International Journal of Mathematical Archive-15(6), 2024, 1-8
Available online through www.ijma.info ISSN 2229 – 5046

TWO NEW MODIFIED ITERATIVE METHODS FOR SOLVING LINEAR SYSTEMS

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(Received On: 17-05-24; Revised & Accepted On: 22-06-24)

ABSTRACT

In order to improve the convergence rate of the iterative methods namely Jacobi method and Gauss-Seidel method, we consider a new preconditioner based on the preconditioner proposed by Zhouji Chen [4]. We discuss some of the convergence property and also provide some comparison results of the new methods. In addition, some simple numerical examples are also introduced to illustrate the theoretical analysis.

Keywords: *Spectral radius, Preconditioning, Preconditioned linear system, Modified Jacobi method, Modified Gauss-Seidel method, M-matrix, Comparison theorem.*

1. INTRODUCTION

Let us consider the following linear system

$$Ax = b$$

Where $A = (a_{ij})_{nxn}$ is a known nonsingular M-matrix, b is a known $nx1$ and x is an unknown $nx1$ vectors. Throughout the present paper, without loss of generality we always assume that the coefficient matrix A has a splitting of the form $A = I - L - U$, where I is the identity matrix, L and U are strictly lower triangular and strictly upper triangular matrices derived from A, respectively.

The basic idea behind an iterative method is first to write the system $Ax = b$ in the equivalent form $x = Tx + c$. After making the system of equations $Ax = b$ in the form $x = Tx + c$, a sequence of approximations $\{x^{(k)}\}_k^\infty$ is obtained by the following scheme:

$$x^{(k+1)} = Tx^{(k)} + c; \quad k = 0, 1, 2, 3, \cdots\cdots$$

starting with an initial approximations $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \cdots\cdots\cdots, x_n^{(0)})^T$ to the true solution vector x and $A = M - N$; M is a nonsingular matrix. Then $T = M^{-1}N$ is called the iteration matrix and $c = M^{-1}b$ is called the iteration vector of the iterative method.

Next, we transform the linear system $Ax = b$ into the preconditioned linear system

$$PAx = Pb,$$

Where P is called a preconditioner. When we apply the Jacobi iterative method or the Gauss-Seidel iterative method to the above preconditioned linear system, then we obtain the preconditioned Jacobi iterative method or the preconditioned Gauss-Seidel iterative method for solving the system of linear equations $Ax = b$.

In 1991, Gunawardena *et al.* [1] first proposed the preconditioner $P_s = I + S$, where S is defined by,

$$S = (s_{ij}) = \begin{cases} -a_{i,i+1}, & 1 \le i \le n-1 \\ 0, & otherwise \end{cases}$$

In 2004, M. Morimoto *et al.* [3] considered the preconditioner P_{sm} as follows:

$$P_{sm} = I + S + S_m\ ;$$

Where S is mentioned above and S_m is defined as

$$S_m = ((s_m)_{ij}) = \begin{cases} -a_{i,k_i}, & i = 1, 2, \cdots\cdots, n-2\ ; \quad j > i+1 \\ 0\ , & otherwise \end{cases}$$

and $$k_i = \min\left\{ \mathrm{j} : \max_{\mathrm{j}} |\mathrm{a_{ij}}|,\ \mathrm{i} < n-1,\ j > i+1 \right\}.$$

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In 2013, Zhouji Chen [4] proposed the preconditioner P_{sm2} as follows:

$$P_{sm2} = I + S + S_m + R_m$$

Where $S,\ S_m$ are given above and R_m is defined as

$$R_m = (r_{ij}^m) = \begin{cases} -a_{n,k_n}, & i = n,\ \ j = k_n \\ 0, & otherwise \end{cases}$$

and $$k_n = \min\{j: |a_{n,j}| = \max\{|a_{n,l}|,\ l = 1, 2, \cdots\cdots, n-1\}\}$$

We propose a new preconditioner P_{sm3} as follows:

$$P_{sm3} = I + S + S_m + R_m + R'$$

Where $S,\ S_m,\ R_m$ are mentioned above and R' is defined by

$$R' = (r_{ij}') = \begin{cases} -a_{nj}, & 1 \le j \le n-1,\ \ j \ne k_n \\ 0, & otherwise \end{cases}$$

