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DUALITY THEORY FOR MULTIOBJECTIVE OPTIMIZATION PROBLEM WITH INVEXITY ASSUMPTIONS

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ABSTRACT

The study of multiobjective optimization problems often involves dealing with interval-valued objective functions, which represent uncertainty in real-world applications. This abstract presents an examination of multiobjective interval-valued optimization problems where the objective functions are invex, a generalization of convexity. The paper investigates the duality theory related with these types of problems, exploring how dual formulations can provide bounds and approximate solutions also insights into the feasibility and optimality of the original problem. By using invexity, we extend traditional convex optimization technique to handle interval-valued objectives, thus expanding the applicability of duality results. This study aims to develop theoretical foundations for solving these difficult problems, with a focus on characterizing optimal solutions and exploring their dual relationships.

Keywords: *Multi-objective, Duality theory, non-linear programming.*

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INTRODUCTION

A class of optimization problem, in which Multiobjective Interval-Valued Optimization Problem (MIVP) is required to be optimized simultaneously where the objective functions or constraints are integral-valued functions (Osuna-Gómez *et al*., 1998). The utility of integral-valued functions is realized when we deal with uncertainties in data as they permit the objective function values to be represented as intervals instead of fixed numbers (Upadhyay *et al.*, 2024).

Invex functions: It extends the idea of convex functions in optimization problem, invex function generalized the idea which says convex function are characterized by the property that any line segment between two points on the function lies above the function (Pini, 1991). It ensures that under weaker condition optimality conditions for a problem are necessary as well as sufficient in comparison to convex function, which further makes it more feasible or suitable for a wider category of problems (Bhurjee & Panda, 2016).

Traditionally convexity has played a crucial role in theory of optimization due to its desirable properties, such as guaranteeing global optimality. However there exist many practical situations where convexity conditions are not satisfied. A generalized convexity concept like invexity has been introduced to address this limitation (Li *et al*., 2015).

In the condition of MIVPs, the uses of invex functions can remarkably increase the solvability of the optimization problem. The invexity of the functions make sure that the optimization environment is well-behaved, even when dealing with interval-valued objectives. (Jaisawal *et al*., 2021)

The combination of invex functions into MIVPs provides a mathematical structure that not only helps the essential uncertainties in interval-valued data but also facilitates the derivation of necessary and sufficient optimality conditions. This makes it possible to develop efficient solution methods that can handle the complexities of multiobjective optimization in uncertain environments. (Gulati *et al*., 2005) (Dubey *et al*., 2022) .

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DUALITY FORMULATIONS UNDER INVEXITY

The study of duality in MIVPs under invexity assumptions involves formulating appropriate dual problems and establishing **weak**, **strong**, and **strict converse duality theorems**. Common dual formulations include: (Ahmad *et al*., 2019) (Chen *et al*., 2019).

1. **Mond-Weir Duality**: Extends the classical duality concepts by considering a specific structure of the primal problem and associated invexity conditions.
2. **Wolfe Duality**: Involves a Lagrangian-based approach, suitable for problems with differentiable functions and inequality constraints.
3. **Mixed Type Duality**: Combines features of different dual formulations to address specific problem structures and requirements.

Under suitable invexity assumptions, these duality formulations enable the derivation of comprehensive duality theorems that guarantee that the optimal values of the primal and dual problems coincide (strong duality) or that the existence of an optimal solution to the dual problem implies optimality for the primal problem (strict converse duality) (Ahmad *et al*., 2019).

SIGNIFICANCE AND APPLICATIONS

The exploration of duality in MIVPs under invexity conditions has significant theoretical and practical implications:

- **Theoretical Advancement**: It extends classical optimization theory, providing new insights and tools for analyzing complex optimization problems involving uncertainty and multiple objectives.
- **Solution Methods**: Duality results facilitate the development of efficient computational algorithms by transforming difficult primal problems into more tractable dual problems.
- **Real-world Applications**: This framework is applicable in various domains such as economics engineering design and decision-making processes where trade-offs between conflicting objectives must be managed under uncertainty (Stewart *et al*., 2008) (Cerda-Flores *et al*., 2022) [12][13].

The remainder of the article is structured as follows: Section 1 offers some background information and initial definitions. Section 2 presents the sufficient optimality conditions. In Section 3, weak, strong, and strict converse duality theorems are examined for a Mond-Weir type dual model. Lastly, Section 4 is dedicated to the conclusion.

