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SUMUDU DECOMPOSITION METHOD FOR SOLVING FRACTIONAL ORDER DIFFUSION AND WAVE EQUATIONS

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ABSTRACT

In this work, we apply the Sumudu decomposition method to derive approximate analytical solutions in series form for the linear and non-linear fractional diffusion and wave equations within the Caputo fractional derivative. This technique is a hybrid of the Sumudu transform method and the Adomian decomposition technique. The procedure is user-friendly and reliable. Some illustrative examples are provided to demonstrate the validity and applicability of the proposed technique.

Keywords: *Sumudu transform, Adomian decomposition method, Caputo fractional derivative, fractional diffusion and wave equations, Mittag-Leffler function.*

2010 Mathematics Subject Classification: *33E12, 34A08, 35A20.*

1. INTRODUCTION

Fractional calculus is a fascinating discipline of mathematical analysis that studies derivatives and integrals of arbitrary orders. It has captivated the attention of scientists and engineers for a very long time, resulting in the development of numerous applications. In recent years, scientists and engineers have rediscovered fractional calculus and applied it to an increasing number of fields, such as chemical physics, electrochemistry of corrosion, mechanics, acoustics, control engineering and signal processing, etc. Fractional differential equations have received increasing attention from researchers due to their remarkable performance in solving real-world problems across various scientific and technological fields. Various numerical and analytical methods have been suggested for solving linear and non-linear fractional order differential equations, such as the Adomian decomposition method (ADM) [2,13,17], the Laplace decomposition method (LDM) [15], the iterative Laplace transform method (ILTM) [25,26], the modified homotopy perturbation method (MHPM)[14], the homotopy perturbation Sumudu transform method (HPSTM) [32], the Fractional Explicit Adams Method (FEAM) [33], the Sumudu transform iterative method (STIM) [4,5,7,27], the Haar wavelet operational matrix method (HWOMM)[24], the homotopy analysis method [3], and the q-homotopy analysis transform method (q-HATM) [6], etc.

In 1993, Watugala introduced the Sumudu transform method (STM) [28] to solve engineering problems. Weerakoon [30] used this method to solve partial differential equations. Later, Weerakoon [31] derived the formula for the inverse of this transform. Demiray *et al.* [11] utilized the STM technique to discover exact solutions for fractional differential equations. Recently, Kumar *et al.* [18] have discovered a new method for solving nonlinear equations by combining the Sumudu transform with the Adomian decomposition method, termed the Sumudu decomposition method (SDM).The Sumudu decomposition method has been proven successful in solving various types of differential equations such as fractional Bratu-type differential equations [20], fractional Delay differential equations [12], fractional Riccati equations [19], fractional integro-differential equations [1] as well as time-fractional PDEs and systems of time-fractional PDEs [16].

In the present study, we will use the Sumudu decomposition method to solve the following linear and non-linear time-fractional diffusion and wave equations as follows

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(i). The linear time-fractional diffusion equation with the specified initial condition, is given by [13,14,15]

$$D_t^{\alpha}u(x,t)=\sum_{i=0}^{n}\psi_i(x,t)\frac{\partial^2 u(x,t)}{\partial x_i^2}+\phi(x,t)u^{(m)}(x,t),\ 0<\alpha\le 1,\ m=0,1, \tag{1}$$

$$u(x,0)=f(x)\,, \tag{2}$$

(ii). The linear time-fractional wave equation with the specified initial conditions, is given by [13,14,15]

$$D_t^{\alpha}u(x,t)=\sum_{i=0}^{n}\psi_i(x,t)\frac{\partial^2 u(x,t)}{\partial x_i^2}+\phi(x,t)u^{(m)}(x,t),\ 1<\alpha\le 2,\ m=0,1, \tag{3}$$

$$u(x,0)=f(x)\,,\ \frac{\partial}{\partial t}u(x,0)=g(x)\,, \tag{4}$$

(iii). The non-linear time-fractional wave equation with the specified initial conditions, is given by [13,14,15]

$$D_t^{\alpha}u(x,t)=\sum_{i=0}^{n}\psi_i(x,t)\frac{\partial^2 u(x,t)}{\partial x_i^2}+\phi(x,t)u^{(m)}(x,t),\ 1<\alpha\le 2,\ m=2,3,.\ . \tag{5}$$

$$u(x,0)=f(x)\,,\ \frac{\partial}{\partial t}u(x,0)=g(x)\,, \tag{6}$$

where $x=(x_1,x_2,...,x_n)$, $\psi_i(x,t)\in C_\alpha$, $\phi(x,t)$ is a source term, and $D_t^{\alpha}(x,t)$ denotes Caputo fractional derivative of order α given in equation (7).

The main objective of this investigation is to expand the work of the SDM in order to derive approximate analytical solutions for the linear and non-linear fractional diffusion and wave equations with initial conditions. The SDM presents the solution in a quick convergent series, which may lead to a closed-form solution. This approach has the benefit of integrating two strong methods for acquiring accurate and approximate solutions to differential equations of fractional order.

2. BASIC DEFINITIONS AND NOTATIONS

In this part, we give some important definitions and properties pertaining to the fractional calculus and Sumudu transform, which are further used in this paper.

