International Journal of Mathematical Archive-17(4), 2026, 1- 6
Available online through www.ijma.info ISSN 2229 – 5046

MODULO THEORY IN YAGER'S FERMATEAN FUZZY STRUCTURES

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(Received On: 13-04-26; Revised & Accepted On: 30-04-26)

ABSTRACT

In this article, we introduce the notion of a fermatean fuzzy sub-module. We prove various characteristics of fermatean fuzzy sub-module. We provide a relation between a fermatean fuzzy small module and a basic small module. Finally, some important properties regarding fermatean fuzzy small modules are investigated.

Keywords: *fuzzy set, intuitionistic fuzzy set, pythagorean fuzzy set, fermatean fuzzy set, sub-module, homomorphism.*

AMS subject classification (2020): *08A72, 03E72, 03B52, 94D05.*

1. INTRODUCTION

Zadeh introduced the concept of fuzzy set which was generalization of the classical set in 1965. This encourages many researchers to investigate set theory in fuzzy setting. Pythagorean fuzzy set is one of the most important fuzzy sets. It's importance ties behind the fact that this set can be applied in order to characterized uncertain data accurately. In order to tackle confusion and unpredictability in decision-making processes more effectively, Senapati.T introduced fermatean fuzzy sets in 2020, an extension of intuitionistic fuzzy sets (IFSs) and pythagorean fuzzy sets (PFSs). AVs and NAVs of fermatean fuzzy sets reveal their dependence on greater powers with sum of cubes is less than 1.Buyukozkan.G have done a systematic review on fermatean fuzzy sets. Revathy.A applied fermatean fuzzy normalized Bonferroni mean operator in MCDM for selection of electric bike and also investigated, generalization of fermatean fuzzy sets and applied fermatean fuzzy Promethae II method for decision making in. Ibrahim. H.Z introduced and applied n, m-rung orthopair fuzzy sets in MCDM. Kakati used fermatean fuzzy Archimedean Heronian Meon-Based model for MCDM.The concepts of fuzzy modules and fuzzy sub-modules was introduced by Nagotia and Ralescu in 1975. Since then, several authors have studied fuzzy modules [3, 7, 10]. The concept of essential fuzzy modules was introduced by Hadi in 2000. Using this idea. Abbas established the concept of essential fuzzy sub-modules and uniform fuzzy modules in 2012.

In this paper, we introduce the notion of a fermatean fuzzy sub-module. We prove various characteristics of fermatean fuzzy sub-module. We provide a relation between a fermatean fuzzy small module and a basic small module. Finally, some important properties regarding fermatean fuzzy small modules are investigated.

2. PRELIMINARIES

Definition 2.1: A fuzzy set 'A' in X is a set of ordered pairs $A = \{(x,\ J_A(x) / x \in X)\}$, where J_A is the grade of membership of $x \in A$ and $J_A : X \to [0, 1]$ is the membership function.

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Definition 2.2: Let$A = \{(x,\ J_A(x)/ x \in X)\}$ be a fuzzy set. The complement of A is defined as $A' = \{(x,\ K_A(x)/ x \in$ X=x, KAx=*1*−JAx.

Definition 2.3: A Pythagorean fuzzy set (PFS) 'A' of universe of discourse X is of the form
$A = \{(x,\ J_A(x),\ K_A(x)/ x \in X)\}$, where $J_A(x)$ and $K_A(x)$ are the membership and non-membership of x respectively in which $0 \le J_A(x) \le 1,\ 0 \le K_A(x) \le 1$ and $0 \le J^2{}_A(x) + K^2{}_A(x) \le 1$ for every $x \in X$.

Suppose if the condition $0 \le J^3{}_A(x) + K^3{}_A(x) \le 1$ for every $x \in X$ is called fermatean fuzzy set.