PRELIMINARIES AND NOTATION

Let A be any set of bounded and closed interval in real number R, also let us considerA^r denote the product $A \times A \times \dots r$ times. Let $A = [a^l, a^u]$ here a^u *and* a^lrespectively be the upper bound and lower bound of A. Any operation between two intervals say A and B is denoted and defined as

$$A \circledast B = \{a * b \text{ where } a \in A \text{ and } b \in B\}$$

Any interval can also be expressed as parametric form (Mahapatra & Mandal, 2012) as for any $A = [a^l, a^u]$,

$$a(t') = a^l + t'(a^u - a^l) \;\; t' \in [0,1]$$

i.e $a(0) = a^l$

$a(1) = a^u$

and $a^l < a(t') < a^u \;\; \forall\, t' \in (0,1)$

then, if $A = \{a(t'): t' \in [0,1]\}$

$B = \{a(t''): t'' \in [0,1]\}$

Then in parametric form

$$A \circledast B = \{a(t') * b(t'') \text{ where } a(t') \in A \text{ and } b(t'') \in B, t' \text{ and } t'' \in [0,1]\}$$

As the set of intervals are partially ordered, and is defined as

For $P = [a^l, a^u]$, $Q = [b^l, b^u]$

then $P \leq_{LU} Q \text{ provided } a^l \leq b^l \text{ and } a^u \leq b^u$

$P <_{LU} Q \text{ provided } P \leq_{LU} Q \text{ and } P \neq Q$

In parametric form

$P \leq_{LU} Q$ if and only if $a(t') \leq b(t') \;\forall\, t' \in [0,1]$

and $P <_{LU} Q \text{ if} P \leq_{LU} Q$ and $a(t') \neq b(t')$ for atleast one $t' \in [0,1]$ (Sun et al. (2014))

Let us consider two tuples of R^n

$u = (u'_1, u'_2, \dots, u'_n)$

$v = (v'_1, v'_2, \dots, v'_n)$

$u = v \Leftrightarrow u'_i = v'_i \quad \forall\, i \in (1,2, \dots, n)$

$u > v \Leftrightarrow u'_i > v'_i \quad \forall\, i \in (1,2, \dots, n)$

$u \geq v \Leftrightarrow u'_i \geq v'_i \quad \forall\, i \in (1,2, \dots, n)$

$u \neq v \Leftrightarrow u'_i \neq v'_i \text{ for at least one } i \in (1,2, \dots, n)$

The product of Interval A with any scalar k is

$$kA = \{ka(t^{'}): a(t^{'}) = a^l + t'(a^u - a^l) \quad t' \in [0,1]\}$$

Let $A^r = (A_1, A_2, \dots, A_r)$ be r tupples of intervals where $A_i = a_i(t_i), t_i \in [0,1] \forall\, i$ and $A_i = [a_i^l, a_i^u]$, $i.e \quad A_i = a_i^l + t_i(a_i^u - a_i^l), \quad A^r\, can\, also\, be\, written\, as\, A^r = (a_1(t_1), a_2(t_2) \dots a_r(t_r))$ then we define operation between real vector $q \in R^r\, and\, A^r$ as $A^r \circ q = \sum_{i=1}^r A_i q_i$.

Let $X \subseteq R^n$ and let us define an interval valued function $\phi_{A^r}(x) = \{f_{a(t')}(x): f_{a(t')}: R^n \to R, a(t') \in A^r\}$

If $f_{a(t')}(x) is\, continous\, in\, t'\, for\, fixed\, x$ then $min_{t' \in [0,1]} f_{a(t')}(x) max_{t' \in [0,1]} f_{a(t')}(x)$ exist and if $f_{a(t')}(x)$

Is increasing then $\phi_{A^r}(x) = [f_{a(0)}(x),\ f_{a(1)}(x)]$

Definition 1: An interval valued function $\phi_{A^r}: X \to I$ is differentiable at x if $f_{a(t)}(x)$ is differentiable at $x \forall\, t_i \in [0,1]$, also the gradient of $\phi_{A^r}(x)$ at $x \in X$ is defined as

$$\nabla \phi_{A^r}(x) = \left(\frac{\partial \phi_{A^r}(x)}{\partial x_1}, \frac{\partial \phi_{A^r}(x)}{\partial x_2}, \dots \frac{\partial \phi_{A^r}(x)}{\partial x_n}\right) \quad (1)$$

