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GPR-Separation Axioms – I

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ABSTRACT

In this paper by using gpr-open sets we define almost gpr-normality and mild gpr-normality also we continue the study of further properties of gpr-normality. We show that these three axioms are regular open hereditary. We also define the class of almost gpr-irresolute mappings and show that gpr-normality is invariant under almost gpr-irresolute M-gpr-open continuous surjection.

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Key words and Phrases: *gpr*-open, semiopen, semipreopen, almost normal, midly normal, M-*gpr*-closed, M-*gpr*-open, rc-continuous.

1. INTRODUCTION

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C.E. Aull studied some separation axioms between T_1 and T_2 spaces, namely, S_1 and S_2. Next, in 1982, S.P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of s-convergence, sequentially semi-closed sets, sequentially s-compact notions. G.B. Navalagi studied P-Normal Almost-P-Normal, Mildly-P-Normal and Pre-US spaces. Recently S. Balasubramanian and P.Aruna Swathi Vyjayanthi studied *v*-Normal Almost- *v*-Normal, Mildly-*v*-Normal and *v*-US spaces. Inspired with these we introduce *gpr*-Normal Almost- *gpr*-Normal, Mildly- *gpr*-Normal, *gpr*-US, *gpr*-S_1 and *gpr*-S_2. Also we examine *gpr*-convergence, sequentially *gpr*-compact, sequentially *gpr*-continuous maps, and sequentially sub *gpr*-continuous maps in the context of these new concepts. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper X and Y denote Topological spaces on which no separation axioms are assumed explicitly stated.

2. PRELIMINARIES

Definition 2.1: $A\subset X$ is called
(i) r-open if $A = (cl(A))^o$.
(ii) rα-open [*v*-open] if $\exists\ U\in \alpha O(X)[RO(X)]$ such that $U\subset A\subset \alpha cl(U)[\ U\subset A\subset cl(U)]$.
(iii) r-closed[α-closed; pre-closed; β−closed] if $A = cl(A^o)[(cl(A^o))^o \subseteq A;\ cl(A^o) \subseteq A;\ cl((cl(A))^o)\subseteq A]$.
(iv) Semi closed [*v*-closed] if its complement if semi open[*v*-open].
(v) g-closed [rg-closed] if $cl\ A\subseteq U$ whenever $A\subseteq U$ and U is open in X.
(vi) pg-closed[gp-closed; gpr-closed] if $pcl(A) \subseteq U$ whenever $A\subseteq U$ and U is pre-open[open; regular-open] in X.
(vii) αg-closed if $\alpha cl(A) \subseteq U$ whenever $A\subseteq U$ and U is open in X.

Definition 2.2: A function *f* is said to be almost–*v*-irresolute if for each x in X and each *v*-neighborhood V of *f*(x), $v cl(f^{-1}(V))$ is a *v*-neighborhood of x.

Definition 2.3: A space X is said to be
(i) *gpr*-T_1 (*gpr*-T_2) if for any $x \neq y$ in X, there exist (disjoint) U; $V\in GPRO(X)$ such that $x\in U$ and $y\in V$.
(ii) weakly Hausdorff if each point of X is the intersection of regular closed sets of X.
(iii) *gpr*-normal if for any pair of disjoint closed sets F_1 and F_2, there exist disjoint *gpr*-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

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(iv) R_0 if for any point x and a closed set F with $x \notin F$ in X, there exists a open set G containing F but not x.
(v) R_1[*gpr*- R_1] iff for x, y $\in$ X with cl{x} $\neq$ cl{y}[*gpr*cl{x} $\neq$ *gpr*cl{y}], there exist disjoint open[*gpr*-open] sets U and V such that cl{x}$\subset$ U, cl{y}$\subset$V.[*gpr*cl{x} $\subset$ U and *gpr*cl{y} $\subset$ V.]

Definition 2.4: Let A$\subset$ X. Then a point x is said to be a
(i) limit point of *A* if each open set containing x contains some point y of *A* such that x $\neq$ y.
(ii) T_0–limit point of *A* if each open set containing x contains some point y of *A* such that cl{x} $\neq$ cl{y}, or equivalently, such that they are topologically distinct.

Note 1: Recall that two points are topologically distinguishable or distinct if there exists an open set containing one of the points but not the other; equivalently if they have disjoint closures. In fact, the T_0–axiom is precisely to ensure that any two distinct points are topologically distinct.

Example 1: Let X = {a, b, c, d} and τ = {{a}, {b, c}, {a, b, c}, X, ϕ}. Then b and c are the limit points but not the T_0–limit points of the set {b, c}. Further d is a T_0–limit point of {b, c}.

Example 2: Let X = (0, 1) and τ = {ϕ, X, and U_n = (0, 1–1⁄n), n = 2, 3, 4 . . .}. Then every point of X is a limit point of X. Every point of X~U_2 is a T_0–limit point of X, but no point of U_2 is a T_0–limit point of X.

Definition 2.5: A set *A* together with all its T_0–limit points will be denoted by T_0–cl*A*.

Note 2: (i). Every T_0–limit point of a set *A* is a limit point of the set but the converse is not true in general.
(ii) In T_0–space both are same.

Note 3: R_0–axiom is weaker than T_1–axiom. It is independent of the T_0–axiom. However $T_1 = R_0+T_0$

Note 4: Every countable compact space is weakly countable compact but converse is not true in general. However, a T_1–space is weakly countable compact iff it is countable compact.

Note 5: We have the following interrelation among different closed sets.

closed → g-closed → αg-closed → gp-closed → gpr-closed
↑ ↑
regular-closed → → → → → → → → → → → rg-closed none is reversible

3. *gpr*-T_0 LIMIT POINT:

Definition 3.01: In X, a point x is said to be a *gpr*-T_0–limit point of *A* if each *gpr*-open set containing x contains some point y of *A* such that *gpr*cl{x} $\neq$ *gpr*cl{y}, or equivalently; such that they are topologically distinct with respect to *gpr*-open sets.

Example 3: Since every regular open set is open set and every open set is gpr-open set, every r-T_0–limit point is a T_0–limit point and every T_0–limit point is a gpr-T_0–limit point of the set.

Definition 3.02: A set A together with all its *gpr*-T_0–limit points is denoted by T_0-*gpr*cl (A)

Lemma 3.01: If x is a gpr-T_0–limit point of a set A then x is gpr-limit point of A.

Lemma 3.02: If X is gpr-T_0–space then every *gpr*-T_0–limit point and every gpr-limit point are equivalent.

Corollary 3.03: If X is r-T_0–space then every *gpr*-T_0–limit point and every gpr-limit point are equivalent.

Theorem 3.04: For x $\neq$ y $\in$ X,
(i) x is a gpr-T_0–limit point of {y} iff $x \notin$ gprcl{y} and $y \notin$ gprcl{x}.
(ii) x is not a gpr-T_0–limit point of {y} iff either $x \in$ gprcl{y}or gprcl{x}= gprcl{y}.
(iii) x is not a gpr-T_0–limit point of {y} iff either $x \in$ gprcl{y}or $y \in$ gprcl{x}.