Then the preconditioned system matrices for the preconditioners P_s, P_{sm}, P_{sm2} and P_{sm3} are respectively as follows:

$$A_s = M_s - N_s = I - D - L - E - U + S - SU$$

$$A_{sm} = M_{sm} - N_{sm} = I - D - D' - L - E - E' - U + S + S_m - SU - S_m U - F'$$

$$A_{sm2} = M_{sm2} - N_{sm2}$$
$$= I - D - D' - L - E - E' - D'' - E'' + R_m - U + S + S_m - SU - S_m U - F'$$

and

$$A_{sm3} = M_{sm3} - N_{sm3}$$
$$= I - D - D' - L - E - E' - D'' - D''' - E'' - E''' + R_m + R' - U + S + S_m - SU - S_m U - F'$$

Where $D,\ E$ are the diagonal and strictly lower triangular parts of SL; $D',\ E'$ and F' are the diagonal, strictly lower and strictly upper triangular parts of $S_m L$; D'' and E'' are the diagonal and strictly lower triangular parts of $R_m(L+U)$ and D''', E''' are the diagonal and strictly lower triangular parts of $R'(L+U)$ respectively.

Now, let we make the following assumption:

$$\text{(A)} \qquad \begin{cases} 0 < a_{i,i+1}a_{i+1,i} + a_{i,K_i}a_{K_i,i} < 1; & i = 1, 2, \cdots\cdots, n-2 \\ 0 < a_{i,i+1}a_{i+1,i} < 1; & i = n-1 \\ 0 < a_{n,K_n}a_{K_n,n} < 1 & \\ 0 \le \sum_{k=1}^{n-1} a_{n,k}a_{k,n} < 1, & k \ne K_n \end{cases}$$

then the preconditioned iteration matrices $T_s,\ T_{sm},\ T_{sm2}$ and T_{sm3} for both the modified Jacobi and modified Gauss-Seidel methods are well-defined. For our convenience, for solving the linear system by modified Jacobi method, we use index J and for solving by modified Gauss-Seidel method, we use index G.

Next, we organize the remaining portion of the paper as follows: Section 2 is the preliminaries. In section 3, we established some comparison theorems. Two simple numerical examples are studied in section 4 to illustrate our theoretical results. Lastly in section 5, conclusion is drawn.

2. PRELIMINARY NOTES

Suppose $A = (a_{ij})_{nxn}$ and $B = (b_{ij})_{nxn}$ then we write $A \ge B$ if $a_{ij} \ge b_{ij}$ holds for all $i, j = 1, 2, \cdots\cdots, n$ and $A \ge 0$ (called nonnegative) if $a_{ij} \ge 0$ for all $i, j = 1, 2, \cdots\cdots, n$; where 0 is an nxn zero matrix. For the $nx1$ vectors a, b; $a \ge b$ and $a \ge 0$ can also be defined in the similar manner.

Definition 2.1: A Z-matrix A is called an M-matrx, if all the diagonal entries of A are positive, all the real eigenvalues of A are positive and the real part of any eigenvalue of A is positive.

Definition 2.2: [5] A matrix $A = (a_{ij})_{nxn}$ is an L-matrix if $a_{ii} > 0, 1 \le i \le n$ and $a_{ij} \le 0; 1 \le i \le n, 1 \le j \le n,\ i \ne j$. A nonsingular L-matrix A is said to be a nonsingular M-matrix if $A^{-1} \ge 0$.

Definition 2.3: Let A be a real matrix. Then the representation $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called

1. convergent if $\rho(M^{-1}N) < 1$;
2. regular if $M^{-1} \ge 0$ and $N \ge 0$;
3. weak regular if $M^{-1} \ge 0$ and $M^{-1}N \ge 0$;
4. nonnegative if $M^{-1}N \ge 0$;
5. M-splitting if M is a nonsingular M-matrix and $N \ge 0$.

Definition 2.4: The splitting $A = M - N$ is called the Jacobi splitting of A if $M = I$ is nonsingular and $N = L + U$. In addition, the splitting is called

(1) Jacobi convergent if $\rho(M^{-1}N) < 1$;
(2) Jacobi regular if $M^{-1} = I^{-1} \geq 0$ and $N = (L + U) \geq 0$.