Definition 2: An interval valued differentiable function $\phi_{A^r}: X \to I$ is an invex function if there exist $\eta: X \times X \to R^n\, at\, y \in X\, if\, for\, all\, x \in X\, if$

$$\phi_{A^r}(x) \odot \phi_{A^r}(y) (>) \geq \eta(x,y)^T . \nabla \phi_{A^r}(y) \quad (2)$$

In parametric form

$$f_{a(t')}(x) - f_{a(t')}(y) (>) \geq \eta(x,y)^T . \nabla f_{a(t')}(y)$$

Example: Let $X = R_+^2$ and let us define an interval valued function $\phi: X \to A$ define as

$$\phi(x) = [2,4]x_1^2 + [3,5]x_2^2 + [2,6]x_1 x_2$$

In parametric form

$$\phi_{A^3}(x) = \{f_{a(t')}(x) = (2+2t_1)x_1^2 + (3+2t_2)x_2^2 + (2+4t_3)x_1 x_2,\ t_i \in [0,1]\ i = 1,2,3\}$$

Let $\eta: X \times X \to R^2$ define as $\begin{bmatrix} -x_1 - \frac{y_1}{2} \\ -x_2 - \frac{y_2}{2} \end{bmatrix}$

Consider $f_{a(t')}(x) - f_{a(t')}(y) - \eta(x,y)^T . \nabla f_{a(t')}(y)$

$$= (2+2t_1)x_1^2 + (3+2t_2)x_2^2 + (2+4t_3)x_1x_2 - (2+2t_1)y_1^2 + (3+2t_2)y_2^2 + (2+4t_3)y_1y_2$$
$$- \left[-x_1 - \frac{y_1}{2}, -x_2 - \frac{y_2}{2}\right] \begin{bmatrix} (2+2t_1)2y_1 + (2+4t_3)y_2 \\ (3+2t_2)2y_2 + (2+4t_3)y_1 \end{bmatrix}$$
$$= (2+2t_1)(x_1^2 + 2x_1y_1) + (3+2t_2)(x_2^2 + 2x_2y_2) + (2+4t_3)(x_1x_2 + y_1y_2 + x_2y_1) \geq 0$$
$$\forall\, x, y \in X \subseteq R_+^2 t_i \in [0,1]$$
$$\therefore f_{a(t')}(x) - f_{a(t')}(y) (>) \geq \eta(x,y)^T . \nabla f_{a(t')}(y)$$
$$i.e \quad \phi_{A^r}(x) \odot \phi_{A^r}(y) (>) \geq \eta(x,y)^T . \nabla \phi_{A^r}(y)$$

Hence $\phi_{A^r}(x)$ is invex.

Definition 3: An interval valued differentiable function $\phi_{A^r}: X \to I$ is quasi-invex if $\exists\, \eta: X \times X \to R^n\, at\, y \in X\, if\, for\, all\, x \in X\, if$

$$\phi_{A^r}(x) \odot \phi_{A^r}(y) \leq 0 \Rightarrow \eta(x,y)^T . \nabla \phi_{A^r}(y) \leq 0 \quad (3)$$

In parametric form

$$f_{a(t')}(x) - f_{a(t')}(y) \leq 0 \Rightarrow \eta(x,y)^T . \nabla f_{a(t')}(y) \leq 0 \forall\, t' \in [0,1]$$

Example: Let us consider an interval valued function $\phi: X \to I$ defined as

$$\phi(x) = [3,6]x^5 + [4,8]$$

In parametric form

$$\phi_{A^2}(x) = (3+3t_1)x^5 + (4+4t_2) t_i \in [0,1]$$

Let $\eta: R \times R \to R$ defined by

$$\eta(x,y) = (x+y)^4 (x-y)$$

We have

$$\phi_{A^r}(x) \odot \phi_{A^r}(y) = f_{a(t)}(x) - f_{a(t)}(y) = (3+3t_1)(x^5 - y^5)$$
$$= (3+3t_1)(x-y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$$

and $\eta(x,y)^T.\nabla f_{a(t)}(y) = 5y^4(3+3t_1)\,(x+y)^4(x-y)$
now let $f_{a(t)}(x) - f_{a(t)}(y) \le 0$
$\Rightarrow x \le y$
$\Rightarrow \eta(x,y)^T.\nabla f_{a(t)}(y) \le 0$
$\Rightarrow \eta(x,y)^T.\nabla \phi_{A^r}(y) \le 0$