Definition 2.4: Let A, B be fermatean fuzzy sets in a fixed set X. Then
(i) A is a subset of B if for all $x \in X$, we have $J^3{}_A(x) \le J^3{}_B(x)$ and $K^3{}_A(x) \ge K^3{}_B(x)$.
(ii) $J^3{}_{A\cap B}(x) = \min\{J^3{}_A(x),\ J^3{}_B(x)\}$, for every $x \in X$ and $K^3{}_{A\cap B}(x) = \max\{K^3{}_A(x),\ K^3{}_B(x)\}$.
(iii) $J^3{}_{A\cup B}(x) = \max\{J^3{}_A(x),\ J^3{}_B(x)\}$, for every $x \in X$ and $K^3{}_{A\cup B}(x) = \min\{K^3{}_A(x),\ K^3{}_B(x)\}$.
(iv) $J^3{}_{A+B}(x) = J^3{}_A(x) + J^3{}_B(x) - J^3{}_A(x)J^3{}_B(x)$ and $K^3{}_{A+B}(x) = K^3{}_A(x).K^3{}_B(x)$.
Now, we are able to introduce the definition of the fermatean fuzzy sub-module.

Definition 2.5: Let M be an R-module and 'A' is called a fermatean fuzzy subset of M. Then 'A' is called a fermatean fuzzy sub-module of M, denoted by $A \le_{FF} M$, if the following conditions are satisfied:
(i) $J^3{}_A(0) = 1$and $J^3{}_A(1) = 0$.
(ii) $J^3{}_A(x+y) \ge \min\{J^3{}_A(x),\ J^3{}_A(y)\}$, for all $x, y \in M$ and
$K^3{}_A(x+y) \le \max\{K^3{}_A(x),\ K^3{}_A(y)\}$, for all $x, y \in M$.
(iii) $J^3{}_A(r\,x) \ge J^3{}_A(x)$ and $K^3{}_A(r\,x) \le K^3{}_A(x)$, for all $x \in M$ and $r \in R$.
Recall that for a module M, we define the fermatean fuzzy set $\chi_M{}^{FF} = (\chi_M,\ \chi_M{}^C)$ in which
$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M \\ 0, & \text{otherwise} \end{cases} \text{and } \chi_M{}^C(x) = \begin{cases} 0, & \text{if } x \in M \\ 1, & \text{otherwise} \end{cases}.$$

Definition 2.6: Let M be a module and A be a fermatean fuzzy subset of M, then
(i) $A^* = J^*{}_A \cap \tilde{J}^*{}_A$, where $J^*{}_A = \{x \in M / J_A(x) > 0\}$, $\tilde{J}^*{}_A = \{x \in M / J_A(x) < 1\}$.
(ii) $A_* = J_{*A} \cap \tilde{J}_{*A}$, where $J_{*A} = \{x \in M / J_A(x) = 1\}$, $\tilde{J}_{*A} = \{x \in M / J_A(x) = 0\}$.

3. STRUCTURES OF FERMATEAN FUZZY SUB-MODULE

Recall that a sub-module N of a module M is called a small subspace of module of M, denoted by $N \ll M$ if $N + S \ne M$ for every proper sub-module S of a module M.
Clearly, the zero sub-module is a small sub-module of any module M.
Moreover, a small sub-module of a module Mshould be a proper sub-module.

Now, we present some well-known structures (properties) regarding the concept of small sub-modules.

Theorem 3.1: Suppose that M is a module and S, T, N are sub-modules of M such that $S < T$. Then,
(i) $S + N \ll M$ if and only if $S \ll M$ and $N \ll M$.
(ii) $T \ll M$ if and only if $S \ll M$ and $\frac{T}{S} \ll \frac{M}{S}$.
(iii) If $S \ll T$, then $S \ll M$.
Now, we are ready to introduce the main concept in this article.
Consider a module M. Then a fermatean fuzzy set, $A = (J_A(x),\ K_A(x))$ is called a fermatean fuzzy small sub-module of M, denote $A \le_{FF} M$, if $A + S \ne \chi_M{}^{FF}$ for any $S \ne \chi_M{}^{FF}$. That is whenever$P + S = \chi_M{}^{FF}$, then$S = \chi_M{}^{FF}$.

Theorem 3.2: Let M be a module and A be a sub-module of M. Then $A \ll M$if $\chi_M{}^{FF} \le_{FF} M$.

