy *n*-normed linear space (briefly F-*n*-NLS).

**Remark 2.4:** From $(n - N_3)$, it follows that in F-*n*-NLS,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$, $N(x_1, x_2, ..., x_i + x_i', ..., x_n, s + t) \ge$
$$\min\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x_i', ..., x_n, t)\}.$$

**Example 2.5:** Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2, ..., x_n, t) = \frac{kt^n}{kt^n + m\|x_1, x_2, ..., x_n\|}, \text{ if } t > 0,\ t \in R,\ k, m, n \in R^+,\ x_1, x_2, ..., x_n \in X$$
$$= 0, \text{ if } t \le 0,\ t \in R,\ x_1, x_2, ..., x_n \in X.$$

Then $(X, N)$ is a fuzzy *n*-normed linear space. In particular if $k = m = n = 1$ we have

$$N(x_1, x_2, ..., x_n, t) = \frac{t}{t + \|x_1, x_2, ..., x_n\|}, \text{ if } t > 0,\ t \in R,\ x_1, x_2, ..., x_n \in X$$
$$= 0, \text{ if } t \le 0,\ t \in R,\ x_1, x_2, ..., x_n \in X.$$

which is called the standard fuzzy *n*-norm induced by the *n*-norm $\|\bullet, \bullet, ..., \bullet\|$.

 

**Definition 2.6:** A sequence $\{x_k\}$ in a fuzzy *n*-normed linear space $(X,N)$ is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that
$N(x_1, x_2, ..., x_{n-1}, x_k - x, t) > 1 - r$, $\forall \ k \geq n_0$.

**Theorem 2.7:** In a fuzzy *n*-normed linear space $(X,N)$, a sequence $\{x_k\}$ converges to $x \in X$ if and only $\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$, $\forall \ t > 0$.

**Definition 2.8:** Let $(X,N)$ be a fuzzy *n*-normed linear space. Let $\{x_k\}$ be a sequence in *X* then $\{x_k\}$ is said to be a Cauchy sequence if $\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_{k+p} - x_k, t) = 1$, $\forall \ t > 0$ and $p = 1,2,3,...$.

**Definition 2.9:** A fuzzy *n*-normed linear space $(X,N)$ is said to be complete if every Cauchy sequence in *X* is convergent.

**Definition 2.10:** A complete fuzzy *n*-normed linear space $(X,N)$ is called a fuzzy *n*-Banach space.

**Definition 2.11:** Let $(X,N)$ be a fuzzy *n*-normed linear space. The open ball
$B(x,r,t)$ and the closed ball $B[x,r,t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x,r,t) = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) > 1 - r\}$$
$$B[x,r,t] = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) \geq 1 - r\}$$

**Definition 2.12:** Let $(X,N)$ be a fuzzy *n*-normed linear space. A subset *A* of *X* is said to be open if there exists $r \in (0,1)$ such that $B(x,r,t) \subset A$ for all $x \in A$ and $t > 0$.

**Definition 2.13:** Let $(X,N)$ be a fuzzy *n*-normed linear space. A subset *A* of *X* is said to be closed if for any sequence $\{x_k\}$ in *A* converges to $x \in A$.
i.e., $\lim_{k \to \infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 1$, for all $t > 0$ implies that $x \in A$.

**Corollary 2.14:** Let $(X,N)$ be a fuzzy *n*-normed linear space. Then *N* is a continuous function on $\underbrace{X \times X \times ... \times X}_{n} \times R$.

## 3. *t*-BEST SIMULTANEOUS APPROXIMATION:

**Definition 3.1:** Let $(X,N)$ be a fuzzy *n*-normed linear space. A subset *A* of *X* is called *F*-bounded if there exists $t > 0$ and $0 < r < 1$ such that $N(x_1, x_2, ..., x_{n-1}, x, t) > 1 - r$, for all $x \in A$.

**Definition 3.2:** Let $(X,N)$ be a fuzzy *n*-normed linear space, *W* be a subset of *X* and *M* be a *F*-bounded subset in *X*. For $t > 0$, we define
$d(M,W,t) = \sup_{w \in W} \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t)$.
An element $w_0 \in W$ is called a *t*-best simultaneous approximation to *M* from *W* if for $t > 0$,
$d(M,W,t) = \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w_0, t)$.