Definition 4: An interval valued differentiable function $\phi_{A^r}: X \to I$ is pseudo invex if $\exists \eta: X \times X \to R^n$ $at\ y \in X\ if\ for\ all\ x \in X\ if$

$$\phi_{A^r}(x) \odot \phi_{A^r}(y) < 0 \Rightarrow \eta(x,y)^T.\nabla \phi_{A^r}(y) < 0 \tag{4}$$

In parametric form

$$f_{a(t)}(x) - f_{a(t)}(y) < 0 \Rightarrow \eta(x,y)^T.\nabla f_{a(t)}(y) < 0 \forall\, t_i \in [0,1]$$

Example: Let us consider an interval valued function $\phi: X \to I$ defined as

$$\phi(x) = [2,5]x^3 + [4,5]x$$

In parametric form

$$\phi_{A^2}(x) = f_{a(t')}(x) = (2+3t'_1)x^3 + (4+t'_2)x \text{ where } t'_i \in [0,1]$$

Let $\eta: R \times R \to R$ defined by

$$\eta(x,y) = (x^2 - y^2)$$

Now $f_{a(t')}(x) - f_{a(t')}(y) = (2+3t'_1)x^3 + (4+t'_2)x - (2+3t'_1)y^3 - (4+t'_2)y$
$= (x-y)\{(2+3t'_1)(x^2+xy+y^2) + (4+t'_2)\}$

and $\eta(x,y)^T.\nabla f_{a(t')}(y) = \{(2+3t'_1)3y^2 + (4+t'_1)\}(x^2-y^2)$
$= \{(2+3t'_1)3y^2 + (4+t'_1)\}(x+y)(x-y)$

If $\eta(x,y)^T.\nabla f_{a(t')}(y) \ge 0$
$\Rightarrow x \ge y$
$\Rightarrow f_{a(t')}(x) - f_{a(t')}(y) \ge 0$
$\phi_{A^2}(x)$ is pseudo invex.

Primal problem

Here, consider multiobjective interval-valued optimization problem (**MIVP**):

$$\text{Min } \phi_{A^r}(x) = \left(\phi^1_{A^r}(x), \phi^2_{A^r}(x), \dots, \phi^s_{A^r}(x)\right)$$
$$= \left(f^1_{a(t')}(x),\ f^2_{a(t')}(x), \dots f^s_{a(t')}(x)\right)$$
$$\text{Subject to } G^j_{B^k}(x) \le 0,\ \ j = 1,2,\dots,m \tag{5}$$

Where $\phi^i_{A^r}: X \to I$ and $G^j_{B^k}: X \to I, j = 1,2,\dots,m, i = 1,2,\dots,s$ and are differentiable interval valued n function and defined by $\phi^i_{A^r}(x) = f^i_{a(t')}(x)$.

Let $\Omega = \{x \in R^n : G^j_{B^k}(x) \le 0,\ \ j = 1,2,\dots,m\}$ i.e $\Omega = \{x \in R^n : g^j_{b(t')}(x) \le 0,\ \ j = 1,2,\dots,m\}$ denote set of all feasible solution to **MIVP** (6)

MOND-WEIR DUALITY

We present Mond-weir dual model for **MIVP**

$$\textbf{(MIVD)}\ \max \phi_{A^r}(y) = \left(\phi^1_{A^r}(y), \phi^2_{A^r}(y), \dots, \phi^s_{A^r}(y)\right)$$
$$= \left(f^1_{a(t')}(y),\ f^2_{a(t')}(y), \dots f^s_{a(t')}(y)\right)$$

Subject to $\sum_{i=1}^{s} \lambda_i \nabla \phi^i_{A^r}(y) + \sum_{j=1}^{m} \mu_j \nabla G^j_{B^k}(y) = 0$
$\mu_j \nabla G^j_{B^k}(y) \ge 0, \lambda_i > 0, \mu_j \ge 0$
i.e $\sum_{i=1}^{s} \lambda_i \nabla f^i_{a(t')}(y) + \sum_{j=1}^{m} \mu_j \nabla g^i_{b(t')}(y) = 0$ (7)
$\mu_j \nabla g^i_{b(t')}(y) \ge 0, \lambda_i > 0, \mu_j \ge 0$

