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SOME RESULTS ON t-BEST APPROXIMATION IN FUZZY ANTI-n-NORMED LINEAR SPACES

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ABSTRACT

The main aim of this paper to give the set of all t-best approximations on fuzzy anti-n-normed linear spaces and prove some theorems in the sense of Vaezpour and Karimi [21].

Key Words: *n-normed space, fuzzy anti-n-normed space, t-best approximation.*

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1. INTRODUCTION:

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by Katsaras in [12]. Felbin [6] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [11]. Cheng and Mordeson [4] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [13].

Bag and Samanta in [1] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [13]. They also studied some properties of the fuzzy norm in [2] and [3]. Bag and Samanta discussed the notion of convergent sequence and Cauchy sequence in fuzzy normed linear space in [1]. They also made in [3] a comparative study of the fuzzy norms defined by Katsaras [12], Felbin [6], and Bag and Samanta [1]. The concept of 2-norm and n-norm on a linear space has been introduced and developed by Gahler in [7,8]. Following Misiak [15], Malceski [14] and Gunawan [9] developed the theory of n-normed space. Narayana and Vijayabalaji [16] introduced the concept of fuzzy n-normed linear space. Vijayabalaji and Thillaigovindan [22] introduced the notion of convergent sequence and Cauchy sequence in fuzzy n-normed linear space and studied the completeness of the fuzzy n-normed linear space. Many authors studied on fuzzy n-normed linear space [5]. Recently, Vaezpour and Karimi [21], studied on the set of all t-best approximations on fuzzy normed spaces and proved several theorems pertaining to this set.

In [10] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [3] and investigated their important properties. In [17,18] Surender Reddy introduced the notion of convergent sequence and Cauchy sequence in fuzzy anti-2-normed linear space and fuzzy anti-n-normed linear spaces. Recently Surender Reddy [19,20] studied on the set of all t-best approximations on fuzzy anti-normed linear spaces and fuzzy anti-2-normed linear spaces.

In the present paper, we give the set of all t-best approximations on fuzzy anti-n-normed spaces and prove some theorems in the sense of Vaezpour and Karimi [21].

2. PRELIMINARIES:

Definition 2.1: Let $n \in N$ and let X be a real linear space of dimension $\geq n$. A real valued function $\|\bullet, \bullet, \ldots, \bullet\|$ on $\underbrace{X \times X \times \ldots \times X}_{n} = X^n$ satisfying the following conditions

nN_1: $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

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nN_2 : $\|x_1, x_2,..., x_n\|$ is invariant under any permutation of $x_1, x_2,..., x_n$,

nN_3: $\|x_1, x_2,..., x_{n-1}, \alpha x_n\| = |\alpha| \|x_1, x_2,..., x_{n-1}, x_n\|$, for every $\alpha \in R$,

nN_4: $\|x_1, x_2,..., x_{n-1}, y + z\| \le \|x_1, x_2,..., x_{n-1}, y\| + \|x_1, x_2,..., x_{n-1}, z\|$ for all $y, z, x_1, x_2,..., x_{n-1} \in X$,

then the function $\|\bullet,\bullet,...,\bullet\|$ is called an *n*-norm on *X* and the pair $(X, \|\bullet,\bullet,...,\bullet\|)$ is called *n*-normed linear space.

Example 2.2: A trivial example of an *n*-normed linear space is $X = R^n$ equipped with the following Euclidean *n*-norm.

$$\|x_1, x_2,..., x_n\|_E = |\det(x_{ij})| = abs\begin{pmatrix} \begin{vmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ ... & ... & ... & ... \\ x_{n1} & x_{n2} & ... & x_{nn} \end{vmatrix} \end{pmatrix},$$

where $x_i = (x_{i1}, x_{i2},..., x_{in}) \in R^n$ for each $i = 1,2,..., n$.