Definition 2.5: A splitting of matrix A i.e. $A = M - N$ is called a Gauss-Seidel splitting if $M = I - L$ is nonsingular and $N = U$. In addition, the splitting is called

(1) Gauss-Seidel convergent if $\rho(M^{-1}N) < 1$;
(2) Gauss-Seidel regular if $M^{-1} = (I - L)^{-1} \geq 0$ and $N = U \geq 0$;
(3) Gauss-Seidel weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$.

Lemma 2.6: [6] Let A be a nonnegative nxn nonzero matrix, then

(1) $\rho(A)$, the spectral radius of A, is an eigenvalue;
(2) A has a nonnegative eigenvector corresponding to $\rho(A)$;
(3) $\rho(A)$ is a simple eigenvalue of A;
(4) $\rho(A)$ increases when any entry of A increases.

Lemma 2.7: [7] Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.

Lemma 2.8: [8] Let A be a nonsingular M-matrix and let $A = M_1 - N_1 = M_2 - N_2$ be two convergence splitting, the first one weak regular and second one regular if $M_1^{-1} \geq M_2^{-1}$, then

$$\rho(M_1^{-1}N_1) \leq \rho(M_2^{-1}N_2) < 1.$$

Lemma 2.9: [9] Let A be a nonsingular L-matrix. Then A is called a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$.

Lemma 2.10: [7] Let A be irreducible, $A = M - N$ be an M-splitting. Then there is a positive vector x such that $M^{-1}Nx = \rho(M^{-1}N)x$ and $\rho(M^{-1}N) > 0$.

Lemma 2.11: [11] Let $A = (a_{ij}) \in R^{nxn}$ be an irreducible M-matrix with $a_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$, and let $A_s = (I + S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. Then $M_s^{-1}N_s$ has a positive perron vector and $\rho(M_s^{-1}N_s) > 0$.

Lemma 2.12: [12] Let A be an M-matrix and let $A_s = (I + S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. If $\rho(M_s^{-1}N_s) > 0$, then $Ax \geq 0$ for any nonnegative perron vector of $M_s^{-1}N_s$.

Lemma 2.13: [1] Let A be a nonnegative matrix. Then

(a) If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
(b) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

3. COMPARISON RESULTS FOR MODIFIED ITERATIVE METHODS

In this section, we derive some comparison theorems of the modified Jacobi and modified Gauss-Seidel methods with the preconditioners $P_s = I + S$ [1], $P_{sm} = I + S + S_m$ [3], $P_{sm2} = I + S + S_m + R_m$ [4] and the proposed preconditioners $P_{sm3} = I + S + S_m + R_m + R'$ and also discuss the convergence property of the two modified iterative methods with the above preconditioners. We assume, $A = (a_{ij})_{nxn}$ is a nonsingular M-matrix with $a_{n,1} \neq 0$ and $a_{i,i+1} \neq 0, i = 1, 2, 3, \cdots\cdots, n-1$.

Theorem 3.1: Let A be a nonsingular M-matrix. Then under the assumption (A), the splitting $A_{sm3_G} = M_{sm3_G} - N_{sm3_G}$ is regular and Gauss-Seidel convergent.

Proof: Under the assumption (A), we can notice that the diagonal elements of A_{sm3_G} all are positive and due to this ${M_{sm3_G}}^{-1}$ exists. We know that (see [9]) an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + S + S_m + R_m + R' \geq 0$ giving $A_{sm3_G}y = (I + S + S_m + R_m + R')Ay > 0$. Thus, we have the L-matrix A_{sm3_G} is a nonsingular M-matrix which means that ${A_{sm3_G}}^{-1} \geq 0$ and therefore by Lemma 2.7., we obtain $\rho({M_{sm3_G}}^{-1}N_{sm3_G}) < 1$ i.e. $\rho(T_{sm3_G}) < 1$.