Corollary 3.05:
(i) If x is a gpr-T_0–limit point of {y}, then y cannot be a gpr-limit point of {x}.
(ii) If gprcl{x}= gprcl{y}, then neither x is a gpr-T_0–limit point of {y} nor y is a gpr-T_0–limit point of {x}.
(iii) If a singleton set A has no gpr-T_0–limit point in X, then gprclA = gprcl{x} for all $x \in$ gprcl{A}.

Lemma 3.06: In X, if x is a gpr-limit point of a set A, then in each of the following cases x becomes gpr-T_0–limit point of A ($\{x\} \neq A$).

(i) gprcl{x} $\neq$ gprcl{y} for $y \in A$, $x \neq y$.
(ii) gprcl{x} = {x}
(iii) X is a gpr-T_0–space.
(iv) A~{x} is gpr-open

Corollary 3.07: In X, if x is a limit point of a set A, then in each of the following cases x becomes gpr-T_0–limit point of A ($\{x\} \neq A$).

(i) gprcl{x} $\neq$ gprcl{y} for $y \in A$, $x \neq y$.
(ii) gprcl{x} = {x}
(iii) X is a gpr-T_0–space.
(iv) A~{x} is gpr-open

4. gpr-$\mathbf{T_0}$ **AND** gpr-$\mathbf{R_i}$ **AXIOMS, i = 0, 1:**

In view of Lemma 3.6(iii), gpr-T_0–axiom implies the equivalence of the concept of limit point of a set with that of gpr-T_0–limit point of the set. But for the converse, if $x \in$ gprcl{y} then gprcl{x} $\neq$ gprcl{y} in general, but if x is a gpr-T_0–limit point of {y}, then gprcl{x} = gprcl{y}

Lemma 4.01: In a space X, a limit point x of {y} is a gpr-T_0–limit point of {y} iff gprcl{x} $\neq$ gprcl{y}.

This lemma leads to characterize the equivalence of gpr-T_0–limit point and gpr-limit point of a set as the gpr-T_0–axiom.

Theorem 4.02: The following conditions are equivalent:

(i) X is a gpr-T_0 space
(ii) Every gpr-limit point of a set A is a gpr-T_0–limit point of A
(iii) Every r-limit point of a singleton set {x} is a gpr-T_0–limit point of {x}
(iv) For any x, y in X, $x \neq y$ if $x \in$ gprcl{y}, then x is a gpr-T_0–limit point of {y}

Note 5: In a *gpr*-T_0–space X if every point of X is a r-limit point of X, then every point of X is *gpr*-T_0–limit point of X. But a space X in which each point is a *gpr*-T_0–limit point of X is not necessarily a *gpr*-T_0–space

Theorem 4.03: The following conditions are equivalent:

(i) X is a gpr-R_0 space
(ii) For any x, y in X, if $x \in$ gprcl{y}, then x is not a gpr-T_0–limit point of {y}
(iii) A point gpr-closure set has no gpr-T_0–limit point in X
(iv) A singleton set has no gpr-T_0–limit point in X.

Since every r-R_0–space is *gpr*-R_0–space, we have the following corollary

Corollary 4.04: The following conditions are equivalent:

(i) X is a r-R_0 space
(ii) For any x, y in X, if $x \in$ gprcl{y}, then x is not a gpr-T_0–limit point of {y}
(iii) A point gpr-closure set has no gpr-T_0–limit point in X
(iv) A singleton set has no gpr-T_0–limit point in X.

Theorem 4.05: In a gpr-R_0 space X, a point x is gpr-T_0–limit point of A iff every gpr-open set containing x contains infinitely many points of A with each of which x is topologically distinct

If *gpr*-R_0 space is replaced by rR_0 space in the above theorem, we have the following corollaries:

Corollary 4.06: In an rR_0–space X,

(i) If a point x is rT_0–limit point of a set then every gpr-open set containing x contains infinitely many points of A with each of which x is topologically distinct.
(ii) If a point x is gpr-T_0–limit point of a set then every gpr-open set containing x contains infinitely many points of A with each of which x is topologically distinct.

Theorem 4.07: X is gpr-R_0 space iff a set A of the form A = $\cup$ gprcl{$x_{i\ i=1\ \text{to}\ n}$}, a finite union of point closure sets has no gpr-T_0–limit point.

Corollary 4.08: If X is rR_0 space and

(i) If A = $\cup$ gprcl{x_i} i =1 to n, a finite union of point closure sets has no gpr-T_0–limit point.

(ii)If $X = \cup$ gprcl$\{x_i\}$ i =1 to n ,then X has no gpr-T_0–limit point.

Theorem 4.09: The following conditions are equivalent:
(i) X is gpr-R_0–space
(ii) For any x and a set A in X, x is a gpr-T_0–limit point of A iff every gpr-open set containing x contains infinitely many points of A with each of which x is topologically distinct.

Various characteristic properties of *gpr*-T_0–limit points studied so far is enlisted in the following theorem for a ready reference.

Theorem 4.10: In a gpr-R_0–space, we have the following:
(i) A singleton set has no gpr-T_0–limit point in X.
(ii) A finite set has no gpr-T_0–limit point in X.
(iii) A point gpr-closure has no set gpr-T_0–limit point in X
(iv) A finite union point gpr-closure sets have no set gpr-T_0–limit point in X.
(v) For x, $y\in$ X, $x\in T_0$– gprcl$\{y\}$ iff x = y.
(vi) For any x, $y\in$ X, $x \neq y$ iff neither x is gpr-T_0–limit point of $\{y\}$nor y is gpr-T_0–limit point of $\{x\}$
(vii)For any x, $y\in$ X, $x \neq y$ iff T_0– gprcl$\{x\}$ $\cap T_0$– gprcl$\{y\}$ = ϕ.
(viii) Any point $x\in X$ is a gpr-T_0–limit point of a set A in X iff every gpr-open set containing x contains infinitely many points of A with each which x is topologically distinct.

Theorem 4.11: X is gpr-R_1 iff for any gpr-open set U in X and points x, y such that $x\in X\sim U$, $y\in U$, there exists a gpr-open set V in X such that $y\in V\subset U$, $x\notin V$.

Lemma 4.12: In gpr-R_1 space X, if x is a gpr-T_0–limit point of X, then for any non empty gpr-open set U, there exists a non empty gpr-open set V such that $V\subset U$, $x\notin$ gprcl(V).

Lemma 4.13: In a gpr- regular space X, if x is a gpr-T_0–limit point of X, then for any non empty gpr-open set U, there exists a non empty gpr-open set V such that gprcl(V)$\subset$U, $x\notin$ gprcl(V).

Corollary 4.14: In a regular space X,
(i) If x is a gpr-T_0–limit point of X, then for any non empty gpr-open set U, there exists a non empty gpr-open set V such that gprcl(V)$\subset$U, $x\notin$ gprcl(V).
(ii) If x is a T_0–limit point of X, then for any non empty gpr-open set U, there exists a non empty gpr-open set V such that gprcl(V)$\subset$U, $x\notin$ gprcl(V).