Proof: Suppose that $\chi_A{}^{FF} \le_{FF} M$ and $P + S = M$ for some proper sub-module S of M. Then for any $a \in M$, there exist $x \in A$ and $y \in S$ such that $x + y = a$. We obtain
$$\left(J^3{}_{\chi_A}{}^{FF} + \chi^3{}_A{}^{FF}\right)(a) = \chi^3{}_A(a) + \chi^3{}_S(a) - \chi^3{}_A(a)\chi^3{}_S(a)$$
$$\ge \min\{\chi^3{}_A(x) + \chi^3{}_S(x) - \chi^3{}_A(x)\chi^3{}_S(x),\ \chi^3{}_A(y) + \chi^3{}_S(y) - \chi^3{}_A(y)\chi^3{}_S(y)\}$$
This means that,
$$\left(J^3{}_{\chi_A}{}^{FF} + \chi^3{}_A{}^{FF}\right)(a) = \chi^3{}_A{}^C(a)\chi^3{}_S{}^C(a) \le \max\left\{\chi^3{}_A{}^C(x)\chi^3{}_S{}^C(x), \chi^3{}_A{}^C(y)\chi^3{}_S{}^C(y)\right\} = 0.$$
This means that, $J^3{}_{\chi_A}{}^{FF} + \chi^3{}_A{}^{FF} = \chi^3{}_M{}^C$.
Thus $\chi_A{}^{FF} + \chi_S{}^{FF} = \chi_M{}^{FF}$, but this contradicts the facts that $\chi_A{}^{FF} \le_{FF} M$ and $\chi_S{}^{FF} \ne \chi_M{}^{FF}$ as S is a proper sub-module of M.
∴ A is a small sub-module of M.

Theorem 3.3: Let M be a module and A be a fermatean fuzzy sub-module of M. If $A \leq_{FF} M$, then $A_* \ll M$.

Proof: Assume that $A \leq_{FF} M$. In order to see that $A_* \ll M$, suppose that $A_* + S = M$ for a sub-module S of M.
We aim to show that $A + \chi_S{}^{FF} = \chi_M{}^{FF}$.
Let $a \in M$. Then $a = x + y$, for some $x \in A$ and $y \in S$. Then

$$J^3{}_{A+\chi_S{}^{FF}}(a) = J^3{}_A(a) + \chi^3{}_S(a) - J^3{}_A(a)\chi^3{}_S(a)$$
$$\geq \min\{J^3{}_A(x) + \chi^3{}_S(x) - J^3{}_A(x)\chi^3{}_S(x),\ J^3{}_A(y) + \chi^3{}_S(y) - J^3{}_A(y)\chi^3{}_S(y)\}$$

Moreover,

$$K^3{}_{A+\chi_S{}^{FF}}(a) = K^3{}_A(a)\chi^3{}_S{}^C(a) \leq \max\left\{K^3{}_A(x)\chi^3{}_S{}^C(x),\ K^3{}_A(y)\chi^3{}_S{}^C(y)\right\}$$

Thus $A + \chi_S{}^{FF} = \chi_M{}^{FF}$.
By hypothesis, $\chi_S{}^{FF} = \chi_M{}^{FF}$. Therefore $S = M$. Therefore $A_* \leq_{FF} M$.

Example 3.4: Consider the Z –module Z_{20} and the sub-module $S = \langle 5 \rangle$.
Let A be a fermatean fuzzy sub-module of Z_{20} defined as follows.

$$J_A(a) = \begin{cases} 1, & \text{if } a \in S \\ 1/4, & \text{otherwise} \end{cases} \text{and } K_A(a) = \begin{cases} 1, & \text{if } a \in S \\ 1/6, & \text{otherwise} \end{cases}$$

It is clear that A_* is not a small sub-module of Z_{20} as $A_* + \langle 2 \rangle = Z_{20}$.
Thus A_* is not a fermatean fuzzy small sub-module of Z_{20}.

Corollary 3.5: Let A, S be two fermatean fuzzy sub-modules of a module M in which $A \subseteq S$. Then $A \leq_{FF} S$ if and only if $A_* \leq_{FF} S^*$.