We also see that $L + E + E' + E'' + E''' - R_m - R' \geq 0$, since $L \geq R_m + R' \geq 0$. Using the assumption (A), we have $D + D' + D'' + D''' < I$ so that $I - D - D' - D'' - D''' \geq 0$. Thus

$$M_{sm3_G}{}^{-1} = (I - D - D' - D'' - D''' - L - E - E' - E'' - E''' + R_m + R')^{-1}$$
$$= [(I - D - D' - D'' - D''') - (L + E + E' + E'' + E''' - R_m - R')]^{-1}$$
$$= [I - (I - D - D' - D'' - D''')^{-1}(L + E + E' + E'' + E''' - R_m - R')]^{-1}(I - D - D' - D'' - D''')^{-1}$$
$$= \{I + [(I - D - D' - D'' - D''')^{-1}(L + E + E' + E'' + E''' - R_m - R')] + [(I - D - D' - D'' - D''')^{-1}(L + E+E'+E''+E'''-R_m-R')]^2 + \cdots\cdots + [(I-D-D'-D''-D''')^{-1}(L+E+E'+E''+E'''-R_m-R')]^{n-1}\}(I-D-D'-D''-D''')^{-1} \geq 0$$

and $N_{sm3_G} = U - S - S_m + SU + S_mU + F' \geq 0$, since $U \geq S + S_m \geq 0$ and $SU + S_mU + F' \geq 0$.

Hence $A_{sm3_G} = M_{sm3_G} - N_{sm3_G}$ is a regular and Gauss-Seidel convergent splitting by definition 2.5. and Lemma 2.7..

Theorem 3.2: Let A be a nonsingular M-matrix. Then under the assumption (A), the splitting $A_{sm2_G} = M_{sm2_G} - N_{sm2_G}$ is regular and Gauss-Seidel convergent.

Proof: See, Zhouji Chen [4, Theorem 1].

Theorem 3.3: Let A be a nonsingular M-matrix. Then under the assumption(A), the following inequality holds:

$$\rho(T_{sm3_G}) \leq \rho(T_{sm2_G}) < 1.$$

Proof: From Theorem 3.2., we have $A_{sm2_G} = M_{sm2_G} - N_{sm2_G}$, where $M_{sm2_G} = I - D - D' - D'' - L - E - E' - E'' + R_m$ and $N_{sm2_G} = U - S - S_m + SU + S_mU + F'$; is the regular and Gauss-Seidel convergent splitting i.e., $M_{sm2_G}{}^{-1} \geq 0$, $N_{sm2_G} \geq 0$ and $\rho(M_{sm2_G}{}^{-1}N_{sm2_G}) < 1$. Again, from Theorem 3.1., we know that $A_{sm3_G} = M_{sm3_G} - N_{sm3_G}$, where $M_{sm3_G} = I - D - D' - D'' - D''' - L - E - E' - E'' - E''' + R_m + R'$ and $N_{sm3_G} = U - S - S_m + SU + S_mU + F'$; is the regular and Gauss-Seidel convergent splitting i.e., $M_{sm3_G}{}^{-1} \geq 0$, $N_{sm3_G} \geq 0$ and $\rho(M_{sm3_G}{}^{-1}N_{sm3_G}) < 1$.

Since A is a nonsingular M-matrix and $A_{sm2_G} = P_{sm2}A$ is the Gauss-Seidel splitting, thus there exists a positive eigenvector x such that $T_{sm2_G}x = \rho(T_{sm2_G})x$ and $\rho\left(T_{sm2_G}\right) > 0$.

Ovbiously, $$N_{sm3_G} = N_{sm2_G}$$

Also, $$M_{sm3_G} - M_{sm2_G} = R' - D''' - E''' = R' - (D''' + E''')$$
$$= R' - R'(L + U) = R' - R'(I - A) = R'A \quad (1)$$

It follows from (1) that

$$M_{sm2_G}{}^{-1} - M_{sm23_G}{}^{-1} = M_{sm3_G}{}^{-1}R'AM_{sm2_G}{}^{-1} \quad (2)$$

Multiplying on the right of (2) by N_{sm2_G}, we get

$$T_{sm2_G} - T_{sm23_G} = M_{sm3_G}{}^{-1}R'AT_{sm2_G} \quad (3)$$

Multiplying again on the right of (3) by $x > 0$, we have

$$\rho\left(T_{sm2_G}\right)x - T_{sm3_G}x = \rho(T_{sm2_G})M_{sm3_G}{}^{-1}R'Ax \quad (4)$$

Again,

$$A_{sm2_G} = P_{sm2}A = M_{sm2_G} - N_{sm2_G}$$

Or, $$A = P_{sm2}^{-1}M_{sm2_G} - P_{sm2}^{-1}N_{sm2_G}$$

We assume $$M_1 = P_{sm2}^{-1}M_{sm2_G} \text{ and } N_1 = P_{sm2}^{-1}N_{sm2_G}$$

One can easily verify that $M_1^{-1}N_1 = M_{sm2_G}{}^{-1}N_{sm2_G}$ and hence $\rho(M_1^{-1}N_1) < 1$ and $A = M_1 - N_1$ be a regular and Gauss-Seidel convergent splitting and thus there exists a positive vector x such that $\rho(M_1^{-1}N_1)x = M_1^{-1}N_1x$.

