International Journal of Mathematical Archive-14(9), 2023, 8-12
Available online through www.ijma.info ISSN 2229 – 5046

On Distance Divisor Symmetric n-Sigraphs

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(Received On: 25-08-23; Revised & Accepted On: 07-09-23)

ABSTRACT

An n-tuple $(a_1, a_2, \ldots, a_n)$ *is symmetric, if* $a_k = a_{n-k+1}, 1 \leq k \leq n$. *Let* $H_n = \{(a_1, a_2, \ldots, a_n): a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ *be the set of all symmetric n-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair* $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, *where* $G = (V, E)$ *is a graph called the underlying graph of* S_n *and* $\sigma: E \rightarrow H_n$ $(\mu: V \rightarrow H_n)$ *is a function. In this paper, we introduced a new notion distance divisor symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of distance divisor symmetric n-signed graphs.*

***Keywords**: Symmetric n-sigraphs, Symmetric n-marked graphs, Balance, Switching, Distance divisor symmetric n-sigraphs, Complementation.*

***AMS 2020 subject classification**: 05C22.*

1. INTRODUCTION

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \ldots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \ldots, a_n) : a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric n-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \left\lceil \frac{n}{2} \right\rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, where $G = (V, E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric n-tuple / symmetric n-sigraph / symmetric n-marked graph.

An n-tuple $(a_1, a_2, \ldots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an n-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity n-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.

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Further, in an n-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the n-tuples on the edges of A.

In [7], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. S. K. Reddy [3]):

Definition: Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and
(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [7].

Theorem 1.1: (**E. Sampathkumar *et al.* [7]**) An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an *n-sigraph* $\overline{S_n} = (\bar{G}, \sigma^c)$, where for any edge e = $uv \in \bar{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ is defined here is an i-balanced *n-sigraph due to Theorem 1.1.*

In [7], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [2, 4-6, 9–19])

Let $S_n = (G, \sigma)$ and $S_n' = (G', \sigma')$ be two n-sigraphs. Then S_n and S_n' are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if uv is an edge in S_n with label $(a_1, a_2, \ldots, a_n)$ then $\phi(u)\phi(v)$ is an edge in S_n' with label $(a_1, a_2, \ldots, a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The$n$-sigraph obtained in this way is denoted by $S_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S_n' (or that they are *switching equivalent* to each other), written as $S_n \sim S_n'$, whenever there exists an n-marking of S_n such that $S_\mu(S_n) \cong S_n'$.

Two n-sigraphs $S_n = (G, \sigma)$ and $S_n' = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi: G \rightarrow G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\boldsymbol{\sigma}(\Phi(C))$ in S_n'.

We make use of the following known result (see [7]).

Theorem 1.2: (**E. Sampathkumar *et al.* [7]**) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. Distance Divisor *n*-Sigraph of an *n*-Sigraph

Let $G = (V,E)$$ be a graph with $|V|=p$ and $|E|=q$. The shortest path P in G is said to be distance divisor path, if $l(P)|q$, where $l(P)$ denotes the length path P.

Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. The distance divisor graph $DD(G)$ of G = (V, E) is a graph with V$(DD(G)) = V(G)$ and any two vertices u and v in $DD(G)$ are joined by an edge if there exists a distance divisor path between them in G. This concept were introduced by Saravanakumar and Nagarajan [20].

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of distance divisor graphs to n-sigraphs as follows: The distance divisor n-sigraph $DD(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $DD(G)$ and the n-tuple of any edge uv is $DD(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n. Further, an n-sigraph $S_n = (G, \sigma)$ is called distance divisor n-sigraph, if $S_n \cong DD(S_n')$ for some n-sigraph S_n'. The following result indicates the limitations of the notion $DD(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be detour radial n-sigraphs.

Theorem 2.1: *For any n-sigraph $S_n = (G, \sigma)$, its* distance divisor *n-sigraph $DD(S_n)$ is i-balanced.*

Proof: Since the n-tuple of any edge uv in $DD(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n, by Theorem 1.1, $DD(S_n)$ is i-balanced.

For any positive integer k, the k^{th} iterated distance divisor n-sigraph, $DD^k(S_n)$ of S_n is defined as follows:

$$DD^0(S_n) = S_n,\ DD^k(S_n) = DD(DD^{k-1}(S_n)).$$

Corollary 2.2: *For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $DD^k(S_n)$ is i-balanced.*

The following result characterizes n-sigraphs which are distance divisor n-sigraphs.

Theorem 2.3: *An n-sigraph $S_n = (G, \sigma)$ is a distance divisor n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a distance divisor graph.*

Proof: Suppose that S_n is i-balanced and G is a distance divisor graph. Then there exists a graph H such that $DD(H) \cong G$. Since S_n is i-balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge $e = uv$ in S_n satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the n-sigraph $S_n{}' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the n-marking of the corresponding vertex in G. Then clearly, $DD(S_n{}') \cong S_n$. Hence S_n is a distance divisor n-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a distance divisor n-sigraph. Then there exists an n-sigraph$S_n{}' = (H, \sigma')$ such that $DR(S_n{}') \cong S_n$. Hence G is the distance divisor graph of H and by Theorem 2.1, S_n is i-balanced.