Theorem 4.15: If X is a gpr-compact gpr-R_1-space, then X is a Baire Space.

Proof: Let $\{A_n\}$ be a countable collection of gpr-closed sets of X, each A_n having empty interior in X. Take A_1, since A_1 has empty interior, A_1 does not contain any gpr-open set say U_0. Therefore we can choose a point $y\in U_0$ such that $y\notin A_1$. For X is gpr-regular, and $y\in (X\sim A_1)\cap U_0$, a gpr-open set, we can find a gpr-open set U_1 in X such that $y\in U_1$, gprcl(U_1) $\subset(X\sim A_1)\cap U_0$. Hence U_1 is a non empty gpr-open set in X such that gprcl(U_1)$\subset U_0$ and gprcl(U_1)$\cap A_1 = \phi$. Continuing this process, in general, for given non empty gpr-open set U_{n-1} , we can choose a point of U_{n-1} which is not in the gpr-closed set A_n and a gpr-open set U_n containing this point such that gprcl(U_n) $\subset U_{n-1}$ and gprcl(U_n)$\cap A_n = \phi$. Thus we get a sequence of nested non empty gpr-closed sets which satisfies the finite intersection property. Therefore $\cap$ gprcl(U_n) $\neq \phi$. Then some $x\in \cap$ gprcl(U_n) which in turn implies that $x\in U_{n-1}$ as gprcl(U_n)$\subset U_{n-1}$ and $x\notin A_n$ for each n.

Corollary 4.16: If X is a compact gpr-R_1-space, then X is a Baire Space.

Corollary 4.17: Let X be a gpr-compact gpr-R_1-space. If $\{A_n\}$ is a countable collection of gpr-closed sets in X, each A_n having non-empty gpr-interior in X, then there is a point of X which is not in any of the A_n.

Corollary 4.18: Let X be a gpr-compact R_1-space. If $\{A_n\}$ is a countable collection of gpr-closed sets in X, each A_n having non-empty gpr- interior in X, then there is a point of X which is not in any of the A_n.

Theorem 4.19: Let X be a non empty compact gpr-R_1-space. If every point of X is a gpr-T_0–limit point of X then X is uncountable.

Proof: Since X is non empty and every point is a gpr-T_0-limit point of X, X must be infinite. If X is countable, we construct a sequence of gpr- open sets $\{V_n\}$ in X as follows:

Let X = V_1, then for x_1 is a gpr-T_0-limit point of X, we can choose a non empty gpr-open set V_2 in X such that $V_2 \subset V_1$ and $x_1 \notin$ gprclV_2. Next for x_2 and non empty gpr-open set V_2, we can choose a non empty gpr-open set V_3 in X such that $V_3 \subset V_2$ and $x_2 \notin$ gprclV_3. Continuing this process for each x_n and a non empty gpr-open set V_n , we can choose a non empty gpr-open set V_{n+1} in X such that $V_{n+1} \subset V_n$ and $x_n \notin$ gprclV_{n+1}.

Now consider the nested sequence of gpr-closed sets gprcl$V_1 \supset$ gprcl$V_2 \supset$ gprcl$V_3 \supset \ldots \supset$ gprcl$V_n \supset \ldots$ Since X is gpr-compact and {gprclV_n} the sequence of gpr-closed sets satisfies finite intersection property. By Cantors intersection theorem, there exists an x in X such that $x \in$ gprclV_n. Further $x \in X$ and $x \in V_1$, which is not equal to any of the points of X. Hence X is uncountable.

Corollary 4.20: Let X be a non empty gpr-compact gpr-R_1-space. If every point of X is a gpr-T_0–limit point of X then X is uncountable

5. *gpr*–T_0-IDENTIFICATION SPACES AND *gpr*–SEPARATION AXIOMS

Definition 5.01: Let *(X, τ)* be a topological space and let $\Re$ be the equivalence relation on X defined by x$\Re$y iff *gprcl*{x} = *gprcl*{y}

Problem 5.02: show that x$\Re$y iff *gprcl*{x} = *gprcl*{y} is an equivalence relation

Definition 5.03: The space (X_0, Q(X_0)) is called the *gpr*-T_0–identification space of *(X, τ)*, where X_0 is the set of equivalence classes of $\Re$ and Q(X_0) is the decomposition topology on X_0.
Let P_X: $(X, \tau) \rightarrow (X_0, Q(X_0))$ denote the natural map

Lemma 5.04: If $x \in X$ and $A \subset X$, then $x \in$ gpr-clA iff every gpr-open set containing x intersects A.

Theorem 5.05: The natural map $P_X:(X,\tau) \rightarrow (X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O \in PO(X,\tau)$ and (X_0, Q(X_0)) is gpr-T_0

Proof: Let $O \in PO(X, \tau)$ and let $C \in P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then *gprcl*{y} = *gprcl*{x}, which, by lemma, implies $y \in O$. Since $\tau \subset PO(X, \tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \in \tau$, which implies P_X is closed and open.

Let G, H$\in$ X_0 such that G ≠ H and let $x \in G$ and $y \in H$. Then *gprcl*{x} ≠ *gprcl*{y}, which implies that $x \notin$ *gprcl*{y} or $y \notin$ *gprcl*{x}, say $x \notin$ *gprcl*{y}. Since P_X is continuous and open, then $G \in A = P_X\{X\sim$ *gprcl*{y}$\} \notin PO(X_0, Q(X_0))$ and $H \notin A$

Theorem 5.06: The following are equivalent:
(i) X is gprR_0 (ii) X_0 = { gprcl{x}: $x \in X$}, and (iii) (X_0, Q(X_0)) is gprT_1

Proof: (i) ⇒ (ii) Let $C \in X_0$, and let $x \in C$. If $y \in C$, then $y \in$ *gprcl*{y} = *gprcl*{x}, which implies $C \in$ *gprcl*{x}. If $y \in$ *gprcl*{x}, then $x \in$ *gprcl*{y}, since, otherwise, $x \in X\sim$ *gprcl*{y}$\in PO(X, \tau)$ which implies *gprcl*{x}$\subset X\sim$ *gprcl*{y}, which is a contradiction. Thus, if $y \in$ *gprcl*{x}, then $x \in$ *gprcl*{y}, which implies *gprcl*{y} = *gprcl*{x} and $y \in C$. Hence X_0 = {*gprcl*{x}: $x \in X$}

(ii) ⇒ (iii) Let A ≠ B$\in X_0$. Then ∃ x, y$\in$X s.t A = *gprcl*{x}; B = *gprcl*{y}, and *gprcl*{x}$\cap$*gprcl*{y} = ϕ. Then $A \in C = P_X(X\sim$*gprcl*{y}$) \in PO(X_0, Q(X_0))$ and $B \notin C$. Thus $(X_0, Q(X_0))$ is *gpr*-T_1