The set of all *t*-best simultaneous approximations to *M* from *W* will be denoted by $S_W^t(M)$ and we have
$S_W^t(M) = \{w \in W : \inf_{m \in M} N(x_1, x_2, ..., x_{n-1}, m - w, t) = d(M,W,t)\}$

**Definition 3.3:** Let *W* be a subset of a fuzzy *n*-normed linear space $(X,N)$ then *W* is called a simultaneous *t*-proximinal subset of *X* if for each *F*-bounded set *M* in *X*, there exists at least one *t*-best simultaneous approximation from *W* to *M*. Also *W* is called a simultaneous *t*-Chebyshev subset of *X* if for each *F*-bounded set *M* in *X*, there exists a unique *t*-best simultaneous approximation from *W* to *M*.

**Definition 3.4:** Let $(X, N)$ be a fuzzy *n*-normed linear space. A subset *E* of $X$ is said to be convex if $(1-\lambda)x+\lambda y\in E$ whenever $x, y\in E$ and $0<\lambda<1$.

**Lemma 3.5:** Every open ball in a fuzzy *n*-normed linear space $(X, N)$ is convex.

**Theorem 3.6:** Suppose that *W* is a subset of a fuzzy *n*-normed linear space $(X, N)$ and *M* is *F*-bounded in *X*. Then $S_W^t(M)$ is a *F*-bounded subset of *X* and if *W* is convex and is a closed subset of *X* then $S_W^t(M)$ is closed and is convex for each *F*-bounded subspace *M* of *X*.

**Proof:** Since *M* is *F*-bounded, there exists $t>0$ and $0<r<1$ such that $N(x_1, x_2, ..., x_{n-1}, x, t)>1-r$, for all $x\in M$. If $w\in S_W^t(M)$, then

$$\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m-w, t)=d(M, W, t).$$

Now, for all $m\in M$ and $w\in S_W^t(M)$,

$$\begin{aligned}
N(x_1, x_2, ..., x_{n-1}, w, 2t) &= N(x_1, x_2, ..., x_{n-1}, w-m+m, 2t)\\
&\geq \min\{N(x_1, x_2, ..., x_{n-1}, w-m, t), N(x_1, x_2, ..., x_{n-1}, m, t)\}\\
&\geq \inf_{m\in M}\min\{N(x_1, x_2, ..., x_{n-1}, w-m, t), N(x_1, x_2, ..., x_{n-1}, m, t)\}\\
&\geq \min\{\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, w-m, t), \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m, t)\}\\
&\geq \min\{d(M, W, t), (1-r)\}\geq (1-r_0), \text{ for some } 0<r_0<1.
\end{aligned}$$

Then $S_W^t(M)$ is *F*-bounded. Suppose that $W$ is convex and is a closed subset of *X*. We show that $S_W^t(M)$ is convex and closed. Let $x, y\in S_W^t(M)$ and $0<\lambda<1$. Since *W* is convex, there exists $z_\lambda\in W$ such that $z_\lambda=\lambda x+(1-\lambda)y$, for each $0<\lambda<1$. Now for $t>0$ we have,

$$\begin{aligned}
\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, (\lambda x+(1-\lambda)y)-m, t) &= \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, z_\lambda-m, t)\\
&\leq d(M, W, t).
\end{aligned}$$

On the other hand, for a given $t>0$, take the natural number *n* such that $t>\frac{1}{n}$, we have

$$\begin{aligned}
\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, (\lambda x+(1-\lambda)y)-m, t) &= \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, \lambda(x-y)+y-m, t)\\
&\geq \inf_{m\in M}\min\{N(x_1, x_2, ..., x_{n-1}, x-y, \frac{1}{\lambda n}), N(x_1, x_2, ..., x_{n-1}, y-m, t-\frac{1}{n})\}\\
&= \min\{N(x_1, x_2, ..., x_{n-1}, x-y, \frac{1}{\lambda n}), \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, y-m, t-\frac{1}{n})\}\\
&\geq \lim_{n\to\infty}\left(\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, y-m, t-\frac{1}{n})\right)=d(M, W, t).
\end{aligned}$$

