International Journal of Mathematical Archive-14(8), 2023, 8-12
IJMA Available online through www.ijma.info ISSN 2229 – 5046

On Detour Radial Symmetric *n*-Sigraphs

JEPHRY RODRIGUES K
Department of Mathematics,
Dr. P. Dayananda Pai-P. Satisha Pai Govt. First Grade College
Car Street, Mangalore - 575 001, India.

K. B. MAHESH*
Department of Mathematics,
Dr. P. Dayananda Pai-P. Satisha Pai Govt. First Grade College,
Car Street, Mangalore - 575 001, India.

P. SOMASHEKAR
Department of Mathematics,
Government First Grade College, Nanjangud-571 301, India.

(Received On: 23-06-23; Revised & Accepted On: 02-08-23)

ABSTRACT

A*n n-tuple* $(a_1, a_2, \ldots, a_n)$ *is symmetric, if* $a_k = a_{n-k+1}, 1 \leq k \leq n$. *Let* $H_n = \{(a_1, a_2, \ldots, a_n): a_k \in \{+, -\}, a_k = a_{n-k+1}, 1 \leq k \leq n$ *be the set of all symmetric n-tuples. A symmetric n-sigraph (symmetric n-marked graph) is an ordered pair* $S_n = (G, \sigma)$ $(S_n = (G, \mu))$, *where* $G = (V, E)$ *is a graph called the underlying graph of* S_n *and* $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ *is a function. In this paper, we introduced a new notion detour radial symmetric n-sigraph of a symmetric n-sigraph and its properties are obtained. Also, we obtained the structural characterization of detour radial symmetric n-signed graphs.*

Keywords: *Symmetric n-sigraphs, Symmetric n-marked graphs, Balance, Switching, Detour radial symmetric n-sigraphs, Complementation.*

AMS 2020 subject classification: *05C22.*

1. INTRODUCTION

Unless mentioned or defined otherwise, for all terminology and notion in graph theory the reader is refer to [1]. We consider only finite, simple graphs free from self-loops.

Let $n \geq 1$ be an integer. An *n-tuple* $(a_1, a_2, \ldots, a_n)$ is *symmetric*, if $a_k = a_{n-k+1}, 1 \leq k \leq n$. Let $H_n = \{(a_1, a_2, \ldots, a_n) : a_k \in \{+,-\}, a_k = a_{n-k+1}, 1 \leq k \leq n\}$ be the set of all symmetric *n*-tuples. Note that H_n is a group under coordinate wise multiplication, and the order of H_n is 2^m, where $m = \left\lceil \frac{n}{2} \right\rceil$.

A *symmetric n-sigraph (symmetric n-marked graph)* is an ordered pair $S_n = (G,\sigma)$ $(S_n = (G,\mu))$, where $G = (V,E)$ is a graph called the *underlying graph* of S_n and $\sigma : E \rightarrow H_n$ $(\mu : V \rightarrow H_n)$ is a function.

In this paper by an *n-tuple/n-sigraph/n-marked graph* we always mean a symmetric *n*-tuple/symmetric *n*-sigraph / symmetric *n*-marked graph.

An *n*-tuple $(a_1, a_2, \ldots, a_n)$ is the *identity n-tuple*, if $a_k = +$, for $1 \leq k \leq n$, otherwise it is a *non-identity n-tuple*. In an *n*-sigraph $S_n = (G, \sigma)$ an edge labelled with the identity *n*-tuple is called an *identity edge*, otherwise it is a *non-identity edge*.
Further, in an *n*-sigraph $S_n = (G, \sigma)$, for any $A \subseteq E(G)$ the *n-tuple* $\sigma(A)$ is the product of the *n*-tuples on the edges of *A*.

Corresponding Author: K. B. Mahesh*
Department of Mathematics, Dr. P. Dayananda Pai-P. Satisha Pai Govt. First Grade College,
Car Street, Mangalore - 575 001, India.