(iii) ⇒ (i) Let $x \in U \in$ *gprO(X)*. Let $y \notin U$ and C_x, $C_y \in X_0$ containing x and y respectively. Then $x \notin$ *gprcl*{y}, which implies $C_x \neq C_y$ and there exists *gpr*-open set A such that $C_x \in A$ and $C_y \notin A$. Since P_X is continuous and open, then $y \in B = P_X^{-1}(A) \in x \in$ *gprO(X)* and $x \notin B$, which implies $y \notin$ *gprcl*{x}. Thus *gprcl*{x}$\subset U$. This is true for all *gprcl*{x} implies $\cap$*gprcl*{x}$\subset U$. Hence *X* is *gpr*-R_0

Theorem 5.07: *(X, τ) is gpr-R_1 iff (X_0, Q(X_0)) is gpr-T_2*

The proof is straight forward from using theorems 5.05 and 5.06 and is omitted

6. *gpr*–OPEN FUNCTIONS AND *gpr*–T_i SPACES, i = 0, 1, 2.

Theorem 6.01: X is gpr-T_i ; i = 0,1,2. iff there exists a gpr-continuous, almost–open, 1–1 function from (X, τ) into a gpr-T_i space ; i = 0,1,2. respectively.

Proof: If X is *gpr*-T_i; i = 0, 1, 2, then the identity function on X satisfies the desired properties. The converse is (ii) part of Theorem 2.13.

The following example shows that if f: $(X, \tau) \to (Y, \sigma)$ is continuous, *gpr*-open, bijective, $A \in PO(Y, \sigma)$, and (Y, σ) *gpr*- T_i ; i = 0,1,2, then $f^{-1}(A)$ need not be *gpr*–open and (X, τ) need not be *gpr*-T_i ; i = 0,1,2

Theorem 6.02: *If f: $(X,\tau) \to (Y,\sigma)$ is gpr-continuous, gpr-open, and x, y∈X such that gprcl{x} = gprcl{y}, then gprcl{f(x)} = gprcl{f(y)}.*

Proof: Suppose $gpr\mathrm{cl}\{f(x)\} \neq gpr\mathrm{cl}\{f(y)\}$. Then $f(x) \notin gpr\mathrm{cl}\{f(y)\}$ or $f(y) \notin gpr\mathrm{cl}\{f(x)\}$, say $f(x) \notin gpr\mathrm{cl}\{f(y)\}$. Then $f(x) \in A = Y \sim gpr\text{-}\mathrm{cl}\{f(y)\} \in gprO(Y)$. If $B = Y \sim \mathrm{cl}(\mathrm{int}(A)) \neq \phi$, then $f(x) \notin B$, and $B \cap gpr\mathrm{cl}\{f(y)\} \neq \phi$, which implies $f(y) \in B$, $y \in f^{-1}(B) \in gprO(X)$, and $x \notin f^{-1}(B)$ which is a contradiction. Thus $\mathrm{cl}(\mathrm{int}(A)) = Y$. Since $f(y) \notin A$, then $y \notin \mathrm{int} f^{-1}(A)$. If $x \in \mathrm{cl}(\mathrm{int}(f^{-1}(A)))$, then $\{x\} \cup \mathrm{int}(f^{-1}(A))$ is *gpr*-open containing x and not y, which is a contradiction.

Hence $x \in U = X \sim \mathrm{cl}(\mathrm{int}(f^{-1}(A))) \in \tau$ and $\phi \neq f(U) \in gprO(Y)$. Then $C = (\mathrm{int} f(U)) \cap \mathrm{int}(A) = \phi$, for suppose not. Then $f^{-1}(C) \in gprO(X)$, which implies $f^{-1}(C) \subset \mathrm{cl}(\mathrm{int}(f^{-1}(C))) \subset \mathrm{cl}(\mathrm{int}(f^{-1}(A)))$, which is a contradiction. Hence $C = \phi$, which contradicts $\mathrm{cl}(\mathrm{int}(A)) = Y$.

Theorem 6.03: The following are equivalent
(i) (X,τ) is gpr-T_0
(ii) Elements of X_0 are singleton sets and
(iii)There exists a gpr-continuous, gpr-open, 1–1 function f: $(X, \tau) \to (Y, \sigma)$, where (Y, σ) is gpr-T_0

Proof: (i) is equivalent to (ii) and (i) ⇒ (iii) are straight forward and is omitted.

(iii) ⇒ (i) Let x, y∈X such that $f(x) \neq f(y)$, which implies $gpr\text{-}\mathrm{cl}\{f(x)\} \neq gpr\text{-}\mathrm{cl}\{f(y)\}$. Then by theorem 6.02, $gpr\text{-}\mathrm{cl}\{x\} \neq gpr\text{-}\mathrm{cl}\{y\}$. Hence (X, τ) is *gpr*-T_0

Corollary 6.04: A space (X, τ) is gpr-T_i ; i = 1,2 iff (X, τ) is gpr-T_{i-1} ; i = 1,2, respectively, and there exists a gpr-continuous , gpr-open, 1–1 function f: (X, τ) into a gpr-T_0 space.

Definition 6.05: $f: X \to Y$ is point–*gpr*-closure 1–1 iff for x, y∈X such that $gpr\mathrm{cl}\{x\} \neq gpr\mathrm{cl}\{y\}$, $gpr\mathrm{cl}\{f(x)\} \neq gpr\mathrm{cl}\{f(y)\}$.

Theorem 6.06:
(i) If f: $(X, \tau) \to (Y, \sigma)$ is point– gpr-closure 1–1 and (X, τ) is gpr-T_0, then f is 1–1
(ii) If f: $(X, \tau) \to (Y, \sigma)$, where (X, τ) and (Y, σ) are gpr-T_0 then f is point– gpr-closure 1–1 iff f is 1–1

Proof: omitted

The following result can be obtained by combining results for *gpr*-T_0– identification spaces, *gpr*-induced functions and *gpr*-T_i spaces; i = 1,2.

Theorem 6.07: X is gpr-R_i ; i = 0,1 iff there exists a gpr-continuous, almost–open point– gpr-closure 1–1 function f: (X, τ) into a gpr-R_i space; i = 0,1 respectively.

7. *gpr*-Normal; Almost *gpr*-normal and Mildly *gpr*-normal spaces

Definition 7.1: A space X is said to be *gpr*-normal if for any pair of disjoint closed sets F_1 and F_2 , there exist disjoint *gpr*-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Note 6: From the above definition we have the following implication diagram.

$$\begin{array}{ccccccccc} T_4 & \to & g\text{-}T_4 & \to & \alpha g\text{-}T_4 & \to & gp\text{-}T_4 & \to & gpr\text{-}T_4 \\ \uparrow & & & & & & & & \uparrow \\ r\text{-}T_4 & \to & \to \to \to & \to & \to \to \to & \to & \to \to & & rg\text{-}T_4 \end{array}$$

none is reversible

Example 4: Let X = {a, b, c} and τ = {φ, {a}, {b, c}, X}. Then X is *gpr*-normal.