So $S_W^t(M)$ is convex. Finally let $\{w_n\}\subset S_W^t(M)$ and suppose $\{w_n\}$ converges to some *w* in $X$. Since $\{w_n\}\subset W$ and $W$ is closed so $w\in W$. Therefore by Corollary 2.14, for $t>0$ we have

$$\begin{aligned}
\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, m-w, t) &= \inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, \lim_{n\to\infty} w_n-m, t)\\
&= \lim_{n\to\infty}\inf_{m\in M} N(x_1, x_2, ..., x_{n-1}, w_n-m, t)=d(M, W, t).
\end{aligned}$$

**Theorem 3.7:** The following assertions are hold for $t>0$,

(i) $d(M+x,W+x,t)=d(M,W,t)$, $\forall\ x\in X$,

(ii) $d(\lambda M,\lambda W,t)=d(M,W,\frac{t}{|\lambda|})$, $\forall\ \lambda\in C$,

(iii) $S^{t}_{W+x}(M+x)=S^{t}_{W}(M)+x$, $\forall\ x\in X$,

(iv) $S^{|\lambda|t}_{\lambda W}(\lambda M)=\lambda S^{t}_{W}(M)+x$, $\forall\ \lambda\in C$,

**Proof:** (i) $d(M+x,W+x,t)=\sup_{w\in W}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},(m+x)-(w+x),t)$

$$=\sup_{w\in W}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},m-w,t)=d(M,W,t)$$

(ii) Clearly equality holds for $\lambda=0$, so suppose that $\lambda\neq 0$. Then,

$$d(\lambda M,\lambda W,t)=\sup_{w\in W}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},\lambda(m-w),t)$$

$$=\sup_{w\in W}\inf_{m\in M}N(x_1,x_2,...,x_{n-1},m-w,\frac{t}{|\lambda|})=d(M,W,\frac{t}{|\lambda|})$$

(iii) $x+W\in S^{t}_{W+x}(M+x)$ if and only if,

$$\inf_{m+x\in M+x}N(x_1,x_2,...,x_{n-1},m+x-w-x,t)=d(M+x,W+x,t)$$

and by (i), the above equality holds if and only if,

$$\inf_{m\in M}N(x_1,x_2,...,x_{n-1},m-w,t)=d(M,W,t)$$

for all $w\in W$ and this shows that $w\in S^{t}_{W}(M)$. So $x+w\in S^{t}_{W}(M)+x$.

(iv) $y_0\in S^{|\lambda|t}_{\lambda W}(\lambda M)$ if and only if $y_0\in\lambda W$ and,

$$d(\lambda W,\lambda M,|\lambda|t)=\inf_{\lambda m\in\lambda M}N(x_1,x_2,...,x_{n-1},y_0-\lambda m,|\lambda|t)$$

$$=\inf_{m\in M}N(x_1,x_2,...,x_{n-1},\frac{y_0}{\lambda}-m,t)$$

But by (ii), we have $d(\lambda M,\lambda W,|\lambda|t)=d(W,M,t)$. So we have $\frac{y_0}{\lambda}\in W$ and $d(M,W,t)=\inf_{m\in M}N(x_1,x_2,...,x_{n-1},\frac{y_0}{\lambda}-m,t)$ or equivalently $\frac{y_0}{\lambda}\in S^{t}_{W}(M)$ and the proof is completed.

**Corollary 3.8:** Let $A$ be a nonempty subset of a fuzzy *n*-normed linear space $(X,N)$ then the following statements are hold.

(i) $A$ is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if $A+y$ is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev), for each $y\in X$,

(ii) $A$ is simultaneous *t*-proximinal (respectively simultaneous *t*-Chebyshev) if and only if $\alpha A$ is simultaneous $|\alpha|t$-proximinal (respectively simultaneous $|\alpha|t$-Chebyshev), for each $\alpha\in C$.