Definition 2.3: Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times ... \times X}_{n} \times R$ is called a fuzzy *n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2,..., x_n, y \in X$

$(n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2,..., x_n, t) = 0$,

$(n - N_2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2,..., x_n, t) = 1$ if and only if $x_1, x_2,..., x_n$ are linearly dependent,

$(n - N_3)$: $N(x_1, x_2,..., x_n, t)$ is invariant under any permutation of $x_1, x_2,..., x_n$,

$(n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2,..., x_{n-1}, cx_n, t) = N(x_1, x_2,..., x_{n-1}, x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(n - N_5)$: For all $s, t \in R$, $N(x_1, x_2,..., x_{n-1}, x_n + y, s + t) \ge \min\{N(x_1, x_2,..., x_{n-1}, x_n, s), N(x_1, x_2,..., x_{n-1}, y, t)\}$,

$(n - N_6)$: $N(x_1, x_2,..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2,..., x_n, t) = 1$.

Then the pair (X, N) is called a fuzzy *n*-normed linear space (briefly F-n-NLS).

Example 2.4: Let $(X, \|\bullet,\bullet,...,\bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2,..., x_n, t) = \frac{t}{t + \|x_1, x_2,..., x_n\|}, \text{ if } t > 0, \ t \in R, \ x_1, x_2,..., x_n \in X,$$
$$= 0, \text{ if } t \le 0, \ t \in R, \ x_1, x_2,..., x_n \in X.$$

Then (X, N) is a fuzzy *n*-normed linear space.

Definition 2.5: Let *X* be a linear space over a real field *F*. A fuzzy subset *N* of $\underbrace{X \times X \times ... \times X}_{n} \times R$ is called a fuzzy anti-*n*-norm on *X* if the following conditions are satisfied for all $x_1, x_2,..., x_n, y \in X$.

$(a - n - N_1)$ For all $t \in R$ with $t \le 0$, $N(x_1, x_2,..., x_n, t) = 1$,

$(a - n - N_2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2,..., x_n, t) = 0$ if and only if $x_1, x_2,..., x_n$ are linearly dependent,

$(a - n - N_3)$: $N(x_1, x_2,..., x_n, t)$ is invariant under any permutation of $x_1, x_2,..., x_n$,

$(a - n - N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2,..., x_{n-1}, cx_n, t) = N(x_1, x_2,..., x_{n-1}, x_n, \frac{t}{|c|})$, if $c \ne 0$, $c \in F$,

$(a-n-N_5)$: For all $s,t \in R$, $N(x_1, x_2, ..., x_{n-1}, x_n + y, s+t) \leq$

$$\max\{N(x_1, x_2, ..., x_{n-1}, x_n, s), N(x_1, x_2, ..., x_{n-1}, y, t)\},$$

$(a-n-N_6)$: $N(x_1, x_2, ..., x_n, t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 0$.

Then the pair (X, N) is called a fuzzy anti-*n*-normed linear space (briefly Fa-*n*-NLS).

Remark 2.6:

From $(a-2-N_3)$, it follows that in Fa-*n*-NLS,

$(a-n-N_4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., cx_i, ..., x_n, t) = N(x_1, x_2, ..., x_i, ..., x_n, \frac{t}{|c|})$, if $c \neq 0$, $c \in F$,

$(a-n-N_5)$: For all $s,t \in R$, $N(x_1, x_2, ..., x_i + x_i', ..., x_n, s+t) \leq$

$$\max\{N(x_1, x_2, ..., x_i, ..., x_n, s), N(x_1, x_2, ..., x_i', ..., x_n, t)\}.$$

Example 2.7: Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a *n*-normed linear space. Define

$$N(x_1, x_2, ..., x_n, t) = \frac{\|x_1, x_2, ..., x_n\|}{t + \|x_1, x_2, ..., x_n\|}, \text{ if } t > 0, \ t \in R, \ x_1, x_2, ..., x_n \in X,$$

$$= 1, \text{ if } t \leq 0, \ t \in R, \ x_1, x_2, ..., x_n \in X.$$

Then (X, N) is a fuzzy anti-*n*-normed linear space.

Definition 2.8: A sequence $\{x_k\}$ in a fuzzy anti-*n*-normed linear space (X, N) is said to be converges to $x \in X$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that

$$N(x_1, x_2, ..., x_{n-1}, x_k - x, t) < r, \ \forall \ k \geq n_0.$$

Theorem 2.9: In a fuzzy anti-*n*-normed linear space (X, N), a sequence $\{x_k\}$ converges to $x \in X$ if and only $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 0$, $\forall \ t > 0$.