Let W $= \left\{y \in R^n : \sum_{i=1}^{s} \lambda_i \nabla \phi^i_{A^r}(y) + \sum_{j=1}^{m} \mu_j \nabla G^j_{B^k}(y) = 0, \mu_j \nabla G^j_{B^k}(y) \ge 0, \lambda_i > 0, \mu_j \ge 0\ \ i = 1,2,\dots,s\ \ j = 1,2,\dots,m\right\}$ (8)
i.e. $W = \{y \in R^n : \sum_{i=1}^{s} \lambda_i \nabla f^i_{a(t')}(y) + \sum_{j=1}^{m} \mu_j \nabla g^j_{b(t')}(y) = 0, \mu_j \nabla g^j_{b(t')}(y) \ge 0, \lambda_i > 0, \mu_j \ge 0,\ j = 1,2,\dots,m\ i = 1,2,\dots,s$ } denote feasible solution set to **MIVD**

WEAK DUALITY THEOREM

Theorem 1: Let x' and (y',λ,μ) be the feasible solution of **MIVP** and **MIVD** respectively. Assume that $\phi^i_{A^r}$ and $G^j_{B^k} i = 1,2,\dots,s \;\; j = 1,2,\dots,m$, are invex function whose range are intervals with reference toη at y. Then $\phi_{A^r}(x') \geq \phi_{A^r}(y')$.

Proof: Let x' and (y',λ,μ) be the feasible solution of **MIVP** and **MIVD** respectively. Let $\phi^i_{A^r}$ and $G^j_{B^k} i = 1,2,\dots,s \;\; j = 1,2,\dots,m$, are invex function whose range are intervals with reference to η at y'.

Since $\phi^i_{A^r}$ is invex with respect to η at y' then, we have from (2)

$$\phi^i_{A^r}(x') \odot \phi^i_{A^r}(y') \geq \eta(x',y')^T.\nabla\phi^i_{A^r}(y') \forall\, i = 1,2,\dots,s$$

In parametric form

$$f^i_{a(t')}(x') - f^i_{a(t')}(y') \geq \eta(x',y')^T.\nabla f^i_{a(t')}(y') \qquad \forall\, i = 1,2,\dots,s$$

$$\Rightarrow \textstyle\sum_{i=1}^{s} f^i_{a(t')}(x') - f^i_{a(t')}(y') \geq \eta(x',y')^T.\sum_{i=1}^{s}\nabla f^i_{a(t')}(y) \tag{9}$$

Also, since $\mu_j \geq 0,\; j = 1,2,\dots,m$ and from feasibility of x' and (y',λ,μ) to **MIVP** and **MIVD** respectively **we** obtain

$$\textstyle\sum_{j=1}^{m}\mu_j\, G^j_{B^k}(x') \leq \sum_{j=1}^{m}\mu_j\, G^j_{B^k}(y')$$

$$i.e\; \textstyle\sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(x') \leq \sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(y') \tag{10}$$

and the invexity of $G^j_{B^k}$ with respect to η at y' will give

$$\textstyle\sum_{j=1}^{m}\mu_j\, G^j_{B^k}(x') \odot \sum_{j=1}^{m}\mu_j\, G^j_{B^k}(y') \geq \eta(x',y')^T.\sum_{j=1}^{m}\mu_j\, G^j_{B^k}(y')$$

$$i.e\;\; \textstyle\sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(x') - \sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(y') \geq \eta(x',y')^T.\sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(y') \tag{11}$$

On adding (9) and (11), we get

$$\textstyle\sum_{i=1}^{s}\; (f^i_{a(t')}(x') - f^i_{a(t')}(y')) + \sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(x') - \sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(y') \geq \eta(x',y')^T.\left(\sum_{i=1}^{s}\nabla f^i_{a(t')}(y') + \sum_{j=1}^{m}\mu_j\, g^j_{b(t')}(y')\right)$$

From (7) and (10) we get

$$\textstyle\sum_{i=1}^{s}\; (f^i_{a(t')}(x') - f^i_{a(t')}(y')) \geq 0$$

$$\textstyle\sum_{i=1}^{s}\; f^i_{a(t')}(x') \geq \sum_{i=1}^{s} f^i_{a(t')}(y')$$

$i.e.\; \phi_{A^r}(x') \geq \phi_{A^r}(y')$, Hence the theorem.