Now, $$Ax = (M_1 - N_1)x = M_1(I - M_1^{-1}N_1)x$$
$$= M_1[1 - \rho(M_1^{-1}N_1)]x$$
$$= \frac{1 - \rho(M_1^{-1}N_1)}{\rho(M_1^{-1}N_1)} M_1\, \rho(M_1^{-1}N_1)x$$
$$= \frac{1-\rho(M_1^{-1}N_1)}{\rho(M_1^{-1}N_1)} N_1x \geq 0 \quad (5)$$

We can easily observe from (4) and (5) that

$$T_{sm3_G}x \leq \rho(T_{sm2_G})x$$

Therefore, it follows from Lemma 2.13., that

$$\rho(T_{sm3_G}) \le \rho(T_{sm2_G})$$

Hence
$$\rho(T_{sm3_G}) \le \rho(T_{sm2_G}) < 1.$$

This completes the proof of the theorem.

Theorem 3.4: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm2_G}) \le \rho(T_{sm_G}) < 1.$$

Proof: See, Zhouji Chen [4, Theorem 3].

Theorem 3.5: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm_G}) \le \rho(T_{s_G}) < 1.$$

Proof: See, M. Morimoto et al. [3].

Theorem 3.6: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{s_G}) \le \rho(T) < 1.$$

Proof: See, A. D. Gunawardena *et al.* [1].

Theorem 3.7: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm3_G}) \le \rho(T_{sm2_G}) \le \rho(T_{sm_G}) \le \rho(T_{s_G}) \le \rho(T) < 1.$$

Proof: Combining Theorem 3.3., Theorem 3.4., Theorem 3.5., and Theorem 3.6., we obtain

$$\rho(T_{sm3_G}) \le \rho(T_{sm2_G}) \le \rho(T_{sm_G}) \le \rho(T_{s_G}) \le \rho(T) < 1.$$

Theorem 3.8: Let A be a nonsingular M-matrix. Then under the assumption (A), the splittings $A_{sm3_J} = M_{sm3_J} - N_{sm3_J}$ and $A_{sm2_J} = M_{sm2_J} - N_{sm2_J}$ both are regular and Jacobi convergent.

Proof: We have $A^{-1} \ge 0$, as A is a nonsingular M-matrix. Under the assumption (A), the diagonal elements of A_{sm3_J} are positive and so ${M_{sm3_J}}^{-1}$ is well defined. We know that (see [9]) an L-matrix A is a nonsingular M-matrix if and only if there exists a positive vector y such that $Ay > 0$. By taking such y, the fact that $I + S + S_m + R_m + R' \ge 0$ gives $A_{sm3_J} y = (I + S + S_m + R_m + R')Ay > 0$. Due to this, the L-matrix A_{sm3_J} is a nonsingular M-matrix that means ${A_{sm3_J}}^{-1} \ge 0$ and hence by Lemma 2.7., we have $\rho({M_{sm3_J}}^{-1} N_{sm3_J}) < 1$ i.e. $\rho(T_{sm3_J}) < 1$.
From the assumption (A) mentioned above, we have $D + D' + D'' + D''' < I$ so that, $I - D - D' - D'' - D''' \ge 0$. Hence

$$\begin{aligned} {M_{sm3_J}}^{-1} &= (I - D - D' - D'' - D''')^{-1} \\ &= [I - (D + D' + D'' + D''')]^{-1} \\ &= I + (D + D' + D'' + D''') + (D + D' + D'' + D''')^2 + \cdots\cdots\cdots \ge 0 \end{aligned}$$

and $N_{sm3_J} = L + E + E' + E'' + E''' - R_m - R' + U - S - S_m + SU + S_m U + F' \ge 0$, since $L \ge R_m + R' \ge 0$, $U \ge S + S_m \ge 0$ and $E + E' + E'' + E''' + SU + S_m U + F' \ge 0$.