**Corollary 3.9:** Let $A$ be a nonempty subspace of a fuzzy *n*-normed linear space $X$ and $M$ be a *F*-bounded subset of $X$. Then for $t>0$,

(i) $d(A,M+y,t)=d(A,M,t)$, $\forall\ y\in A$,

(ii) $S^{t}_{A}(M+y)=S^{t}_{A}(M)+y$, $\forall\ y\in A$,

(iii) $d(A,\alpha M,|\alpha|t)=d(A,M,t)$, for $0\neq\alpha\in C$,

(iv) $S_A^{|\alpha|t}(\alpha M)=\alpha S_A^t(M)$, for $0\neq\alpha\in C$.

## 4. SIMULTANEOUS t-PROXIMINALITY AND SIMULTANEOUS t-CHEBYSHEVITY IN QUOTIENT SPACES:

In this section we give characterization of simultaneous *t*-proximinality and simultaneous *t*-Chebyshevity in quotient spaces.

**Definition 4.1:** Let $(X,N)$ be a fuzzy *n*-normed linear space, $M$ be a linear manifold in *X* and let $Q: X\to X/M$ be the natural map $Qx = x+M$. We define

$$N(x_1,x_2,...,x_{n-1},x+M,t)=\sup\{N(x_1,x_2,...,x_{n-1},x+y,t): y\in M\},\quad t>0$$

**Theorem 4.2:** If $M$ is a closed subspace of a fuzzy *n*-normed linear space $(X,N)$ and $N(x_1,x_2,...,x_{n-1},x+M,t)$ is defined as above then

(a) $N$ is a fuzzy n-norm on $X/M$.

(b) $N(x_1,x_2,...,x_{n-1},Qx,t)\geq N(x_1,x_2,...,x_{n-1},x,t)$.

(c) If $(X,N)$ is a fuzzy *n*-Banach space then so is $(X/M,N)$.

**Proof:** (a) It is clear that $N(x_1,x_2,...,x_{n-1},x+M,t)=0$ for $t\leq 0$.

Let $N(x_1,x_2,...,x_{n-1},x+M,t)=1$ for $t>0$. By definition there is a sequence $\{x_k\}$ in $M$ such that $N(x_1,x_2,...,x_{n-1},x+x_k,t)\to 1$. So $x+x_k\to 0$ or equivalently $x_k\to(-x)$ and since *M* is closed so $x\in M$ and $x+M=M$, the zero element of $X/M$. On the other hand we have,

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},(x+M)+(y+M),t) &= N(x_1,x_2,...,x_{n-1},(x+y)+M,t)\\ &\geq N(x_1,x_2,...,x_{n-1},(x+m)+(y+n),t)\\ &\geq \min\{N(x_1,x_2,...,x_{n-1},x+m,t_1),N(x_1,x_2,...,x_{n-1},y+n,t_2)\}\end{aligned}$$

for $m,n\in M$, $x_1,x_2,...,x_{n-1},x,y\in X$ and $t_1+t_2=t$. Now if we take supremum on both sides, we have

$$N(x_1,x_2,...,x_{n-1},(x+M)+(y+M),t)\geq \min\{N(x_1,x_2,...,x_{n-1},x+M,t_1),N(x_1,x_2,...,x_{n-1},y+M,t_2)\}.$$

Also we have,

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},\alpha(x+M),t) &= N(x_1,x_2,...,x_{n-1},\alpha x+M,t)\\ &=\sup\{N(x_1,x_2,...,x_{n-1},\alpha x+\alpha y,t): y\in M\}\\ &=\sup\{N(x_1,x_2,...,x_{n-1},x+y,\frac{t}{|\alpha|}): y\in M\}\\ &= N(x_1,x_2,...,x_{n-1},x+M,\frac{t}{|\alpha|})\end{aligned}$$

and the remaining properties are obviously true. Therefore *N* is a fuzzy *n*-norm on $X/M$.