Example 5: Let X = {a, b, c, d} and τ = {φ,{b, d},{a, b, d},{b, c, d}, X}. Then X is *gpr*-normal and is not normal.
We have the following characterization of *gpr*-normality.

Theorem 7.1: For a space X the following are equivalent:
(i) X is *gpr*-normal.
(ii) For every pair of open sets U and V whose union is X, there exist *gpr*-closed sets A and B such that $A\subset U$, $B \subset V$ and $A\cup B = X$.
(iii) For every closed set F and every open set G containing F, there exists a *gpr*-open set U such that $F \subset U \subset$ *gprcl*(U) $\subset G$.

Proof: (a)⇒(b): Let U and V be a pair of open sets in a *gpr*-normal space X such that $X = U\cup V$. Then X-U,X-V are disjoint closed sets . Since X is *gpr*-normal there exist disjoint *gpr*-open sets U_1 and V_1 such that $X\text{-}U \subset U_1$ and $X\text{-}V \subset V_1$. Let $A = X\text{-}U_1$, $B = X - V_1$. Then A and B are *gpr*-closed sets such that $A \subset U$, $B \subset V$ and $A\cup B = X$.

(b) ⇒(c): Let F be a closed set and G be an open set containing F. Then X-F and G are open sets whose union is X. Then by (b) , there exist *gpr*-closed sets W_1 and W_2 such that $W_1 \subset X\text{-}F$ and $W_2 \subset G$ and $W_1 \cup W_2 = X$. Then $F\subset X\text{-}W_1$, $X\text{-}G \subset X\text{-}W_2$ and $(X\text{-}W_1)\cap(X\text{-}W_2)= \phi$. Let $U= X\text{-}W_1$ and $V= X\text{-}W_2$. Then U and V are disjoint *gpr*-open sets such that $F\subset U\subset X\text{-}V\subset G$.As X-V is *gpr*-closed set, we have *gprcl*(U) $\subset$X-V and $F\subset U\subset$ *gprcl*(U)$\subset$G.

(c) ⇒ (a): Let F_1 and F_2 be any two disjoint closed sets of X. Put $G = X\text{-}F_2$, then $F_1\cap G = \phi$. $F_1 \subset G$ where G is an open set .Then by (c) , there exists a *gpr*-open set U of X such that $F_1 \subset U \subset$ *gprcl*(U) $\subset$G. It follows that $F_2 \subset X\text{-}$ *gprcl*(U) = V , say, then V is *gpr*-open and $U \cap V = \phi$. Hence F_1 and F_2 are separated by *gpr*-open sets U and V. Therefore X is *gpr*-normal.

Theorem 7.2: A regular open subspace of a *gpr*-normal space is *gpr*-normal.

Proof: Let Y be a regular open subspace of a *gpr*-normal space X .Let A and B be disjoint closed subsets of Y .As Y is regular open , A,B are closed sets of X. By *gpr*-normality of X, there exist disjoint *gpr*-open sets U and V in X such that $A \subset U$ and $B\subset V$, $U\cap Y$ and $V\cap Y$ are *gpr*-open in Y such that $A\subset U\cap Y$ and $B\subset V\cap Y$. Hence Y is *gpr*-normal.

Example 6: Let X = {a, b, c} with $\tau =\{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ is *gpr*-normal and *gpr*-regular.

Now, we define the following.

Definition 7.2: A function f: X → Y is said to be almost –*gpr*-irresolute if for each x in X and each *gpr*-neighborhood V of f(x), *gpr*cl(f^{-1}(V)) is a *gpr*-neighborhood of x.

Clearly every *gpr*-irresolute map is almost *gpr*-irresolute.

The Proof of the following lemma is straightforward and hence omitted.

Lemma 7.1: f is almost *gpr*-irresolute iff $f^{-1}(V) \subset$ *gpr*-int(*gprcl*(f^{-1}(V)))) for every $V\in$ *GPRO*(Y).

Now we prove the following.

Lemma 7.2: f is almost *gpr*-irresolute iff f(*gprcl*(U)) $\subset$ *gprcl*(f(U)) for every $U\in$ *GPRO*(X).

Proof: Let $U\in$ *GPRO*(X).Suppose $y\notin$ *gprcl*(f(U)). Then there exists $V\in$ *GPRO*(y) such that $V\cap f(U) = \phi$.

Hence $f^{-1}(V)\cap U= \phi$. Since $U\in$ *GPRO*(X), we have *gpr*-int(*gprcl*(f^{-1}(V))) $\cap$ *gprcl*(U) = ϕ. Then by lemma 7.1,
$f^{-1}(V)\cap$ *gprcl*(U) = ϕ and hence $V\cap f$(*gprcl*(U)) = ϕ. This implies that $y\notin f$(*gpr*cl(U)).

Conversely, if $V\in$ *GPRO*(Y), then W = X- *gprcl*(f^{-1}(V)))$\in$ *GPRO*(X). By hypothesis, f(*gprcl*(W))$\subset$*gprcl* (f(W))) and hence X- *gpr*-int(*gprcl*(f^{-1}(V))) = *gprcl*(W)$\subset f^{-1}$(*gprcl*(f(W)))$\subset f$(*gprcl*[f(X-f^{-1}(V))])$\subset f^{-1}$[*gprcl*(Y-V)]
$= f^{-1}(Y\text{-}V) = X\text{-}f^{-1}(V)$.

Therefore, $f^{-1}(V)\subset$*gpr*-int(*gprcl*(f^{-1}(V))). By lemma 7.1, f is almost *gpr*-irresolute.

Now we prove the following result on the invariance of *gpr*-normality.

Theorem 7.3: If f is an M-*gpr*-open continuous almost *gpr*-irresolute function from a *gpr*-normal space X onto a space Y, then Y is *gpr*-normal.

Proof: Let A be a closed subset of Y and B be an open set containing A. Then by continuity of f, f^{-1}(A) is closed and f^{-1}(B) is an open set of X such that $f^{-1}(A) \subset f^{-1}(B)$. As X is *gpr*-normal, there exists a *gpr*-open set U in X such that

$f^{1}(A) \subset U \subset gprcl(U) \subset f^{1}(B)$. Then $f(f^{1}(A)) \subset f(U) \subset f(gprcl(U)) \subset f(f^{1}(B))$. Since f is M-*gpr*-open almost *gpr*-irresolute surjection, we obtain $A \subset f(U) \subset gprcl(f(U)) \subset B$. Then again by Theorem 7.1 the space Y is *gpr*-normal.

Lemma 7.3: A mapping f is M-*gpr*-closed if and only if for each subset B in Y and for each *gpr*-open set U in X containing $f^{1}(B)$, there exists a *gpr*-open set V containing B such that $f^{1}(V) \subset U$.