Definition 2.10: Let (X, N) be a fuzzy anti-*n*-normed linear space. Let $\{x_k\}$ be a sequence in *X* then $\{x_k\}$ is said to be a Cauchy sequence if $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_{k+p} - x_k, t) = 0$, $\forall \ t > 0$ and $p = 1,2,3,...$.

Definition 2.11: A fuzzy anti-*n*-normed linear space (X, N) is said to be complete if every Cauchy sequence in *X* is convergent.

Definition 2.12: A complete fuzzy anti-*n*-normed linear space (X, N) is called a fuzzy anti-*n*-Banach space.

3. MAIN RESULTS:

Definition 3.1: Let (X, N) be a fuzzy anti-*n*-normed linear space. The open ball $B(x, r, t)$ and the closed ball $B[x, r, t]$ with the center $x \in X$ and radius $0 < r < 1$, $t > 0$ are defined as follows:

$$B(x, r, t) = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) < r\}$$

$$B[x, r, t] = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x - y, t) \leq r\}$$

Definition 3.2: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of *X* is said to be open if there exists $r \in (0,1)$ such that $B(x, r, t) \subset A$ for all $x \in A$ and $t > 0$.

Definition 3.3: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of X is said to be closed if for any sequence $\{x_k\}$ in *A* converges to $x \in A$.

i.e., $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 0$, for all $t > 0$ implies that $x \in A$.

Definition 3.4: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *B* of X is said to be closure of $A \subset B$ if for any $x \in B$, there exists a sequence $\{x_k\}$ in A such that $\lim_{k\to\infty} N(x_1, x_2, ..., x_{n-1}, x_k - x, t) = 0$, for all $t > 0$. We denote the set *B* by $\overline{A}$.

Definition 3.5: Let (X, N) be a fuzzy anti-*n*-normed linear space. A subset *A* of X is said to be compact if for any sequence $\{x_k\}$ in *A* has a sequence converging to an element of *A*.

Lemma 3.6: If (X, N) be a fuzzy anti-*n*-normed linear space then
(i) the function $(x, y) \to x + y$ is continuous.
(ii) the function $(\alpha, x) \to \alpha x$ is continuous.

Proof: (i) If $x_k \to x$ and $y_k \to y$ then as $k \to \infty$,

$$N(x_1, x_2, ..., x_{n-1}, (x_k + y_k) - (x + y), t) \le \max\{N(x_1, x_2, ..., x_{n-1}, x_k - x, \frac{t}{2}), N(x_1, x_2, ..., x_{n-1}, y_k - y, \frac{t}{2})\} \to 0$$

(ii) If $x_k \to x$, $\alpha_k \to \alpha$ and $\alpha_k \ne 0$ then

$$\begin{aligned} N(x_1, x_2, ..., x_{n-1}, \alpha_k x_k - \alpha x, t) &= N(x_1, x_2, ..., x_{n-1}, \alpha_k (x_k - x) + x(\alpha_k - \alpha), t) \\ &\le \max\{N(x_1, x_2, ..., x_{n-1}, \alpha_k (x_k - x), \frac{t}{2}), N(x_1, x_2, ..., x_{n-1}, x(\alpha_k - \alpha), \frac{t}{2})\} \\ &\le \max\{N(x_1, x_2, ..., x_{n-1}, x_k - x, \frac{t}{2|\alpha_k|}), N(x_1, x_2, ..., x_{n-1}, x, \frac{t}{2|\alpha_k - \alpha|})\} \to 0 \text{ as } k \to \infty. \end{aligned}$$

Definition 3.7: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of X. Let $d(A, x, t) = \inf\{N(x_1, x_2, ..., x_{n-1}, x - y, t) : y \in A\}$, where $x \in X$, $t > 0$. An element $y_0 \in A$ is said to be a *t*-best approximation of *x* from *A* if $N(x_1, x_2, ..., x_{n-1}, y_0 - x, t) = d(A, x, t)$.