In [7], the authors defined two notions of balance in n-sigraph $S_n = (G, \sigma)$ as follows (See also R. Rangarajan and P. S. K. Reddy [3]):

Definition: Let $S_n = (G, \sigma)$ be an n-sigraph. Then,

(i) S_n is *identity balanced* (or *i-balanced*), if product of n-tuples on each cycle of S_n is the identity n-tuple, and
(ii) S_n is *balanced*, if every cycle in S_n contains an even number of non-identity edges.

Note: An i-balanced n-sigraph need not be balanced and conversely.

The following characterization of i-balanced n-sigraphs is obtained in [7].

Theorem 1.1: (E. Sampathkumar *et al.* [7]) An n-sigraph $S_n = (G, \sigma)$ is i-balanced if, and only if, it is possible to assign n-tuples to its vertices such that the n-tuple of each edge uv is equal to the product of the n-tuples of u and v.

Let $S_n = (G, \sigma)$ be an n-sigraph. Consider the n-marking μ on vertices of S_n defined as follows: each vertex $v \in V$, $\mu(v)$ is the n-tuple which is the product of the n-tuples on the edges incident with v. *Complement* of S_n is an *n-sigraph* $\overline{S_n} = (\bar{G}, \sigma^c)$, where for any edge e = $uv \in \bar{G}$, $\sigma^c(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S_n}$ is defined here is an i-balanced *n-sigraph due to Theorem 1.1.*

In [7], the authors also have defined switching and cycle isomorphism of an n-sigraph $S_n = (G, \sigma)$ as follows: (See also [2, 4-6, 9–19])

Let $S_n = (G, \sigma)$ and $S_n' = (G', \sigma')$ be two n-sigraphs. Then S_n and S_n' are said to be *isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that if uv is an edge in S_n with label $(a_1, a_2,\dots ,a_n)$ then $\phi(u)\phi(v)$ is an edge in S_n' with label $(a_1, a_2,\dots ,a_n)$.

Given an n-marking μ of an n-sigraph $S_n = (G, \sigma)$, *switching* S_n with respect to μ is the operation of changing the n-tuple of every edge uv of S_n by $\mu(u)\sigma(uv)\mu(v)$. The n-sigraph obtained in this way is denoted by $S_\mu(S_n)$ and is called the *μ-switched n-sigraph* or just *switched n-sigraph*.

Further, an n-sigraph S_n *switches* to n-sigraph S_n' (or that they are *switching equivalent* to each other), written as $S_n \sim S_n'$, whenever there exists an n-marking of S_n such that $S_\mu(S_n) \cong S_n'$.

Two n-sigraphs $S_n = (G,\sigma)$ and $S_n' = (G', \sigma')$ are said to be *cycle isomorphic*, if there exists an isomorphism $\phi : G \rightarrow G'$ such that the n-tuple $\sigma(C)$ of every cycle C in S_n equals to the n-tuple $\boldsymbol{\sigma}(\Phi(C))$ in S_n'.

We make use of the following known result (see [7]).

Theorem 1.2: (E. Sampathkumar *et al.* [7]) *Given a graph G, any two n-sigraphs with G as underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

2. Detour Radial *n*-Sigraph of an *n*-Sigraph

Let $G=(V, E)$ be a connected graph. For any two vertices $u, v \in V(G)$, the detour distance $D(u, v)$ is the length of the longest u-v path in G. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(G)$ of G is defined by $r(G) = \min\{e(u): u \in G\}$.

For any vertex u in G, the detour eccentricity $D_e(u)$ of u is the detour distance to a vertex farthest from u. The detour radius $D_r(G)$ of G is defined by $D_r(G) = \min\{D_e(u): u \in G\}$. The diameter $d(G)$ of G is defined by $d(G) = \max\{e(u): u \in G\}$ and the detour diameter $D_d(G)$ of G is max $\{D_e(u): u \in G\}$.