Now we prove the following:

Theorem 7.4: If f is an M-*gpr*-closed continuous function from a *gpr*-normal space onto a space Y, then Y is *gpr*-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 [19] and lemma 7.3, we prove that the following result.

Theorem 7.5: If f is an M-*gpr*-closed map from a weakly Hausdorff *gpr*-normal space X onto a space Y such that $f^{1}(y)$ is S-closed relative to X for each $y \in Y$, then Y is *gpr*-T_2.

Proof: Let y_1 and y_2 be any two distinct points of Y. Since X is weakly Hausdorff, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are disjoint closed subsets of X by lemma 2.2 [19]. As X is *gpr*-normal, there exist disjoint *gpr*-open sets V_1 and V_2 such that $f^{-1}(y_i) \subset V_I$, for i = 1, 2. Since f is M-*gpr*-closed, there exist *gpr*-open sets U_1 and U_2 containing y_1 and y_2 such that $f^{-1}(U_i) \subset V_i$ for i = 1, 2. Then it follows that $U_1 \cap U_2 = \phi$. Hence Y is *gpr*-T_2.

Theorem 7.6: For a space *X* we have the following:
(a) If *X* is normal then for any disjoint closed sets A and B, there exist disjoint *gpr*-open sets U, V such that $A \subset U$ and $B \subset V$;
(b) If *X* is normal then for any closed set A and any open set V containing A, there exists an *gpr*-open set U of X such that $A \subset U \subset gprcl(U) \subset V$.

Definition 7.2: X is said to be almost *gpr*-normal if for each closed set A and each regular closed set B such that $A \cap B = \phi$, there exist disjoint *gpr*-open sets U and V such that $A \subset U$ and $B \subset V$.

Clearly, every *gpr*-normal space is almost *gpr*-normal, but not conversely in general.

Note 7: From the above definition we have the following implication diagram.

$$\begin{array}{ccccccccc} \text{Al-}T_4 & \rightarrow & \text{Al-g-}T_4 & \rightarrow & \text{Al-}\alpha\text{g-}T_4 & \rightarrow & \text{Al-gp-}T_4 & \rightarrow & \text{Al-gpr-}T_4 \\ \uparrow & & & & & & & & \uparrow \\ \text{Al-r-}T_4 & \rightarrow & \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow & \rightarrow & \rightarrow \rightarrow \rightarrow & & & & \text{Al-rg-}T_4 \end{array}$$

none is reversible

Example 7: Let X = {a, b, c} and τ = {ϕ,{a}, {a, b}, {a, c}, X}.Then X is almost *gpr*-normal and not *gpr*-normal.

Now, we have characterization of almost *gpr*-normality in the following.

Theorem 7.7: For a space X the following statements are equivalent:
(i) X is almost *gpr*-normal
(ii) For every pair of sets U and V , one of which is open and the other is regular open whose union is X, there exist *gpr*-closed sets G and H such that $G \subset U$,$H \subset V$ and $G \cup H = X$.
(iii) For every closed set A and every regular open set B containing A, there is a *gpr*-open set V such that $A \subset V \subset gprcl(V) \subset B$.

Proof: (a)$\Rightarrow$(b) Let U be an open set and V be a regular open set in an almost *gpr*-normal space X such that $U \cup V = X$. Then (X-U) is closed set and (X-V) is regular closed set with $(X-U) \cap (X-V) = \phi$. By almost *gpr*-normality of X, there exist disjoint *gpr*-open sets U_1 and V_1 such that $X-U \subset U_1$ and $X-V \subset V_1$. Let $G = X- U_1$ and $H = X-V_1$ .Then G and H are *gpr*-closed sets such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious.

One can prove that almost *gpr*-normality is also regular open hereditary.

Almost *gpr*-normality does not imply almost *gpr*-regularity in general. However, we observe that every almost *gpr*-normal *gpr*-R_0 space is almost *gpr*-regular.

Next, we prove the following.

Theorem 7.8: Every almost regular, *v*-compact space X is almost *gpr*-normal.

Recall that a function *f*: X→ Y is called rc-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost *gpr*-normality in the following.

Theorem 7.9: If *f* is continuous M-*gpr*-open rc-continuous and almost *gpr*-irresolute surjection from an almost *gpr*-normal space X onto a space Y, then Y is almost *gpr*-normal.

Definition 7.3: A space X is said to be mildly *gpr*-normal if for every pair of disjoint regular closed sets F_1 and F_2 of X, there exist disjoint *gpr*-open sets U and V such that $F_1 \subset U$ and $F_2 \subset V$.

Note 8: From the above definition we have the following implication diagram.

$$\begin{array}{ccccccccc} \text{Mild-}T_4 & \to & \text{Mild-g-}T_4 & \to & \text{Mild-}\alpha\text{g-}T_4 & \to & \text{Mild-gp-}T_4 & \to & \text{Mild-gpr-}T_4 \\ \uparrow & & & & & & & & \uparrow \\ \text{Mild-r-}T_4 & \to & \to & \to & \to & \to & \to & \to & \text{Mild-rg-}T_4 \end{array}$$

none is reversible

We have the following characterization of mild *gpr*-normality.

Theorem 7.10: For a space X the following are equivalent.

(i) X is mildly *gpr*-normal.

(ii) For every pair of regular open sets U and V whose union is X, there exist *gpr*-closed sets G and H such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(iii) For any regular closed set A and every regular open set B containing A, there exists a *gpr*-open set U such that $A \subset U \subset gprcl(U) \subset B$.

(iv) For every pair of disjoint regular closed sets, there exist *gpr*-open sets U and V such that $A \subset U$, $B \subset V$ and $gprcl(U) \cap gprcl(V) = \phi$.

This theorem may be proved by using the arguments similar to those of Theorem 7.7.

Also, we observe that mild *gpr*-normality is regular open hereditary.

We define the following

Definition 7.4: A space X is weakly *gpr*-regular if for each point x and a regular open set U containing {x}, there is a *gpr*-open set V such that $x \in V \subset clV \subset U$.

Theorem 7.11: If *f*: X → Y is an M-*gpr*-open rc-continuous and almost *gpr*-irresolute function from a mildly *gpr*-normal space X onto a space Y, then Y is mildly *gpr*-normal.

Proof: Let A be a regular closed set and B be a regular open set containing A. Then by rc-continuity of *f*, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since X is mildly *gpr*-normal, there exists a *gpr*-open set V such that $f^{-1}(A) \subset V \subset gprcl(V) \subset f^{-1}(B)$ by Theorem 7.10. As *f* is M-*gpr*-open and almost *gpr*-irresolute surjection, it follows that $f(V) \in GPRO(Y)$ and $A \subset f(V) \subset gprcl(f(V)) \subset B$. Hence Y is mildly *gpr*-normal.

Theorem 7.12: If *f*: X → Y is rc-continuous, M-*gpr*-closed map from a mildly *gpr*-normal space X onto a space Y, then Y is mildly *gpr*-normal.