Definition 3.8: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*. For $x \in X$, $t > 0$, we shall denote the set of all elements of *t*-best approximation of *x* from *A* by $P_A^t(x)$ and is defined as

$$P_A^t(x) = \{y \in A : d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, y - x, t)\}.$$

If each $x \in X$ has at least (respectively exactly) one *t*-best approximation in *A* then *A* is called a *t*-proximinal (respectively t-chebyshev) set.

Definition 3.9: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*. For $t > 0$, *A* is said to be *t*-boundedly compact if for each $x \in X$ and $0 < r < 1$, $B[x, r, t] \cap A$ is a compact subset of *X*.

Definition 3.10: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of X then
(i) $d(A + y, x + y, t) = d(A, x, t)$, for all $x, y \in X$ and $t > 0$,
(ii) $P_A^t(x + y) = P_A^t(x) + y$, for all $x, y \in X$ and $t > 0$,
(iii) $d(\alpha A, \alpha x, t) = d(A, x, \frac{t}{|\alpha|})$, for all $x \in X$, $t > 0$ and $\alpha \in R - \{0\}$,

(iv) $P_{\alpha A}^{|\alpha|t}(\alpha x) = \alpha P_A^t(x)$, for all $x \in X$, $t > 0$ and $\alpha \in R - \{0\}$,

(v) A is *t*-proximinal (respectively *t*-chebyshev) if and only if *A+y* is *t*-proximinal (respectively *t*-chebyshev), for any given $y \in X$,

(vi) A is *t*-proximinal (respectively *t*-chebyshev) if and only if αA is $|\alpha|t$ -proximinal (respectively $|\alpha|t$ -chebyshev), for any given each $\alpha \in R - \{0\}$.

Proof: (i) For $x, y \in X$ and $t > 0$,

$$d(A+y, x+y, t) = \inf\{N(x_1, x_2, ..., x_{n-1}, (z+y)-(x+y), t) : z \in A\}$$
$$= \inf\{N(x_1, x_2, ..., x_{n-1}, z-x, t) : z \in A\} = d(A, x, t).$$

(ii) On using (i), $y_0 \in P_{A+y}^t(x+y)$ if and only if $y_0 \in A+y$ and $d(A+y, x+y, t) = N(x_1, x_2, ..., x_{n-1}, x+y-y_0, t)$ if and only if $y_0 - y \in A$ and $d(A, x, t) = N(x_1, x_2, ..., x_{n-1}, x-(y_0-y), t)$ if and only if $y_0 - y \in P_A^t(x)$

i.e., $y_0 \in P_A^t(x) + y$.

(iii) We have $d(\alpha A, \alpha x, t) = \inf\{N(x_1, x_2, ..., x_{n-1}, \alpha x - \alpha z, t) : z \in A\}$

$$= \inf\{N(x_1, x_2, ..., x_{n-1}, \alpha(x-z), t) : z \in A\}$$
$$= \inf\{N(x_1, x_2, ..., x_{n-1}, x-z, \frac{t}{|\alpha|}) : z \in A\} = d(A, x, \frac{t}{|\alpha|}).$$

(iv) On using (iii), it follows that $y_0 \in P_{\alpha A}^{|\alpha|t}(\alpha x)$ if and only if $y_0 \in \alpha A$ and

$d(\alpha A, \alpha x, t) = N(x_1, x_2, ..., x_{n-1}, \alpha x - y_0, t)$ if and only if $\frac{y_0}{\alpha} \in A$ and

$N(x_1, x_2, ..., x_{n-1}, x - \frac{y_0}{\alpha}, t) = d(A, x, t)$. However, this is equivalent to $\frac{y_0}{\alpha} \in P_A^t(x)$.

i.e., $y_0 \in \alpha P_A^t(x)$.

(v) The proof of (v) is an immediate consequence of (ii).

(vi) The proof of (vi) follows from (iv).