Proof: It is obvious.

Theorem 3.6: Let M be a module and S be a sub-module of M and A is a fermatean fuzzy sub-module of a module M in which $A \subseteq \chi_S{}^{FF}$. If $A/_S$ is a fermatean fuzzy small sub-module of S, then A is a fermatean fuzzy small sub-module of M.

Proof: Assume that Q is a fermatean fuzzy sub-module of M such that $A + Q = \chi_M{}^{FF}$.
In order to see that $A/_S + \left(Q/_S \cap \chi_S{}^{FF}\right)$.
Let $x \in S$. Then we obtain

$$J^3{}_{A/_S+\left(Q/_S \cap \chi_S{}^{FF}\right)}(x) = J^3{}_{A/_S}(x) + J^3{}_{Q/_S \cap \chi_S{}^{FF}}(x) - J^3{}_{A/_S}(x)J^3{}_{Q/_S \cap \chi_S{}^{FF}}(x)$$
$$= J^3{}_{A/_S}(x) + \min\left\{J^3{}_{Q/_S}(x),\ \chi^3{}_S(x)\right\} - J^3{}_{A/_S}(x)\min\left\{J^3{}_{Q/_S}(x),\ \chi^3{}_S(x)\right\}$$
$$= \min\{J^3{}_A(x),\ \chi^3{}_S(x)\} + \min\left\{J^3{}_Q(x),\ \chi^3{}_S(x)\right\} - \min\{J^3{}_A(x),\ \chi^3{}_S(x)\}\min\left\{J^3{}_Q(x),\ \chi^3{}_S(x)\right\}$$
$$= J^3{}_A(x) + J^3{}_Q(x) - J^3{}_A(x)J^3{}_Q(x) = J^3{}_{A+Q}(x) = \chi^3{}_M(x) = 1 = \chi^3{}_S(x) \text{ and}$$
$$K^3{}_{A/_S+\left(Q/_S \cap \chi_S{}^{FF}\right)}(x) = K^3{}_{A/_S}(x)K^3{}_{Q/_S \cap \chi_S{}^{FF}}(x)$$
$$= K^3{}_{A/_S}(x)\max\left\{K^3{}_{Q/_S}(x),\ \chi^3{}_S{}^C(x)\right\}$$
$$= \max\left\{K^3{}_A(x),\ \chi^3{}_S{}^C FF(x)\right\}\max\left\{K^3{}_Q(x),\ \chi^3{}_S{}^C FF(x)\right\} = K^3{}_A(x)K^3{}_Q(x) = K^3{}_{A+Q}(x) = \chi^3{}_M{}^C(x) = 0 = \chi^3{}_S{}^C(x)$$

This implies that $A/_S + \left(Q/_S \cap \chi_S{}^{FF}\right) = \chi_S{}^{FF}$.
By hypothesis, we conclude that $Q/_S \cap \chi_S{}^{FF} = \chi_S{}^{FF}$.
Thus $\chi_S{}^{FF} \subseteq Q/_S$. Then $\chi_M{}^{FF} = A + T \subseteq T \subseteq \chi_M{}^{FF}$.
Therefore, $Q = \chi_M{}^{FF}$ and A is fermatean fuzzy small sub-module of M.
As a consequence of the above theorem, we have the following corollary.

Corollary 3.7: Let M be a module. Also A and S are fermatean fuzzy sub-modules of M in which A⊆S. If A is a fermatean fuzzy small sub-module of S, then A is s fermatean fuzzy small sub-module of M.

Proof: It is obvious.

Note 3.8: The converse of the above theorem need not be true in general. That is if M is a module, S be a sub-module of M and A is a fermatean fuzzy sub-module of M in which $A \subseteq \chi_S{}^{FF}$, then it is not true in general that $A/_S$ is a fermatean fuzzy small sub-module of S. For example take $A/_S = S$.