Therefore, it follows from definition 2.4. and Lemma 2.7., $A_{sm3_J} = M_{sm3_J} - N_{sm3_J}$ is a regular and Jacobi convergent splitting.

Similarly, it can be proved that $A_{sm2_J} = M_{sm2_J} - N_{sm2_J}$ is a regular and Jacobi convergent splitting i.e. ${M_{sm2_J}}^{-1} = (I - D - D' - D'')^{-1} \ge 0$, $N_{sm2_J} = L + E + E' + E'' - R_m + U - S - S_m + SU + S_m U + F' \ge 0$ and $\rho({M_{sm2_J}}^{-1} N_{sm2_J}) < 1$ i.e. $\rho(T_{sm2_J}) < 1$.

Theorem 3.9: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm3_J}) \le \rho(T_{sm2_J}) < 1.$$

Proof: By Theorem 3.8., we have the splittings $A_{sm3_J} = M_{sm3_J} - N_{sm3_J}$ and $A_{sm2_J} = M_{sm2_J} - N_{sm2_J}$ are regular and Jacobi convergent i.e., ${M_{sm3_J}}^{-1} = (I - D - D' - D'' - D''')^{-1} \ge 0$, $N_{sm3_J} = L + E + E' + E'' + E''' - R_m - R' + U - S - S_m + SU + S_m U + F' \ge 0$, $\rho(T_{sm3_J}) < 1$ and ${M_{sm2_J}}^{-1} = (I - D - D' - D'')^{-1} \ge 0$, $N_{sm2_J} = L + E + E' + E'' - R_m + U - S - S_m + SU + S_m U + F' \ge 0$ and $\rho(T_{sm2_J}) < 1$.

In order to develop the required inequality, we consider the following splittings of A:

$$A_{sm3_J} = P_{sm3}A = (I + S + S_m + R_m + R')A = M_{sm3_J} - N_{sm3_J}$$

Then $A = (I + S + S_m + R_m + R')^{-1}M_{sm3_J} - (I + S + S_m + R_m + R')^{-1}N_{sm3_J}$

We let $M_2 = (I + S + S_m + R_m + R')^{-1}M_{sm3_J}$

and $N_2 = (I + S + S_m + R_m + R')^{-1}N_{sm3_J}$

Again, $A_{sm2_J} = P_{sm2}A = (I + S + S_m + R_m)A = M_{sm2_J} - N_{sm2_J}$

Then $A = (I + S + S_m + R_m)^{-1}M_{sm2_J} - (I + S + S_m + R_m)^{-1}N_{sm2_J}$

We assume $M_3 = (I + S + S_m + R_m)^{-1}M_{sm2_J}$

and $N_3 = (I + S + S_m + R_m)^{-1}N_{sm2_J}$

It can be easily verify that $M_2^{-1}N_2 = M_{sm3_J}{}^{-1}N_{sm3_J}$ and $M_3^{-1}N_3 = M_{sm2_J}{}^{-1}N_{sm2_J}$ and then $A = M_2 - N_2 = M_3 - N_3$ be the two regular and convergent splittings. Thus we get, $\rho(M_2^{-1}N_2) < 1$ and $\rho(M_3^{-1}N_3) < 1$. Also

$$\begin{aligned} M_2^{-1} &= M_{sm3_J}{}^{-1}(I + S + S_m + R_m + R') \\ &= (I - D - D' - D'' - D''')^{-1}(I + S + S_m + R_m + R') \\ &\geq [I - (D + D' + D'' + D''')]^{-1}(I + S + S_m + R_m) \\ &\geq [I - (D + D' + D'')]^{-1}(I + S + S_m + R_m) \\ &= M_{sm2_J}{}^{-1}\,(I + S + S_m + R_m) \\ &= [(I + S + S_m + R_m)^{-1}M_{sm2_J}]^{-1} = M_3^{-1} \end{aligned}$$

i.e. $$M_2^{-1} \geq M_3^{-1}$$

Thus it follows from Lemma 2.8., that $\rho(M_2^{-1}N_2) \leq \rho(M_3^{-1}N_3) < 1$

i.e. $$\rho(M_{sm3_J}{}^{-1}N_{sm3_J}) \leq \rho(M_{sm2_J}{}^{-1}N_{sm2_J}) < 1$$

i.e. $$\rho(T_{sm3_J}) \leq \rho(T_{sm2_J}) < 1.$$

This completes the proof of the theorem.