(b) We have,

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},Qx,t) &= N(x_1,x_2,...,x_{n-1},x+M,t)\\ &=\sup\{N(x_1,x_2,...,x_{n-1},x+y,t): y\in M\}\\ &\geq N(x_1,x_2,...,x_{n-1},x,t)\end{aligned}$$

(c) Let $\{y_k+M\}$ be a Cauchy sequence in $X/M$. Then there exists $\varepsilon_k>0$ such that $\varepsilon_k\to 0$ and $N(x_1,x_2,...,x_{n-1},(y_k+M)-(y_{k+1}+M),t)\geq 1-\varepsilon_k$. Let $z_1=0$. We choose $z_2\in M$ such that,

$$N(x_1, x_2, ..., x_{n-1}, y_1 - (y_2 - z_2), t) \geq \min\{N(x_1, x_2, ..., x_{n-1}, (y_1 - y_2) + M, t), (1-\varepsilon_1)\}.$$

But $N(x_1, x_2, ..., x_{n-1}, (y_1 - y_2) + M, t) \geq (1-\varepsilon_1)$. Therefore,

$$N(x_1, x_2, ..., x_{n-1}, y_1 - (y_2 - z_2), t) \geq \min\{(1-\varepsilon_1), (1-\varepsilon_1)\} = (1-\varepsilon_1).$$

Now suppose $z_{k-1}$ has been chosen, $z_k \in M$ can be chosen such that

$$N(x_1, x_2, ..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \geq \min\{N(x_1, x_2, ..., x_{n-1}, (y_{k-1} - y_k) + M, t), (1-\varepsilon_{k-1})\}$$

and therefore,

$$N(x_1, x_2, ..., x_{n-1}, (y_{k-1} + z_{k-1}) - (y_k + z_k), t) \geq \min\{(1-\varepsilon_{k-1}), (1-\varepsilon_{k-1})\} = (1-\varepsilon_{k-1}).$$

Thus, $\{y_k + z_k\}$ is Cauchy sequence in *X*. Since *X* is complete, there is an $y_0$ in *X* such that $y_k + z_k \to y_0$ in *X*. On the other hand $y_k + M = Q(y_k + z_k) \to Q(y_0) = y_0 + M$. Therefore every Cauchy sequence $\{y_k + M\}$ is convergent in $X/M$ and so $X/M$ is complete and $(X/M, N)$ is a fuzzy *n*-Banach space.

**Definition 4.3:** Let *A* be a nonempty set in a fuzzy *n*-normed linear space $(X, N)$. For $x \in X$ and $t > 0$, we shall denote the set of all elements of *t*-best approximation to *x* from *A* by $P_A^t(x)$;

i.e., $$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, y - x, t)\}.$$

where, $d(A, x, t) = \sup\{N(x_1, x_2, ..., x_{n-1}, y - x, t) : y \in A\} = \sup_{y \in A} N(x_1, x_2, ..., x_{n-1}, y - x, t)$.

If each $x \in X$ has at least (respectively exactly) one *t*-best approximation in *A* then *A* is called a *t*-proximinal (respectively *t*-Chebyshev) set.

**Lemma 4.4:** Let $(X, N)$ be a fuzzy *n*-normed linear space and *M* be a *t*-proximinal subspace of *X*. For each nonempty *F*-bounded set *S* in *X* and $t > 0$,

$$d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t)$$

**Proof:** Since *M* is *t*-proximinal it follows that for each $s \in S$ there exists $m_s \in P_M^t(S)$ such that for $t > 0$,

$$N(x_1, x_2, ..., x_{n-1}, s - m_s, t) = \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t).$$

So,
$$\begin{aligned} d(S, M, t) &= \sup_{m \in M} \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m, t) \\ &\geq \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m_s, t) \\ &= \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t) \\ &\geq \sup_{m \in M} \inf_{s \in S} N(x_1, x_2, ..., x_{n-1}, s - m, t) = d(S, M, t) \end{aligned}$$

This implies that, $d(S, M, t) = \inf_{s \in S} \sup_{m \in M} N(x_1, x_2, ..., x_{n-1}, s - m, t)$.

**Example 4.5:** Let $(X = R^n, \|\bullet, \bullet, ..., \bullet\|)$ be *n*-normed linear space and consider $(X, N)$ as its standard induced fuzzy *n*-normed linear space (Example 2.5). A nonempty subset *S* of *X* is *F*-bounded if and only if *S* is bounded in $(X, \|\bullet, \bullet, ..., \bullet\|)$. If we take $M = R$ we can easily prove that *M* is proximinal in $(X, \|\bullet, \bullet, ..., \bullet\|)$.