Corollary 3.11: Let *M* is a non empty subset of X then

(i) $d(M, x+y, t) = d(M, x, t)$, for all $t > 0$ $x \in X$ and $y \in M$,

(ii) $P_M^t(x+y) = P_M^t(x) + y$, for all $t > 0$ $x \in X$ and $y \in M$,

(iii) $d(M, \alpha x, |\alpha t|) = d(M, x, t)$, for all $t > 0$, $x \in X$ and $\alpha \in R - \{0\}$,

(iv) $P_M^{|\alpha|t}(\alpha x) = \alpha P_M^t(x)$, for all $t > 0$, $x \in X$ and $\alpha \in R - \{0\}$.

Proof: The proof of (i) and (ii) follows from theorem 2(i) and 2(ii) and the fact that if *M* is a subspace and $y \in M$ then $M + y = M$.

The proof of (iii) and (iv) follows from theorem 2(iii) and 2(iv) and the fact that if *M* is a subspace and $\alpha \neq 0$ then $\alpha M = M$.

Definition 3.12: For $x \in X$, $0 < r < 1$, $t > 0$,

$S[x, r, t] = \{y \in X : N(x_1, x_2, ..., x_{n-1}, x-y, t) = r\}$ and $e_A^t(x) = d(A, x, t)$.

Theorem 3.13: Let (X, N) be a fuzzy anti-*n*-normed linear space, $A \subset X$, $x \in X/\overline{A}$ and $t > 0$ then we have

$$P_A^t(x) = A \cap B[x, e_A^t(x), t] = A \cap S[x, e_A^t(x), t] \quad (1)$$

Proof: This inclusions

$$P_A^t(x) \subseteq A \cap S[x, e_A^t(x), t] \subseteq A \cap B[x, e_A^t(x), t] \quad (2)$$

are obvious by the definitions of $P_A^t(x)$ and $e_A^t(x)$.

Conversely, let $y \in A \cap B[x, e_A^t(x), t]$, then we have $y \in A$ and

$$N(x_1, x_2, ..., x_{n-1}, y - x, t) \le e_A^t(x) = d(A, x, t) \le N(x_1, x_2, ..., x_{n-1}, y - x, t).$$

Therefore $y \in A$ and $N(x_1, x_2, ..., x_{n-1}, y - x, t) = d(A, x, t)$, which implies that $y \in P_A^t(x)$. So, $A \cap B[x, e_A^t(x), t] \subset P_A^t(x)$. Hence by (2) we have (1) which completes the proof.

Remark 3.14: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*, $x \in X/\overline{A}$ and $t > 0$ then we have

$$A \cap B(x, e_A^t(x), t) = \Phi, \tag{3}$$

because, if $y_0 \in A \cap B(x, e_A^t(x), t)$ then $d(A, x, t) \le N(x_1, x_2, ..., x_{n-1}, x - y_0, t) < d(A, x, t)$ which is impossible.

Corollary 3.15: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*, $x \in X/\overline{A}$ with $P_A^t(x) \ne \Phi$ and $0 < r < 1$ such that,

$$\Phi \ne A \cap B[x, r, t] \subseteq S[x, r, t] \tag{4}$$

Then we have $r = e_A^t(x)$, and we can write $A \cap B[x, r, t] = P_A^t(x)$.

Proof: If $r < e_A^t(x)$ then by the definition of $e_A^t(x)$ we have $A \cap B[x, r, t] = \Phi$,
which contradicts (4). If $r > e_A^t(x)$, since $P_A^t(x) \ne \Phi$, then by (1) we have
$\Phi \ne P_A^t(x) = A \cap B[x, e_A^t(x), t] \subseteq A \cap B(x, r, t)$, which contradicts (4), and this completes the proof.

Definition 3.16: Let (X, N) be a fuzzy anti-*n*-normed linear space, $0 < r < 1$ and $t > 0$.
We shall say that a set $A \subset X$ supports the cell $B[x, r, t]$, or that *A* is a support set of the cell $B[x, r, t]$, if we have $d(A, B[x, r, t], t) = 1$ and $A \cap B(x, r, t) = \Phi$.

Theorem 3.17: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of X and $x \in X/\overline{A}$, $a_0 \in A$ and $t > 0$. We have $a_0 \in P_A^t(x)$ if and only if the set *A* supports the cell $B = B[x, N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t]$.