8. *gpr*-US spaces:

Definition 8.1: A sequence $<x_n>$ is said to be *gpr*-converges to a point x of X, written as $<x_n> \to^{gpr} x$ if $<x_n>$ is eventually in every *gpr*-open set containing x.

Clearly, if a sequence $<x_n>$ *r*-converges to a point x of X, then $<x_n>$ *gpr*-converges to x.

Definition 8.2: A space X is said to be *gpr*-US if every sequence $<x_n>$ in X *gpr*-converges to a unique point.

Theorem 8.1: Every *gpr*-US space is *gpr*-T_1.

Proof: Let X be *gpr*-US space. Let x and y be two distinct points of X. Consider the sequence $<x_n>$ where $x_n = x$ for every n. Cleary, $<x_n>$ *gpr*-converges to x. Also, since $x \neq y$ and X is *gpr*-US, $<x_n>$ cannot *gpr*-converge to y, i.e, there exists a *gpr*-open set V containing y but not x. Similarly, if we consider the sequence $<y_n>$ where $y_n = y$ for all n, and proceeding as above we get a *gpr*-open set U containing x but not y. Thus, the space X is *gpr*-T_1.

Theorem 8.2: Every *gpr*-T_2 space is *gpr*-US.

Proof: Let X be *gpr*-T_2 space and $<x_n>$ be a sequence in X. If possible suppose that $<x_n>$ *gpr*-converge to two distinct points x and y. That is, $<x_n>$ is eventually in every *gpr*-open set containing x and also in every *gpr*-open set containing y. This is contradiction since X is *gpr*-T_2 space. Hence the space X is *gpr*-US.

Definition 8.3: A set F is sequentially *gpr*-closed if every sequence in F *gpr*-converges to a point in F.

Theorem 8.3: X is *gpr*-US iff the diagonal set is a sequentially *gpr*-closed subset of X x X.

Proof: Let X be *gpr*-US. Let $<x_n , x_n>$ be a sequence in Δ. Then $<x_n>$ is a sequence in X. As X is *gpr*-US, $<x_n> \rightarrow^{gpr} x$ for a unique $x \in X$. i.e., if $<x_n>$ *gpr*-converges to x and y. Thus, x = y. Hence Δ is sequentially *gpr*-closed set.

Conversely, let Δ be sequentially *gpr*-closed. Let a sequence $<x_n>$ *gpr*-converge to x and y. Hence sequence $<x_n , x_n>$ *gpr*-converges to (x,y). Since Δ is sequentially *gpr*-closed, $(x,y) \in \Delta$ which means that x = y implies space X is *gpr*-US.

Definition 8.4: A subset G of a space X is said to be sequentially *gpr*-compact if every sequence in G has a subsequence which *gpr*-converges to a point in G.

Theorem 8.4: In a *gpr*-US space every sequentially *gpr*-compact set is sequentially *gpr*-closed.

Proof: Let X be *gpr*-US space. Let Y be a sequentially *gpr*-compact subset of X. Let $<x_n>$ be a sequence in Y. Suppose that $<x_n>$ *gpr*-converges to a point in X-Y. Let $<x_{np}>$ be subsequence of $<x_n>$ that *gpr*-converges to a point $y \in Y$ since Y is sequentially *gpr*-compact. Also, let a subsequence $<x_{np}>$ of $<x_n>$ *gpr*-converge to $x \in$ X-Y. Since $<x_{np}>$ is a sequence in the *gpr*-US space X, x= y. Thus, Y is sequentially *gpr*-closed set.

Next, we give a hereditary property of *gpr*-US spaces.

Theorem 8.5: Every regular open subset of a *gpr*-US space is *gpr*-US.

Proof: Let X be a *gpr*-US space and $Y \subset X$ be an regular open set. Let $<x_n>$ be a sequence in Y. Suppose that $<x_n>$ *gpr*-converges to x and y in Y. We shall prove that $<x_n>$ *gpr*-converges to x and y in X. Let U be any *gpr*-open subset of X containing x and V be any *gpr*-open set of X containing y. Then, $U \cap Y$ and $V \cap Y$ are *gpr*-open sets in Y. Therefore, $<x_n>$ is eventually in $U \cap Y$ and $V \cap Y$ and so in U and V. Since X is *gpr*-US, this implies that x = y. Hence the subspace Y is *gpr*-US.

Theorem 8.6: A space X is *gpr*-T_2 iff it is both *gpr*-R_1 and *gpr*-US.

Proof: Let X be *gpr*-T_2 space. Then X is *gpr*-R_1 and *gpr*-US by Theorem 8.2.

Conversely, let X be both *gpr*-R_1 and *gpr*-US space. By Theorem 8.1, X is both *gpr*-T_1 and *gpr*-R_1 and, it follows that space X is *gpr*-T_2.

Definition 8.5: A point y is a *gpr*-cluster point of sequence $<x_n>$ iff $<x_n>$ is frequently in every *gpr*-open set containing x.

The set of all *gpr*-cluster points of $<x_n>$ will be denoted by *gpr*-cl(x_n).

Definition 8.6: A point y is *gpr*-side point of a sequence $<x_n>$ if y is a *gpr*-cluster point of $<x_n>$ but no subsequence of $<x_n>$ *gpr*-converges to y.

Now, we define the following.

Definition 8.7: A space X is said to be *gpr*-S_1 if it is *gpr*-US and every sequence $<x_n>$ *gpr*-converges with subsequence of $<x_n>$ *gpr*-side points.

Definition 8.8: A space X is said to be *gpr*-S_2 if it is *gpr*-US and every sequence $<x_n>$ in X *gpr*-converges which has no *gpr*-side point.

Lemma 8.1: Every *gpr*-S_2 space is *gpr*-S_1 and Every *gpr*-S_1 space is *gpr*-US.

Now using the notion of sequentially continuous functions, we define the notion of sequentially *gpr*-continuous functions.

Definition 8.9: A function *f* is said to be sequentially *gpr*-continuous at $x \in X$ if $f(x_n)$ *gpr*-converges to $f(x)$ whenever $<x_n>$ is a sequence *gpr*-converging to x. If *f* is sequentially *gpr*-continuous at all $x\in X$, then *f* is said to be sequentially *gpr*-continuous.

Theorem 8.7: Let *f* and *g* be two sequentially *gpr*-continuous functions. If Y is *gpr*-US, then the set $A = \{x \mid f(x) = g(x)\}$ is sequentially *gpr*-closed.

Proof: Let Y be *gpr*-US and suppose that there is a sequence $<x_n>$ in A *gpr*-converging to $x \in X$. Since *f* and *g* are sequentially *gpr*-continuous functions, $f(x_n) \to^{gpr} f(x)$ and $g(x_n) \to^{gpr} g(x)$. Hence $f(x) = g(x)$ and $x \in A$. Therefore, A is sequentially *gpr*-closed.

Next, we prove the product theorem for *gpr*-US spaces.

Theorem 8.8: Product of arbitrary family of *gpr*-US spaces is *gpr*-US.