Theorem 3.10: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm2_J}) \leq \rho(T_{sm_J}) < 1.$$

Proof: This theorem follows from Theorem 3.7. of the article "A comparative study of modified iterative methods for solving linear systems with M-matrices".

Theorem 3.11: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm_J}) \leq \rho(T_{s_J}) < 1.$$

Proof: The proof of this theorem is similar to the proof of the Theorem 3.10..

Theorem 3.12: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{s_J}) \leq \rho(T_J) < 1.$$

Proof: The proof of this theorem follows from the Theorem 3.1., of the article "A comparative study of modified iterative methods for solving linear systems with M-matrices".

Theorem 3.13: Let A be a nonsingular M-matrix. Then under the assumption (A), the following inequality holds:

$$\rho(T_{sm3_J}) \leq \rho(T_{sm2_J}) \leq \rho(T_{sm_J}) \leq \rho(T_{s_J}) \leq \rho(T_J) < 1.$$

Proof: The proof of this theorem follows from the Theorem 3.9., Theorem 3.10., Theorem 3.11., and Theorem 3.12..

4. NUMERICAL EXAMPLES

In this section, we consider two simple numerical examples in order to confirm our theoretical analysis provided in section 3.

Example 1: Let us consider the following matrix:

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.1 & -0.2 \\ -0.1 & 1 & -0.1 & -0.3 & -0.1 \\ -0.2 & -0.1 & 1 & -0.1 & -0.2 \\ -0.2 & -0.1 & -0.1 & 1 & -0.3 \\ -0.1 & -0.2 & -0.2 & -0.1 & 1 \end{pmatrix}$$

After computation using MATLAB R12, we obtain

$$\rho(T_G) = 0.4608, \quad \rho(T_J) = 0.6551$$
$$\rho(T_{s_G}) = 0.3414, \quad \rho(T_{s_J}) = 0.6044$$
$$\rho(T_{sm_G}) = 0.2217, \quad \rho(T_{sm_J}) = 0.5344$$
$$\rho(T_{sm2_G}) = 0.1996, \quad \rho(T_{sm2_J}) = 0.5195$$
$$\rho(T_{sm3_G}) = 0.1662, \quad \rho(T_{sm3_J}) = 0.4694$$

Clearly, we have

$$\rho(T_{sm3_G}) < \rho(T_{sm2_G}) < \rho(T_{sm_G}) < \rho(T_{s_G}) < \rho(T_G) < 1$$

and

$$\rho(T_{sm3_J}) < \rho(T_{sm2_J}) < \rho(T_{sm_J}) < \rho(T_{s_J}) < \rho(T_J) < 1$$

Example 2: Let us consider the following matrix:

$$A = \begin{pmatrix} 1 & -0.1 & -0.2 & -0.5 \\ -0.1 & 1 & -0.1 & -0.5 \\ -0.3 & -0.1 & 1 & -0.1 \\ -0.4 & -0.3 & -0.1 & 1 \end{pmatrix}$$

By computation in MATLAB R12, we obtain

$$\rho(T_G) = 0.5408, \quad \rho(T_J) = 0.7332$$
$$\rho(T_{s_G}) = 0.4907, \quad \rho(T_{s_J}) = 0.7159$$
$$\rho(T_{sm_G}) = 0.2694, \quad \rho(T_{sm_J}) = 0.5337$$
$$\rho(T_{sm2_G}) = 0.2501, \quad \rho(T_{sm2_J}) = 0.5233$$
$$\rho(T_{sm3_G}) = 0.2432, \quad \rho(T_{sm3_J}) = 0.5081$$

Clearly, we have

$$\rho(T_{sm3_G}) < \rho(T_{sm2_G}) < \rho(T_{sm_G}) < \rho(T_{s_G}) < \rho(T_G) < 1$$

and

$$\rho(T_{sm3_J}) < \rho(T_{sm2_J}) < \rho(T_{sm_J}) < \rho(T_{s_J}) < \rho(T_J) < 1$$

5. CONCLUSION

In the present article, we proposed two modified iterative methods with a new preconditioner for solving system of linear equations. The comparison theorems and numerical experiments show that the proposed methods are superior as compared to the respective classical methods and some of the other respective modified methods.

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Source of support: Nil, Conflict of interest: None Declared.