Proof: Assume that $a_0 \in P_A^t(x)$. Hence $N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) = d(A, x, t)$. Then by (3), we have $A \cap B(x, N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t) = \Phi$, on the other hand, since $a_0 \in A \cap B[x, N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t]$, we have $d(A, B, t) = 1$. Consequently, the set *A* supports the cell *B*. Conversely, suppose $a_0 \notin P_A^t(x)$, hence $N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) > d(A, x, t)$ and let $0 < \varepsilon < 1$ such that $N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) > d(A, x, t) + \varepsilon$. Then there exists an $a \in A$ such that
$N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) > d(A, x, t) + \varepsilon > N(x_1, x_2, ..., x_{n-1}, a - x, t)$, hence
$a \in B(x, N(x_1, x_2, ..., x_{n-1}, a_0 - x, t), t)$. Consequently, *A* does not support the cell *B*.

Remark 3.18: We recall that a set *A* in a topological space τ is said to be countably compact, if every countable open cover of *A* has a finite subcover, or, which is equivalent, if for every decreasing sequence $A_1 \supset A_2 \supset ...$ of non-void closed subset of *A* we have $\bigcap_{n=1}^{\infty} A_n \ne \Phi$.

Theorem 3.19: Let (X, N) be a fuzzy anti-*n*-normed linear space, τ be an arbitrary topology on *X* and $t > 0$. If *A* is a nonempty subset of *X* such that for $A \cap B[x, r, t]$ is τ -countably compact, then *A* is *t*-proximinal.

Proof: For all $n \in N$, $0 < 1 - d(A,x,t) + \frac{d(A,x,t)}{n+1} < 1$. Put

$$A_n^t = A \cap B\left[x, 1 - d(A,x,t) + \frac{d(A,x,t)}{n+1}, t\right], \quad (n = 1,2,...).$$

Since for every $n \in N$, $d(A,x,t)\left(1 - \frac{1}{n+1}\right) > d(A,x,t)$, obviously $A_1^t \supset A_2^t \supset ...$ and each $A_n^t \neq \Phi$. Hence there exists $a_n^t \in A$ such that

$$d(A,x,t)\left(1 - \frac{1}{n+1}\right) > N(x_1, x_2, ..., x_{n-1}, a_n^t - x, t).$$

It follows that $a_n^t \in A_n^t$. Now, since each A_n^t is τ-countably compact and τ-closed, we conclude that there exists an $a_0 \in \bigcap_{n=1}^{\infty} A_n^t$. Then we have $d(A,x,t) \leq N(x_1, x_2, ..., x_{n-1}, a_0 - x, t) \leq d(A,x,t)\left(1 - \frac{1}{n+1}\right), \quad (n = 1,2,...)$, hence
$a_0 \in P_A^t(x)$ which completes the proof.

Definition 3.20: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*. An element $y_0 \in A$ is said to be an *F*-best approximation of $x \in X$ from *A* if it is a *t*-best approximation of *x* from *A*, for every $t > 0$, i.e., $y_0 \in \bigcap_{t \in (0,\infty)} P_A^t(x)$.

The set of all elements of *F*-best approximations of $x \in X$ from *A* is denoted by $FP_A(x)$ and is defined as $FP_A(x) = \bigcap_{t \in (0,\infty)} P_A^t(x)$.

If each $x \in X$ has at least (respectively exactly) one *F*-best approximation in *A* then *A* is called a F-proximinal (respectively *F*-chebyshev) set.

Example 3.21: Let $X = R^3$. Define $N : X \times X \times X \times [0,\infty) \to [0,1]$ by

$$N(x_1, x_2, x_3, t) = \frac{\|x_1, x_2, x_3\|}{t}, \text{ if } t > 0, \ t \in R, \ x_1, x_2, x_3 \in X,$$

$$= 1, \text{ if } t \leq 0, \ t \in R, \ x_1, x_2, x_3 \in X,$$

where $\|x_1, x_2, x_3\| = \min_{1 \leq i \leq 3} \sum_{j=1}^{3} |x_{ij}|$. Then (X, N) is a fuzzy anti-3-normed linear space.