Proposition 3.9: LetM be a module and A, S, Q be fermatean fuzzy sub-modules of M. Then

$$(A \cap S) + (A \cap Q) \subseteq A \cap (S \cap Q).$$

Proof: Let $a \in M$. Then

$$\begin{aligned} J^3{}_{(A\cap S)+(A\cap Q)}(a) &= J^3{}_{(A\cap S)}(a) + J^3{}_{(A\cap Q)}(a) - J^3{}_{(A\cap S)}(a)J^3{}_{(A\cap Q)}(a) \\ &= \min\{J^3{}_A(a),\ J^3{}_S(a)\} + \min\left\{J^3{}_A(a),\ J^3{}_Q(a)\right\} - \min\{J^3{}_A(a),\ J^3{}_S(a)\}\min\left\{J^3{}_A(a),\ J^3{}_Q(a)\right\} \\ &\le \min\left\{J^3{}_A(a),\ \left[J^3{}_S(a) + J^3{}_Q(a) - J^3{}_S(a)J^3{}_Q(a)\right]\right\} \\ &= \min\left\{J^3{}_A(a),\ J^3{}_{S+Q}(a)\right\} \\ &= J^3{}_{A\cap(S+Q)}(a) \end{aligned}$$

Moreover,

$$\begin{aligned} K^3{}_{(A\cap S)+(A\cap Q)}(a) &= K^3{}_{(A\cap S)}(a)K^3{}_{(A\cap Q)}(a) = \max\{K^3{}_A(a), K^3{}_S(a)\}\max\{K^3{}_A(a), K^3{}_Q(a)\} \\ &\ge \max\{K^3{}_A(a), K^3{}_S(a), K^3{}_Q(a)\} = K^3{}_{A\cap(S+Q)}(a) \end{aligned}$$

Proposition 3.10: Let M be a module, A and S are fermatean fuzzy sub-modules of M, in which $\chi_M{}^{FF} = A \oplus_{FF} S$. Then$M = A^* \oplus S^* = A_* \oplus S_*$.

Proof: Let $a \in M$. Then

$$1 = \chi^3{}_M(a) = J^3{}_{A+S}(a) = J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a).$$

This implies that, $J^3{}_A(a) = 1$ or $J^3{}_S(a) = 1$. So that $J^3{}_A(a) = 0$ or $J^3{}_S(a) = 0$.
Hence $a \in A_*$ or $a \in S_*$, so that $M = A + S_*$.
Hence $\qquad M = A_* + S_*$
Our aim is to show that $A^* \cap S^*$.
Assume that $a \in A^* \cap S^*$. Then $J^3{}_P(a)J^3{}_S(a) > 0$.
Since $\qquad \chi_M{}^{FF} = A \oplus_{FF} S$, we obtain $0 < \min\{J^3{}_P(a),\ J^3{}_S(a)\} = \chi^3{}_0(a)$.
Which means that $a = 0$ and hence, $A_* \cap S_* \subseteq A^* \cap S^* = 0$.
Hence the proof.

Now, we able to clear that the converse of corollary-3.7 is true if S is a fermatean fuzzy direct summand of Mas follow.

Proposition 3.11: Let M be a module, A and S are fermatean fuzzy sub-modules of M, in which A⊆S and S is a fermatean fuzzy direct summand of M. Then A is a fermatean fuzzy small sub-module of S if and only if A is a fermatean fuzzy sub-module of M.

Proof: Assume that A is a fermatean fuzzy small sub-module of M. Applying theorem-3.3, A_* is a small sub-module of M. That S is a fermatean fuzzy direct summand of M and $A_* \subseteq S^*$, implies that A_* is a small sub-module of S^*. Applying corollary 3.7, this result hold.

Theorem 3.12: Let M be a module and S be a fermatean fuzzy sub-module of M such that $A \cap S = \chi_0{}^{FF}$. Then

(i) $(A \oplus_{FF} S)^* = A^* \oplus S^*$,
(ii) $(A \oplus_{FF} S)_* = A_* \oplus S_*$.