**Lemma 4.6:** Let $(X, N)$ be a fuzzy *n*-normed linear space, *M* be a *t*-proximinal subspace of *X* and *S* be an arbitrary subset of *X* then the following assertions are equivalent:

(i) *S* is a *F*-bounded subset of *X*.

(ii) $S/M$ is a *F*-bounded subset of $X/M$.

**Proof:** Suppose that *S* be a *F*-bounded subset of *X*. Then there exist $t>0$, $0<r<1$ such that, $N(x_1,x_2,...,x_{n-1},x,t)>1-r$, for all $x\in S$. But,
$N(x_1,x_2,...,x_{n-1},x+M,t)=\sup_{y\in M} N(x_1,x_2,...,x_{n-1},x+y,t)\geq N(x_1,x_2,...,x_{n-1},x,t)\geq 1-r$.

So, $(i)\Rightarrow(ii)$ is proved. Now to prove that $(ii)\Rightarrow(i)$. Let $S/M$ be a *F*-bounded subset of $X/M$. Since *M* is *t*-proximinal, then for each $s\in S$ there exists $m_s\in M$ such that $m_s\in P_M^t(S)$. So for each $s\in S$,

$$N(x_1,x_2,...,x_{n-1},s-m_s,t)=\sup_{m\in M} N(x_1,x_2,...,x_{n-1},s-m,t) \tag{1}$$

Now from Lemma 4.4, we conclude that for $t>0$,

$$\begin{aligned}\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m_s,t)&=\inf_{s\in S}\sup_{m\in M} N(x_1,x_2,...,x_{n-1},s-m,t)\\ &=\sup_{m\in M}\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m,t).\end{aligned}$$

Then for $0<r<1$ such that $\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m_s,t)\geq r$ and $t>0$ there exists $m_r\in M$ such that

$$\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m_r,t)\geq \inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m_s,t)-r\geq 0.$$

So by (1), for all $s\in S$ we have

$$\begin{aligned}N(x_1,x_2,...,x_{n-1},s,t)&=N(x_1,x_2,...,x_{n-1},s-m_r+m_r,t)\\ &\geq \min\{N(x_1,x_2,...,x_{n-1},s-m_r,\tfrac{t}{2}),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\ &\geq \inf_{s\in S}\min\{N(x_1,x_2,...,x_{n-1},s-m_r,\tfrac{t}{2}),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\ &\geq \min\{(\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s-m_s,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\ &=\min\{(\inf_{s\in S}\sup_{m\in M} N(x_1,x_2,...,x_{n-1},s-m,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}\\ &\geq \min\{(\inf_{s\in S} N(x_1,x_2,...,x_{n-1},s+M,\tfrac{t}{2})-r),N(x_1,x_2,...,x_{n-1},m_r,\tfrac{t}{2})\}.\end{aligned} \tag{2}$$

Since $S/M$ is *F*-bounded, by its definition we can find $0<r_0<1$ such that in the right hand side of (2) be greater than or equal to $(1-r_0)$ and this completes the proof.

**Lemma 4.7:** Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space $(X,N)$ and $W\supseteq M$ a subspace of *X*. Let *K* be *F*-bounded in *X*. If $w_0\in S_W^t(K)$, then
$w_0+M\in S_{W/M}^t(K/M)$.