The detour radial graph $DR(G)$ of $G=(V, E)$ is a graph with $V(DR(G))=V(G)$ and any two vertices u and v in $DR(G)$ are joined by an edge if and only if $D(u, v)=D_r(G)$. This concept was introduced by Ganeshwari and Pethanachi Selvam [1].

Motivated by the existing definition of complement of an n-sigraph, we extend the notion of detour radial graphs to n-sigraphs as follows: The detour radial n-sigraph $DR(S_n)$ of an n-sigraph $S_n = (G, \sigma)$ is an n-sigraph whose underlying graph is $DR(G)$ and the n-tuple of any edge uv is $DR(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical n-marking of S_n.

Further, an n-sigraph $S_n = (G, \sigma)$ is called detour radial n-sigraph, if $S_n \cong DR(S_n')$ for some n-sigraph S_n'. The following result indicates the limitations of the notion $DR(S_n)$ as introduced above, since the entire class of i-unbalanced n-sigraphs is forbidden to be detour radial n-sigraphs.

Theorem 2.1: *For any n-sigraph $S_n = (G, \sigma)$, its* detour radial *n-sigraph $DR(S_n)$ is i-balanced.*

Proof: Since the *n*-tuple of any edge uv in $DR(S_n)$ is $\mu(u)\mu(v)$, where μ is the canonical *n*-marking of S_n, by Theorem 1.1, $DR(S_n)$ is *i*-balanced.

For any positive integer k, the k^{th} iterated detour radial *n*-sigraph, $DR^k(S_n)$ of S_n is defined as follows:

$$DR^0(S_n) = S_n,\ DR^k(S_n) = DR(DR^{k-1}(S_n)).$$

Corollary 2.2: *For any n-sigraph $S_n = (G, \sigma)$ and for any positive integer k, $DR^k(S_n)$ is i-balanced.*

The following result characterizes *n*-sigraphs which are detour radial *n*-sigraphs.

Theorem 2.3: *An n-sigraph $S_n = (G, \sigma)$ is a detour radial n-sigraph if, and only if, S_n is i-balanced n-sigraph and its underlying graph G is a detour radial graph.*

Proof: Suppose that S_n is *i*-balanced and G is a detour radial graph. Then there exists a graph H such that $DR(H) \cong G$. Since S_n is *i*-balanced, by Theorem 1.1, there exists a marking ζ of G such that each edge $e = uv$ in S_n satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the *n*-sigraph $S_n' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the *n*-marking of the corresponding vertex in G. Then clearly, $DR(S_n') \cong S_n$. Hence S_n is a detour radial *n*-sigraph.

Conversely, suppose that $S_n = (G, \sigma)$ is a detour radial *n*-sigraph. Then there exists an *n*-sigraph$S_n' = (H, \sigma')$ such that $DR(S_n') \cong S_n$. Hence G is the detour radial graph of H and by Theorem 2.1, S_n is *i*-balanced.

In [1], the authors characterizes the graphs $G = (V, E)$ such that $G \cong DR(G)$.

Theorem 2.4: (Ganeshwari and Selvam [1])
Let G=(V, E) be a graph with atleast one cycle which covers all the vertices of G. Then G and the detour radial graph DR(G) are isomorphic if and only if G is isomorphic to either (i) K_n or (ii) C_n or (iii) $K_{m,n}$ with m=n.

In view of the above result, we now characterize the *n*-sigraphs such that the detour radial *n*-sigraph and its corresponding *n*-sigraph are switching equivalent.

Theorem 2.5: *For any n-sigraph $S_n = (G, \sigma)$ and its underlying graph G contains atleast one cycle which covers all the vertices. Then S_n and the detour radial n-sigraph $DR(S_n)$ are cycle isomorphic if and only if the underlying of S_n satisfies the conditions of Theorem 2.4 and S_n is i-balanced.*

Proof: Suppose $DR(S_n) \sim S_n$. This implies, $DR(G) \cong G$ and hence by Theorem 2.4, we see that the graph G satisfies the conditions in Theorem 2.4. Now, if S_n is any *n*-sigraph with underlying graph contains at least one Hamilton cycle and satisfies the conditions of Theorem 2.4. Then $DR(S_n)$ is *i*-balanced and hence if S_n is *i*-unbalanced and its detour radial *n*-sigraph $DR(S_n)$ being *i*-balanced cannot be switching equivalent to S_n in accordance with Theorem 1.2. Therefore, S_n must be *i*-balanced.