Proof:

(i) Since $A \cap S = \chi_0{}^{FF}$, we need to show that $(A + S)^* = A^* + S^*$.
Suppose that $a \in (A + S)^*$. By definition, we have $J^3{}_{A+S}(a) > 0$.
This implies that $0 < J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a)$
This means that $J^3{}_A(a) \ne 0$ or $J^3{}_S(a) \ne 0$.
Moreover, $1 > K^3{}_{A+S}(a) = K^3{}_A(a)K^3{}_S(a)$.
This implies that $K^3{}_A(a) < 1$ or $K^3{}_S(a) < 1$.
Thus $a \in A^*$or $a \in S^*$, so that $a \in A^* + S^*$. Then $(A + S)^* \subseteq A^* + S^*$.
Now, suppose that $a = x_1 + y_1 \in A^* + S^*$, where $x_1 \in A^*$ and $y_1 \in S^*$.
By definition, $J^3{}_A(x_1), J^3{}_S(y_1) > 0$. Thus,

$$\begin{aligned} 0 &< \min\left\{J^3{}_A(x_1)\left(1 - J^3{}_S(x_1) + J^3{}_S(x_1)\right),\ J^3{}_A(y_1)\left(1 - J^3{}_S(y_1) + J^3{}_S(y_1)\right)\right\} \\ &= \min\{J^3{}_A(x_1) + J^3{}_S(x_1) - J^3{}_A(x_1)J^3{}_S(x_1), J^3{}_A(y_1) + J^3{}_S(y_1) - J^3{}_A(y_1)J^3{}_S(y_1)\} \\ &\le J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_{A+S}(a). \end{aligned}$$

Moreover, $K^3{}_A(x_1), K^3{}_S(y_1) < 1$, which implies that

$$1 > \max\{K^3{}_A(x_1)K^3{}_S(x_1),\ K^3{}_A(y_1)K^3{}_S(y_1)\} \ge K^3{}_A(a)K^3{}_S(a) = K^3{}_{A+S}(a).$$

Thus $a \in (A + S)^*$. Then $A^* + S^* \subseteq (A + S)^*$.
Therefore, we have $(A + S)^* = A^* + S^*$.

(ii) Since $A \cap S = \chi_0^{FF}$, we need to show that $(A+S)_* = A_* + S_*$.
Suppose that $a \in (A+S)_*$. By definition, we have $J^3{}_{A+S}(a) = 1$.
This implies that $1 = J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a)$
which means that $J^3{}_A(a) = 1$ or $J^3{}_S(a) = 1$.
Moreover, $0 = K^3{}_{A+S}(a) = K^3{}_A(a)K^3{}_S(a)$,
which implies that $J^3{}_A(a) = 0$ or $J^3{}_S(a) = 0$.
Thus $a \in A_*$ or $a \in S_*$, so that $a \in A_* + S_*$. Then $(A+S)_* \subseteq A_* + S_*$.
Now, suppose that $a = x_1 + y_1 \in A_* + S_*$, where $x_1 \in A_*$ and $y_1 \in S_*$.
By definition-2.6, $J^3{}_A(x_1), J^3{}_S(y_1) = 1$. Thus,

$$1 = \min\{J^3{}_A(x_1) + J^3{}_S(x_1) - J^3{}_A(x_1)J^3{}_S(x_1), J^3{}_A(y_1) + J^3{}_S(y_1) - J^3{}_A(y_1)J^3{}_S(y_1)\}$$
$$\leq J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_{A+S}(a)$$

Moreover, $K^3{}_A(x_1), K^3{}_S(y_1) \leq 0$, which implies that

$$0 = \max\{K^3{}_A(x_1)K^3{}_S(x_1),\ K^3{}_A(y_1)K^3{}_S(y_1)\} \geq K^3{}_A(a)K^3{}_S(a) = K^3{}_{A+S}(a).$$

Thus $a \in (A+S)_*$. Then $A_* + S_* \subseteq (A+S)_*$
Therefore, we have $(A+S)_* = A_* + S_*$.