**Proof:** Since *K* is *F*-bounded by Lemma 4.6, $K/M$ is F-bounded in $X/M$. Assume that $w_0\in S_W^t(K)$ and $w_0+M\notin S_{W/M}^t(K/M)$. Thus there exists $w'\in M$ such that for $t>0$,

$$\begin{aligned}\inf_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w'+M),t)&>\inf_{k\in K} N(x_1,x_2,...,x_{n-1},k-(w_0+M),t)\\ &\geq \inf_{k\in K} N(x_1,x_2,...,x_{n-1},k-w_0,t)=d(K,W,t)\end{aligned} \tag{3}$$

On the other hand for each $k\in K$ and for $t>0$,

$$N(x_1,x_2,...,x_{n-1},k-(w'+M),t)=\sup_{m\in M} N(x_1,x_2,...,x_{n-1},k-(w'+m),t)$$

 

Then for each $0<\varepsilon<1$ and $k\in K$ there exists $m_k \in M$ such that for $t>0$,

$$N(x_1, x_2,..., x_{n-1}, k-(w'+m_k), t) \geq N(x_1, x_2,..., x_{n-1}, k-(w'+M), t)-\varepsilon .$$

Since $w'+m_k \in M$ we conclude that

$$\begin{aligned} d(K,W,t) &\geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w'+m_k), t) \\ &\geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w'+M), t)-\varepsilon \end{aligned}$$

Thus, $$d(K,W,t) \geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w'+M), t) \tag{4}$$

By (3) and (4) we get,

$$d(K,W,t) \geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w'+M), t) > d(K,W,t),$$

and this is a contradiction. Therefore $w_0+M \in S^t_{W/M}(K/M)$ and the proof is completed.

**Corollary 4.8:** Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space $(X,N)$ and $W\supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then $W/M$ is a simultaneous *t*-proximinal subspace of $X/M$ .

**Corollary 4.9:** Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space $(X,N)$ and $W\supseteq M$ a subspace *X*. If *W* is simultaneous *t*-proximinal then for each *F*-bounded set *K* in *X*,

$$Q(S^t_W(K)) \subseteq S^t_{W/M}(K/M).$$

**Theorem 4.10:** Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space $(X,N)$ and $W\supseteq M$ subspace of *X*. If *K* is *F*-bounded set in *X* such that $w_0+M \in S^t_{W/M}(K/M)$ and $m_0 \in S^t_M(K-w_0)$, then $w_0+m_0 \in S^t_W(K)$.

**Proof:** In view of Lemma 4.4, for $t>0$ we have,

$$\begin{aligned} \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, (k-w_0)-m_0, t) &= \sup_{m\in M}\inf_{k\in K} N(x_1, x_2,..., x_{n-1}, (k-w_0)-m, t) \\ &= \inf_{k\in K}\sup_{m\in M} N(x_1, x_2,..., x_{n-1}, k-(w_0+m), t) \\ &= \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w_0+M), t) \\ &\geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w+M), t) \quad \forall\ w\in W \\ &\geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-w, t) \quad \forall\ w\in W . \end{aligned}$$

Hence, $\inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-(w_0+m_0), t) \geq \inf_{k\in K} N(x_1, x_2,..., x_{n-1}, k-w, t)\ \ \forall\ w\in W$ .

But $w_0+m_0 \in W$ . Then $w_0+m_0 \in S^t_W(K)$ and so the proof is completed.

**Theorem 4.11:** Let *M* be a *t*-proximinal subspace of a fuzzy *n*-normed linear space $(X,N)$ and $W\supseteq M$ a simultaneous *t*-proximinal subspace of *X*. Then for each *F*-bounded set *K* in *X*,

$$Q(S^t_W(K)) = S^t_{W/M}(K/M)$$

**Proof:** By Corollary 4.9, we obtain

$$Q(S^t_W(K)) \subseteq S^t_{W/M}(K/M).$$

Also by Lemma 4.6, $W/M$ is simultaneous *t*-proximinal in $X/M$. Now let, $w_0 + M \in S^t_{W/M}(K/M)$, where $w_0 \in W$. By simultaneous *t*-proximinality of *M* there exists $m_0 \in M$ such that $m_0 \in S^t_M(K - w_0)$. Then in view of Theorem 4.10, we conclude that $w_0 + m_0 \in S^t_W(K)$.

Therefore $w_0 + M \in Q(S^t_W(K))$ and the proof is completed.

**Corollary 4.12:** Let *W* and *M* be subspaces of a fuzzy *n*-normed linear space $(X, N)$. If $M$ is simultaneous *t*-proximinal then the following assertions are equivalent:

(i) $W/M$ is simultaneous *t*-proximinal in $X/M$.