Consider a graph $G = (V, E)$ with $|V| = p$ and $|E| = q$. Let $k_1, k_2, \ldots, k_\tau$ denote the positive divisors of q with $k_1 = 1, k_2 = 2, \ldots, k_\tau = q$ and $k_1 < k_2 < \ldots < k_\tau$. In [20], the authors characterizes the graphs such that G and $DD(G)$ are isomorphic.

Theorem 2.4: *Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$, where q is a composite number. Then G and $DD(G)$ are isomorphic if and only if the diameter of G is less than or equal to k_2-1.*

In view of the above, we have the following result:

Theorem 2.5: *For any n-sigraph $S_n = (G, \sigma)$ with $|V| = p$ and $|E| = q$, where q is a composite number. Then S_n and $DD(S_n)$ are cycle isomorphic if and only if S_n is i-balanced and the diameter of G is less than or equal to k_2-1.*

Proof: Suppose $DD(S_n) \sim S_n$. This implies, $DD(G) \cong G$ and hence by Theorem 2.4, we see that the diameter of G is less than or equal to k_2-1. Now, if S_n is any n-sigraph with the diameter of G is less than or equal to k_2-1. Then $DD(S_n)$ is i-balanced and hence if S_n is i-unbalanced and its distance divisor n-sigraph $DD(S_n)$ being i-balanced cannot be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be i-balanced.

Conversely, suppose that S_n is i-balanced n-sigraph with the underlying graph G satisfies the conditions of Theorem 2.4. Then, since $DD(S_n)$ is i-balanced as per Theorem 2.1 and since $DD(G) \cong G$ by Theorem 2.4, the result follows from Theorem 1.2 again.

Theorem 2.6: *For any two n-sigraphs S_n and $S_n{}'$ with the same underlying graph, their* distance divisor *n-sigraphs are switching equivalent.*

Proof: Suppose $S_n = (G, \sigma)$ and $S_n{}' = (G', \sigma'))$ be two n-sigraphs with $G \cong G'$. By Theorem 2.1, $DD(S_n)$ and $DD(S_n{}')$ are i-balanced and hence, the result follows from Theorem 1.2.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2,.., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written $(S_n{}^m)$, is the same graph but with each edge label $a = (a_1, a_2,\ldots ,a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $DD(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $DD(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.7: *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $DD(G)$ is bipartite then $(DD(S_n))^m$ is i-balanced.*

Proof: Since, by Theorem 2.1, $DD(S_n)$ is *i*-balanced, for each k, $1 \leq k \leq n$, the number of *n*-tuples on any cycle C in $DD(S_n)$ whose k^{th} co-ordinate are − is even. Also, since $DD(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of *n*-tuples on any cycle C in $DD(S_n)$ whose k^{th} co-ordinate are + is also even. This implies that the same thing is true in any *m*-complement, where for any $m \in H_n$. Hence $(DD(S_n))^t$ is *i*-balanced.

Theorem 2.6 provides easy solutions to other *n*-sigraph switching equivalence relations, which are given in the following results.

Corollary 2.8: *For any two n-sigraphs S_n and $S_n{}'$ with the same underlying graph, $DD(S_n)$ and $DD((S_n{}')^m)$ are switching equivalent.*

Corollary 2.9: *For any two n-sigraphs S_n and $S_n{}'$ with the same underlying graph, $DD((S_n)^m)$ and $DD(S_n{}')$ are switching equivalent.*

Corollary 2.10: *For any two n-sigraphs S_n and $S_n{}'$ with the same underlying graph, $DD((S_n)^m)$ and $DD((S_n{}')^m)$ are switching equivalent.*

Corollary 2.11: *For any two n-sigraphs $S_n = (G, \sigma)$ and $S_n{}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(DD(S_n))^m$ and $DD(S_n{}')$ are switching equivalent.*

Corollary 2.12: *For any two n-sigraphs $S_n = (G, \sigma)$ and $S_n{}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $DD(S_n)$ and $DD((S_n{}')^m)$ are switching equivalent.*

Corollary 2.13: *For any two n-sigraphs $S_n = (G, \sigma)$ and $S_n{}' = (G', \sigma')$ with the $G \cong G'$ and G, G' are bipartite, $(DD(S_1))^m$ and $(DD(S_2))^m$ are switching equivalent.*

3. CONCLUSION

We have introduced a new notion for *n*-signed graphs called distance divisor *n*-sigraph of an *n*-signed graph. We have proved some results and presented the structural characterization of distance divisor *n*-signed graph. There is no structural characterization of distance divisor graph, but we have obtained the structural characterization of distance divisor *n*-signed graph.

ACKNOWLEDGMENTS

The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

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Source of support: Nil, Conflict of interest: None Declared.