4. HOMOMORPHISM OF MODULES

Definition 4.1: Let A, S be two R-modules and $U \leq_{FF} A$ and $V \leq_{FF} S$. Consider an R-homomorphism $\varphi: A \to S$.
For $s \in S$, we define

$$J_{\varphi(U)}(s) = \begin{cases} \max\{J_U(a): s = \varphi(a)\}, & \text{if } s \in Im(\varphi) \\ 0, & \text{otherwise} \end{cases} \text{ and } K_{\varphi(U)}(s) = \begin{cases} \min\{J_U(a): s = \varphi(a)\}, & \text{if } s \in Im(\varphi) \\ 0, & \text{otherwise} \end{cases}.$$

Now, we are ready to prove the following:

Theorem 4.2: Let $\varphi: A \to S$ be a homomorphism of modules. If 'D' is a fermatean fuzzy small sub-module of A, then $\varphi(D)$ is a fermatean fuzzy sub-module of S.

Proof: Suppose that $\varphi(D) + U = \chi_S{}^{FF}$. Our aim is to prove that $U = \chi_S{}^{FF}$.

Let $s \in S$, then $1 = J^3{}_{\varphi(D)+U}(s) = J^3{}_{\varphi(D)}(s) + J^3{}_U(s) - J^3{}_{\varphi(D)}(s)J^3{}_U(s)$.

In this case that $s \notin Im(\varphi)$, we obtain

$$1 = J^3{}_{\varphi(D)}(s) + J^3{}_U(s) - J^3{}_{\varphi(D)}(s)J^3{}_U(s) = J^3{}_U(s) \text{ and so} 1 = J^3{}_U(s) \text{ and } K^3{}_U(s) = 0.$$

If $s \in Im(\varphi)$, we have

$$1 = J^3{}_{\varphi(D)}(s) + J^3{}_U(s) - J^3{}_{\varphi(D)}(s)J^3{}_U(s)$$
$$= \max\{J^3{}_D(a)/\varphi(a) = s\} + J^3{}_U(s) - \max\{J^3{}_D(a)/\varphi(a) = s\}J^3{}_U(s)$$
$$= J^3{}_D(a) + J^3{}_U(s) - J^3{}_D(a)J^3{}_U(s), \text{ for some a in which } \varphi(a) = s.$$
$$= J^3{}_D(a)\left(1 - J^3{}_U(s)\right) + J^3{}_U(s).$$

If $J^3{}_D(a) = 1$, then $D = \chi_A$ and this is a contradiction with the fact that D is a fermatean fuzzy small sub-module of A.
Thus $J^3{}_U(s) = 1$ and $D = \chi_S$.

Moreover, $0 = K^3{}_{\varphi(D)+U}(s) = K^3{}_{\varphi(D)}(s)K^3{}_U(s) = K^3{}_D(a)K^3{}_D(s)$, for some a in which $\varphi(a) = s$.

Note that φ is one to one and so 'a' is unique.

By hypothesis $K^3{}_D(a) \neq 0$, so that $K^3{}_U(a) = 0$.

Hence $D = \chi_S{}^{FF}$.

Note 4.3:

(i) If φ is not one to one, then the above theorem need not be true. For instance, take S is the zero module and φ the zero homomorphism.

(ii) The converse of the above theorem need not be true. That is $\varphi: A \to S$ is a monomorphism of modules, D is a fermatean fuzzy sub-module of A and $\varphi(D)$ is a fermatean fuzzy small sub-module of S, then it is not true in general that D is a fermatean fuzzy sub-module of A. For example, Let A be a fermatean fuzzy small sub-module of S and consider the inclusion $A \to S$. Then $\varphi(A) = A$ is a fermatean fuzzy sub-module of S but A is not fermatean fuzzy sub-module of A.

CONCLUSION

In this article, we introduce the notion of a fermatean fuzzy sub-module. In addition, we discuss the concept fermatean fuzzy sub-modules and investigate some results regarding this idea. Moreover, we find a relationship between small sub-module and fermatean fuzzy sub-module. We also study homomorphism between fermatean fuzzy modules.

FUTURE DIRECTION

This work can be extended in generalized orthopair fuzzy sets. It can be applied in order to solve multi-criteria decision making problems.

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Source of support: Nil, Conflict of interest: None Declared.