Let $A = \{(a,b,c) \in R^3 : a^2 + b^2 \leq 1, \ 0 \leq c \leq a^2 + b^2\}$
and $x_1 = (1,0,0)$, $x_2 = (0,1,0)$, $x = (0,0,4)$ are in *X*. Let $a_0 = (0,-1,1)$ and $a_1 = (0,1,1)$ are in *A*, Then for every $t > 0$,

$$N(x_1, x_2, a_0 - x, t) = N(x_1, x_2, (0,-1,1) - (0,0,4), t) = \frac{1}{t}$$

$$N(x_1, x_2, a_1 - x, t) = N(x_1, x_2, (0,1,1) - (0,0,4), t) = \frac{1}{t}.$$

On the other hand

$$d(A,x,t) = d(A,(0,0,4),t) = \inf\{N(x_1, x_2, u - (0,0,4), t) : u \in A\}$$
$$= \inf\{N(x_1, x_2, (a,b,c) - (0,0,4), t) : a^2 + b^2 \leq 1, \ 0 \leq c \leq a^2 + b^2\}$$

$$= \inf\left\{\frac{\min\left(|x_{11}|+|x_{12}|+|x_{13}|,|x_{21}|+|x_{22}|+|x_{23}|,|x_{31}|+|x_{32}|+|x_{33}-4|\right)}{t}\right\}$$

$$= \frac{1}{t}$$

So, for every $t > 0$, $a_0 = (0,-1,1)$ and $a_1 = (0,1,1)$ are *t*-best approximations of $(0,0,4)$ from A. Hence $a_0 = (0,-1,1)$ and $a_1 = (0,1,1)$ are *F*-best approximations of $x = (0,0,4)$ from *A*. Therefore *A* is not an *F*-chebyshev set.

Example 3.22: Let $X = R^3$. Define $N : X \times X \times X \times R \to [0,1]$ by

$$N(x_1, x_2, x_3, t) = \frac{\|x_1, x_2, x_3\|}{t + \|x_1, x_2, x_3\|}, \text{ if } t > 0,\ t \in R,\ x_1, x_2, x_3 \in X,$$

$$= 1, \text{ if } t \le 0,\ t \in R,\ x_1, x_2, x_3 \in X,$$

where $\|x_1, x_2, x_3\| = \min_{1 \le i \le 3} \sum_{j=1}^{3} |x_{ij}|$. Then (X, N) is a fuzzy anti-3-normed linear space.

Let $A = \{(x, y, z) \in R^3 : x^2 + y^2 + z^2 \ge 1\}$.

Then, for every $a = (x, y, z) \in R^3$ where $x^2 + y^2 + z^2 < 1$, there exists a unique $a_0 = (x_0, y_0, z_0) \in A$ (especially in ∂A) which is an *F*-best approximation of *a* from *A*.

So *A* is an *F*-proximinal set.

Remark 3.23: For an arbitrary set $A \subset X$ we shall denote by ∂A the boundary of *A*, and by M_A the set of all elements of the *F*-best approximation of the elements $x \in X$ from *A*.

i.e., $\mathrm{M}_A = \bigcup_{x \in X} FP_A(x)$.

Theorem 3.24: Let (X, N) be a fuzzy anti-*n*-normed linear space and A is a non empty subset of *X*, and *A* be a *F*-best proximinal set in *X* then $\partial A \subset \overline{\mathrm{M}}_A$.

Proof: If $\partial A = \Phi$, the proof is obvious. If $\partial A \ne \Phi$, let $a_0 \in \partial A$, $0 < \varepsilon < 1$ and $t > 0$ be arbitrary. Then there exists $0 < \varepsilon' < 1$ such that $\varepsilon' < \varepsilon$ and the cell $B(a_0, \varepsilon', \frac{t}{2})$ contains at least one element $x \in X/A$. Let $\pi_A(x) \in FP_A(x)$ (it exists, since by hypothesis, *A* is *F*-proximinal). Then we have,