Proof: Let $X = \prod_{\lambda\in\wedge} X_\lambda$ where X_λ is *gpr*-US. Let a sequence $<x_n>$ in X *gpr*-converges to x $(= x_\lambda)$ and y $(= y_\lambda)$. Then the sequence $<x_{n\lambda}>$ *gpr*-converges to x_λ and y_λ for all $\lambda \in \wedge$. For suppose there exists a $\mu \in \wedge$ such that $<x_{n\mu}>$ does not *gpr*-converges to x_μ. Then there exists a τ_μ-*gpr*-open set U_μ containing x_μ such that $<x_{n\mu}>$ is not eventually in U_μ. Consider the set $U = \prod_{\lambda\in\wedge} X_\lambda \times U_\mu$. Then U is a *gpr*-open subset of X and $x \in U$. Also, $<x_n>$ is not eventually in U, which contradicts the fact that $<x_n>$ *gpr*-converges to x. Thus we get $<x_{n\lambda}>$ *gpr*-converges to x_λ and y_λ for all $\lambda \in \wedge$.

Since X_λ is *gpr*-US for each $\lambda\in\wedge$. Thus x = y. Hence X is *gpr*-US.

9. Sequentially sub-*gpr*-continuity:

In this section we introduce and study the concepts of sequentially sub-*gpr*-continuity, sequentially nearly *gpr*-continuity and sequentially *gpr*-compact preserving functions and study their relations and the property of *gpr*-US spaces.

Definition 9.1: A function *f* is said to be sequentially nearly *gpr*-continuous if for each point $x\in X$ and each sequence $<x_n>$ in X *gpr*-converging to x, there exists a subsequence $<x_{nk}>$ of $<x_n>$ such that $<f(x_{nk})>\to^{gpr} f(x)$.

Definition 9.2: A function *f* is said to be sequentially sub-*gpr*-continuous if for each point $x \in X$ and each sequence $<x_n>$ in X *gpr*-converging to x, there exists a subsequence $<x_{nk}>$ of $<x_n>$ and a point $y \in Y$ such that $<f(x_{nk})> \to^{gpr} y$.

Definition 9.3: A function *f* is said to be sequentially *gpr*-compact preserving if $f(K)$ is sequentially *gpr*-compact in Y for every sequentially *gpr*-compact set K of X.

Lemma 9.1: Every function *f* is sequentially sub-*gpr*-continuous if Y is a sequentially *gpr*-compact.

Proof: Let $<x_n>$ be a sequence in X *gpr*-converging to a point x of X. Then $\{f(x_n)\}$ is a sequence in Y and as Y is sequentially *gpr*-compact, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ *gpr*-converging to a point $y\in Y$. Hence *f* is sequentially sub-*gpr*-continuous.

Theorem 9.1: Every sequentially nearly *gpr*-continuous function is sequentially *gpr*-compact preserving.

Proof: Suppose *f* is a sequentially nearly *gpr*-continuous function and let K be any sequentially *gpr*-compact subset of X. Let $<y_n>$ be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially *gpr*-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ *gpr*-converging to a point $x \in K$. By hypothesis, *f* is sequentially nearly *gpr*-continuous and hence there exists a subsequence $<x_j>$ of $<x_{nk}>$ such that $f(x_j)\to^{gpr} f(x)$. Thus, there exists a subsequence $<y_j>$ of $<y_n>$ *gpr*-converging to $f(x)\in f(K)$. This shows that $f(K)$ is sequentially *gpr*-compact set in Y.

Theorem 9.2: Every sequentially *sg*-continuous function is sequentially *gpr*-continuous.

Proof: Let f be a sequentially *gp*-continuous and $<x_n>$ be a sequence in X which *gp*-converges to a point $x\in X$. Then $<x_n>$ *gp*-converges to x. Since f is sequentially *gp*-continuous, $f(x_n)\to^{gp} f(x)$. But we know that $<x_n>$ *gp*-converges to x implies $<x_n>$ *gpr*-converges to x and hence $f(x_n)\to^{gpr} f(x)$ implies f is sequentially *gpr*-continuous.

Note 9: From the above Theorem we have the following implication diagram.

$$\begin{array}{ccccccccc} \text{seq.c} & \to & \text{seq.g.c} & \to & \text{seq.}\alpha\text{g.c} & \to & \text{seq.gp.c} & \to & \text{seq.gpr.c} \\ \uparrow & & & & & & & & \uparrow \\ \text{Seq.r.c} & \to & \to \to \to \to & \to & \to \to \to & \to & \to & \to & \text{seq.rg.c} \end{array}$$

none is reversible

Theorem 9.3: Every sequentially *gpr*-compact preserving function is sequentially sub-*gpr*-continuous.

Proof: Suppose f is a sequentially *gpr*-compact preserving function. Let x be any point of X and $<x_n>$ any sequence in X *gpr*-converging to x. We shall denote the set $\{x_n \mid n= 1,2,3, \dots\}$ by A and $K = A \cup \{x\}$. Then K is sequentially *gpr*-compact since $x_n \to^{gpr} x$. By hypothesis, f is sequentially *gpr*-compact preserving and hence $f(K)$ is a sequentially *gpr*-compact set of Y. Since $\{f(x_n)\}$ is a sequence in $f(K)$, there exists a subsequence $\{f(x_{nk})\}$ of $\{f(x_n)\}$ *gpr*-converging to a point $y\in f(K)$. This implies that f is sequentially sub-*gpr*-continuous.

Theorem 9.4: A function f: X→ Y is sequentially *gpr*-compact preserving iff $f_{/K}$: K → f (K) is sequentially sub-*gpr*-continuous for each sequentially *gpr*-compact subset K of X.

Proof: Suppose f is a sequentially *gpr*-compact preserving function. Then $f(K)$ is sequentially *gpr*-compact set in Y for each sequentially *gpr*-compact set K of X. Therefore, by Lemma 9.1 above, $f_{/K}$: K→ f (K) is sequentially *gpr*-continuous function.

Conversely, let K be any sequentially *gpr*-compact set of X. Let $<y_n>$ be any sequence in $f(K)$. Then for each positive integer n, there exists a point $x_n\in K$ such that $f(x_n) = y_n$. Since $<x_n>$ is a sequence in the sequentially *gpr*-compact set K, there exists a subsequence $<x_{nk}>$ of $<x_n>$ *gpr*-converging to a point $x \in K$. By hypothesis, $f_{/K}$: K→ f(K) is sequentially sub-*gpr*-continuous and hence there exists a subsequence $<y_{nk}>$ of $<y_n>$ *gpr*-converging to a point $y\in f(K)$.This implies that f(K) is sequentially *gpr*-compact set in Y. Thus, f is sequentially *gpr*-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-*gpr*-continuous function to be sequentially *gpr*-compact preserving.

Corollary 9.1: If f is sequentially sub-*gpr*-continuous and f (K) is sequentially *gpr*-closed set in Y for each sequentially *gpr*-compact set K of X, then f is sequentially *gpr*-compact preserving function.

Proof: Omitted.

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