(ii) $W + M$ is simultaneous *t*-proximinal in *X*.

**Proof:** $(i) \Rightarrow (ii)$. Let *K* be an arbitrary *F*-bounded set in *X*. Then by Lemma 4.6, $K/M$ is a *F*-bounded set in $X/M$. Since $(W+M)/M = W/M$ and *M* are simultaneous *t*-proximinal it follows that there exists $w_0 + M \in (W+M)/M$ and $m_0 \in M$ such that $w_0 + M \in S^t_{(W+M)/M}(K/M)$ and $m_0 \in S^t_M(K - w_0)$. By Theorem 4.10, we have $w_0 + m_0 \in S^t_{W+M}(K)$. This shows that $W + M$ is simultaneous *t*-proximinal in *X*.

$(ii) \Rightarrow (i)$. Since $W + M$ is simultaneous *t*-proximinal and $W + M \supseteq M$, by Corollary 4.8, we have $(W+M)/M = W/M$ is simultaneous *t*-proximinal.

**Theorem 4.13:** Let *W* and *M* be subspaces of a fuzzy *n*-normed linear space $(X, N)$. If $M$ is simultaneous *t*-Chebyshev then the following assertions are equivalent:

(i) $W/M$ is simultaneous *t*-Chebyshev in $X/M$.

(ii) $W + M$ is simultaneous *t*-Chebyshev in *X*.

**Proof:** $(i) \Rightarrow (ii)$, By hypothesis $(W+M)/M = W/M$ is simultaneous *t*-Chebyshev. Assume that $(ii)$ is false. Then some *F*-bounded subset *K* of *X* has two distinct simultaneous *t*-best approximations such as $l_0$ and $l_1$ in $W + M$. Thus we have,

$$l_0,\ l_1 \in S^t_{W+M}(K). \tag{5}$$

Since $W + M \supseteq M$ by lemma 4.6, $l_0 + M$, $l_1 + M \in S^t_{(W+M)/M}(K/M) = S^t_{W/M}(K/M)$.

Since $W/M$ is simultaneous *t*-Chebyshev, $l_0 + M = l_1 + M$. So there exists $0 \neq m_0 \in M$ such that $l_1 = l_0 + m_0$.

By (5) for all $t > 0$,

$$\begin{aligned}\inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, (k - l_0) - m_0, t) &= \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_1, t)\\ &= \inf_{k\in K} N(x_1, x_2, ..., x_{n-1}, k - l_0, t)\\ &= d(K, W + M, t)\\ &= d(K - l_0, W + M, t) \geq d(K - l_0, M, t)\end{aligned}$$

This shows that both *m* and zero are simultaneous *t*-best approximations to $S - l_0$ from *M* and this is a contradiction.

$(ii) \Rightarrow (i)$. Assume that $(i)$ does not hold. Then for some *F*-bounded subset *K* of *X*, $K/M$ has two distinct simultaneous *t*-best approximations such as $w + M$ and $w' + M$ in $W/M$. Thus $w - w' \notin M$. Since *M* is simultaneous *t*-proximinal there exists simultaneous *t*-best approximations *m* and $m'$ to $K - w$ and $K - w'$ from *M* respectively. Therefore $m \in S^t_M(K - w)$ and $m' \in S^t_M(K - w')$. Since $W + M \supseteq M$, $w + M$ and $w' + M$ are in $S^t_{W/M}(K/M) = S^t_{(K+M)/M}(K/M)$, by Theorem 4.10, $w + m$ and $w' + m' \in S^t_{W+M}(K)$. But $W + M$ is simultaneous *t*-Chebyshev. Thus $w + m = w' + m'$ and so $w - w' \in M$, which is a contradiction.

 

**Corollary 4.14:** Let $M$ be simultaneous $t$-Chebyshev subspace of a fuzzy $n$-normed linear space $(X, N)$. If $W \supseteq M$ is a simultaneous $t$-Chebyshev subspace in $X$, then the following assertions are equivalent:
(i) $W$ is simultaneous $t$-Chebyshev in $X$.
(ii) $W/M$ is simultaneous $t$-Chebyshev in $X/M$.

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