$$\begin{aligned}
N(x_1, x_2, ..., x_{n-1}, a_0 - \pi_A(x), t) &\le \max\{N(x_1, x_2, ..., x_{n-1}, a_0 - x, \tfrac{t}{2}), N(x_1, x_2, ..., x_{n-1}, x - \pi_A(x), \tfrac{t}{2})\} \\
&= \max\{N(x_1, x_2, ..., x_{n-1}, a_0 - x, \tfrac{t}{2}), N(x_1, x_2, ..., x_{n-1}, A - x, \tfrac{t}{2})\} \\
&\le \max\{N(x_1, x_2, ..., x_{n-1}, a_0 - x, \tfrac{t}{2}), N(x_1, x_2, ..., x_{n-1}, a_0 - x, \tfrac{t}{2})\} \\
&\le \max\{\varepsilon', \varepsilon'\} = \varepsilon' < \varepsilon
\end{aligned}$$

So, $B(a_0, \varepsilon, t) \cap \mathrm{M}_A \ne \Phi$ and since $\varepsilon > 0$ is arbitrary, we obtain $a_0 \in \overline{\mathrm{M}}_A$ which completes the proof.

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REFERENCES:

[1] T. Bag and T.K. Samanta, Finite dimensional fuzzy normed linear spaces, The Journal of Fuzzy Mathematics, Vol. 11(3) (2003), 687-705.

[2] T. Bag and T.K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets and Systems, Vol. 151 (2005), 513-547.

[3] T. Bag and T.K. Samanta, A comparative study of fuzzy norms on a linear space, Fuzzy Sets and Systems, Vol. 159 (2008), 670-684.

[4] S.C. Cheng and J.N. Mordesen, Fuzzy linear operators and fuzzy normed linear spaces, Bull. Cal. Math. Soc., Vol. 86 (1994), 429-436.

[5] H. Efe and C. Yildiz, some results in fuzzy compact linear operators, Journal of computational Analysis Applications, Vol. 12(1-B) (2010), 251-262.

[6] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets and Systems, Vol. 48 (1992), 239-248.

[7] S.Gahler, Lineare 2-normierte Raume, Math. Nachr., Vol. 28 (1964), 1-43.

[8] S. Gahler, Unter Suchungen uber verallagemeinerte *m*-mertische raume I, Math. Nachr., Vol. 40 (1969), 165-189.

[9] H. Gunawan and Mashadi, On n-normed spaces, Int. J. Math. Math. Sci., Vol. 27(10) (2001), 631-639.

[10] Iqbal H. Jebril and T.K. Samanta, Fuzzy Anti-Normed space, Journal of Mathematics and Technology, February (2010), 66-77.

[11] O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, Vol. 12 (1984), 215-229.

[12] A.K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems, Vol. 12 (1984), 143-154.

[13] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric space, Kybernetica, Vol. 11 (1975), 326-334.

[14] R. Malceski, Strong n-convex *n*-normed spaces, Math. Bulletin, Vol. 21(47) (1997), 81-102.

[15] A. Misiak, *n*-inner product spaces, Math. Nachr., Vol. 140 (1989), 299-319.

[16] I. Narayana and S. Vijayabalaji, Fuzzy *n*-normed linear space, Int. J. Math. Math. Sci., Vol. 24 (2005), 3963-3977.

[17] B. Surender Reddy, Fuzzy anti-2-normed linear space, Journal of Mathematics Research, Vol. 3(2) May (2011), 137-144.

[18] B. Surender Reddy, Fuzzy anti-*n*-normed linear space, Journal of Mathematics and Technology, Vol. 2(1) February (2011), 14-26.

[19] B. Surender Reddy, Some results on *t*-best approximation in fuzzy anti-normed linear spaces. (Communicated)

[20] B. Surender Reddy, Some results on *t*-best approximation in fuzzy anti-2-normed linear spaces. (Communicated)

[21] S.M. Vaezpour and F. Karimi, *t*-best approximation in fuzzy normed spaces, Iranian Journal of fuzzy systems, Vol. 5(2) (2008), 93-99.

[22] S. Vijayabalaji and N. Thillaigovindan, Complete fuzzy n-normed linear space, Journal of Fundamental Sci., Vol. 3(1) (2007), 119-126.

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