Conversely, suppose that S_n is *i*-balanced *n*-sigraph with the underlying graph G satisfies the conditions of Theorem 2.4. Then, since $DR(S_n)$ is *i*-balanced as per Theorem 2.1 and since $DR(G) \cong G$ by Theorem 2.4, the result follows from Theorem 1.2 again.

Theorem 2.6: *For any two n-sigraphs S_n and S_n' with the same underlying graph, their detour radial n-sigraphs are switching equivalent.*

Proof: Suppose $S_n = (G, \sigma)$ and $S_n' = (G', \sigma'))$ be two *n*-sigraphs with $G \cong G'$. By Theorem 2.1, $DR(S_n)$ and $DR(S_n')$ are *i*-balanced and hence, the result follows from Theorem 1.2.

In [3], Harshavardhana *et al.* introduced the notion radial *n*-sigraph of an *n*-sigraph and proved some results.

Theorem 2.7: *For any n-sigraph $S_n = (G, \sigma)$, its radial n-sigraph $R(S_n)$ is i-balanced.*

In [1], the authors remarked that $DR(G)$ and $R(G)$ are isomorphic, if G is any cycle of odd length. We now characterize the *n*-sigraphs S_n such that $DR(S_n) \sim R(S_n)$.

Theorem 2.8: *For any n-sigraph $S_n = (G, \sigma)$, $DR(S_n) \sim R(S_n)$ if, and only if, $G \cong C_n$, where n is odd.*

Proof: Suppose $DR(S_n) \sim R(S_n)$. This implies, $DR(G) \cong R(G)$. Hence, G is any cycle of odd length.

Conversely, suppose that S_n is an n-sigraph whose underlying graph G is C_n. Then $DR(G) \cong R(G)$. Since for any n-sigraph S_n, both $DR(S_n)$ and $R(S_n)$ are i-balanced, the result follows by Theorem 1.2.

For any $m \in H_n$, the *m-complement* of $a = (a_1, a_2,.., a_n)$ is: $a^m = am$. For any $M \subseteq H_n$, and $m \in H_n$, the *m-complement* of M is $M^m = \{a^m : a \in M\}$.

For any $m \in H_n$, the *m-complement* of an n-sigraph $S_n = (G, \sigma)$, written $(S_n{}^m)$, is the same graph but with each edge label $a = (a_1, a_2,... ,a_n)$ replaced by a^m.

For an n-sigraph $S_n = (G, \sigma)$, the $DR(S_n)$ is i-balanced. We now examine, the condition under which m-complement of $DR(S_n)$ is i-balanced, where for any $m \in H_n$.

Theorem 2.9: *Let $S_n = (G, \sigma)$ be an n-sigraph. Then, for any $m \in H_n$, if $DR(G)$ is bipartite then $(DR(S_n))^m$ is i-balanced.*

Proof: Since, by Theorem 2.1, $DR(S_n)$ is i-balanced, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $DR(S_n)$ whose k^{th} co-ordinate are $-$ is even. Also, since $DR(G)$ is bipartite, all cycles have even length; thus, for each k, $1 \leq k \leq n$, the number of n-tuples on any cycle C in $DR(S_n)$ whose k^{th} co-ordinate are + is also even. This implies that the same thing is true in any m-complement, where for any $m \in H_n$. Hence $(DR(S_n))^t$ is i-balanced.

ACKNOWLEDGMENTS

The authors thank the anonymous reviewers for their careful reading of our manuscript and their many insightful comments and suggestions.

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Source of support: Nil, Conflict of interest: None Declared.