STRICT CONVERSE DUALITY

Theorem 2: Let x' and (y',λ',μ') be the feasible solution to **MIVP** and **MIVD** respectively. Assume that $\phi^i_{A^r}\; i = 1,2,\dots,s$ are strictly invex function whose range are intervals and $G^j_{B^k}$ are also invex function whose range are intervals with reference same η at y' and $\phi_{A^r}(x') \leq \phi_{A^r}(y')$ then $x' = y'$.

Proof: Let x' and (y',λ',μ') be the feasible solution to **MIVP** and **MIVD** respectively. and also let $\phi^i_{A^r}\; i = 1,2,\dots,s$ are strictly invex function whose range are intervals and $G^j_{B^k}$ are also invex function whose range are intervals with reference same η at y' From the hypothesis

$$\phi_{A^r}(x') \leq \phi_{A^r}(y')$$

$$\Rightarrow \phi^i_{A^r}(x') \leq \phi^i_{A^r}(y') \;\;\forall\;\; i = 1,2,\dots,s$$

$$i.e\;\; f^i_{a(t')}(x') \leq f^i_{a(t')}(y') \;\;\forall\, i = 1,2,\dots,s \tag{12}$$

Let if possible $x' \neq y'$.

Also, since $\phi^i_{A^r},\;\; i = 1,2,\dots,s$ arestrictly invex function whose range are intervalswith referenceη at y' we get

$$\phi^i_{A^r}(x') \odot \phi^i_{A^r}(y') > \eta(x',y')^T.\nabla\phi^i_{A^r}(y') \forall\, i = 1,2,\dots,s$$

$$i.e\;\; f^i_{a(t')}(x') - f^i_{a(t')}(y') > \eta(x',y')^T.\nabla f^i_{a(t')}(y') \qquad \forall\, i = 1,2,\dots,s$$

$$\Rightarrow \textstyle\sum_{i=1}^{s}\;\; (f^i_{a(t')}(x') - f^i_{a(t')}(y')) > \eta(x',y')^T \sum_{i=1}^{s}\nabla f^i_{a(t')}(t') \;\forall\, i = 1,2,\dots,s \tag{13}$$

Now, since $\mu'_j \geq 0,\; j = 1,2,\dots,m$ and from feasibility of $x'$and (y',λ',μ') to **MIVP** and **MIVD** respectivelywe obtain

$$\textstyle\sum_{j=1}^{m}\mu'_j\, G^j_{B^k}(x') \leq \sum_{j=1}^{m}\mu'_j\, G^j_{B^k}(y') \tag{14}$$

and the invexity of $G^j_{B^k}\;\; j = 1,2,\dots m$ with reference toη at y' yields

$$\textstyle\sum_{j=1}^{m}\mu'_j\, G^j_{B^k}(x') \odot \sum_{j=1}^{m}\mu'_j\, G^j_{B^k}(y') \geq \eta(x',y')^T.\sum_{j=1}^{m}\mu'_j\, G^j_{B^k}(y')$$

$$i.e\;\; \textstyle\sum_{j=1}^{m}\mu'_j\, g^j_{b(t')}(x') - \sum_{j=1}^{m}\mu'_j\, g^j_{b(t')}(y') \geq \eta(x',y')^T.\sum_{j=1}^{m}\mu'_j\, g^j_{b(t')}(y') \tag{15}$$

From (14) and (15), we get

$$(x', y')^T . \sum_{j=1}^{m} \mu_j' \, g_{b(t')}^{j}(y') \leq 0 \tag{16}$$

From (7) and (16), we get

$$\eta(x', y')^T \sum_{i=1}^{s} \nabla f_{a(t')}^{i}(y') \geq 0 \tag{17}$$

From (13) and (17), we get

$$\sum_{i=1}^{s} \left(f_{a(t')}^{i}(x') - f_{a(t')}^{i}(y')\right) > 0$$

$$\sum_{i=1}^{s} f_{a(t')}^{i}(x') > \sum_{i=1}^{s} f_{a(t')}^{i}(y') \tag{18}$$

Which contradict (12)

Hence $x' = y'$. This completes the proof.

CONCLUSION

The study of duality in multiobjective interval-valued optimization problems under invexity assumptions represents a flexible framework for addressing complex optimization scenarios characterized by multiple, uncertain objectives. By extending traditional convexity and duality concepts, this approach provides comprehensive tools for both theoretical analysis and practical solution of advanced optimization problems; it is the way for more effective decision-making processes across various disciplines.

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