Definition 1: The Caputo fractional derivative of function $u(x,t)$ of order $\alpha>0$, is defined as [21, 23]

$$\begin{aligned} D_t^{\alpha}u(x,t)&=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-\eta)^{m-\alpha-1}u^{(m)}(x,\eta)d\eta, m-1<\alpha\le m, m\in N\\ &=I_t^{m-\alpha}D^m u(x,t). \end{aligned} \tag{7}$$

Here $D^m=\dfrac{d^m}{dt^m}$ and I_t^{α} stands for the Riemann-Liouville fractional integral operator of order $\alpha>0$, defined as [21]

$$I_t^{\alpha}u(x,t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\eta)^{\alpha-1}u(x,\eta)d\eta,\eta>0,(m-1<\alpha\le m),m\in N. \tag{8}$$

Definition 2: The Sumudu transform is defined over the set of functions

$$\left\{f(t)\,|\,\exists\, M,\rho_1,\ \rho_2>0\,,\ |\,f(t)\,|<Me^{|t|/\rho_j}\ \text{ if }\ t\in(-1)^j\times\left[\,0,\infty\right),\ j=1,2\right\}$$

by the following formula [8,28]

$$S\,[\,f(t)]=F(\omega)=\int_0^{\infty}e^{-t}\,f(\omega t)\,dt,\quad \omega\in(-\rho_1,\rho_2). \tag{9}$$

One of the basic properties of the Sumudu transform is given as

$$S\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right]=\omega^{\alpha},\ \alpha>-1. \tag{10}$$

The inverse Sumudu transform of ω^{α} is defined as

$$S^{-1}\left[\,\omega^{\alpha}\,\right]=\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ \alpha>-1. \tag{11}$$

Definition 3: The Sumudu transform of Caputo fractional derivative of $u(x,t)$ of order $\alpha>0$, is defined as [11, 27]

$$S\left[D_t^{\alpha}u(x,t)\right]=\omega^{-\alpha}S\left[u(x,t)\right]-\sum_{k=0}^{m-1}\left[\omega^{-\alpha+k}\ u^{(k)}(x,0\ \right],\ m-1<\alpha\le m,m\in N. \tag{12}$$

Definition 4: The Mittag-Leffler function $E_\alpha(z)$ with $\alpha > 0$ is defined by the following series representation, valid in the whole complex plane [22]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}, \ \alpha \in C, \ \text{Re}(\alpha) > 0, \tag{13}$$

where $\Gamma(.)$ is the well-known Gamma function.

3. BASIC IDEA OF SUMUDU DECOMPOSITION METHOD

In order to illustrate the key concept of this method [12], we consider the general fractional partial differential equation with initial conditions of the type

$$D_t^\alpha u(x,t) + R\,u(x,t) + N\,u(x,t) = g(x,t), \ m-1 < \alpha \le m, \ m \in N, \tag{14}$$

$$u^{(k)}(x,0) = h_k(x), \ k = 0,1,2,...,m-1\ , \tag{15}$$

where $D_t^\alpha u(x,t)$ is the Caputo fractional derivative of order α, $m-1<\alpha \le m, m \in N,$ defined by equation (7), R is a linear operator and may include other fractional derivatives of order less than α , N is a non-linear operator which may include other fractional derivatives of order less than α and $g(x,t)$ is a known function.

Applying the Sumudu transform on both sides of equation (14), we have

$$S\left[D_t^\alpha u(x,t)\right] + S[R\,u\,(x,t) + Nu(x,t)] = S[g(x,t)]. \tag{16}$$

By using the equation (12), we get

$$S[u(x,t)] = \sum_{k=0}^{m-1} \omega^k u^{(k)}(x,0) + \omega^\alpha S[g(x,t)] - \omega^\alpha S[R\,u(x,t) + N\,u(x,t)]. \tag{17}$$

On taking inverse Sumudu transform on equation (17), we have

$$u(x,t) = S^{-1}\left[\omega^\alpha \left(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S[g(x,t)]\right)\right] - S^{-1}\left[\omega^\alpha S[Ru(x,t) + Nu(x,t)]\right]. \tag{18}$$

Furthermore, we apply the Adomian decomposition method (ADM) [2], which represents a solution $u(x,t)$ in components of infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{19}$$

and the non-linear term is decomposed as follows

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n, \tag{20}$$

where A_n are the Adomian polynomials given by

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{n=0}^{\infty} \lambda^i u_i\right)\right]_{\lambda=0}, n = 0,1,2,... \tag{21}$$

Substituting equations (19), (20) and (21) into equation (18), we get

$$\sum_{n=0}^{\infty} u_n(x,t) = S^{-1}\left[\omega^\alpha \left(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S\left[g(x,t)\right]\right)\right] - S^{-1}\left[\omega^\alpha S\left(R\left(\sum_{n=0}^{\infty} u(x,t)\right) + \sum_{n=0}^{\infty} A_n\right)\right]. \tag{22}$$

Using the Adomian technique, we determine the formal recurrence relations in the elegant form as follows as

$$u_0(x,t) = S^{-1}\left[\omega^\alpha \left(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S\left[g(x,t)\right]\right)\right], \tag{23}$$

$$u_{n+1}(x,t) = -S^{-1}\left[\omega^\alpha S\left[R\left(u_n(x,t)\right) + A_n\right]\right], \ n = 0,1,2,..., \tag{24}$$

Therefore, the approximate analytical solution of equations (14) and (15) in truncated series form is provided by

$$u(x,t) \cong \lim_{N \to \infty} \sum_{m=0}^{N} u_m(x,t). \tag{25}$$

In general, the solutions in the preceding series converge quickly. The classical approach to convergence of this type of series was described already by Cherruault and Adomian [9] and Cherruault *et al.* [10].

4. ILLUSTRATIVE EXAMPLES

In this section, to give a clear overview of this method, we present some illustrative examples to demonstrate the accuracy and easy implementation of this methodology.

Example 1: Consider the following linear fractional diffusion equation [13,14,15]

$$D_t^{\alpha} u\left(x,t\right) = -\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}, -\infty < x_i < \infty, t > 0,\ 0 < \alpha \le 1, \tag{26}$$

with the initial condition

$$u(x,0) = \exp\left(-x_1 - x_2 - x_3\right), \tag{27}$$

where $x = (x_1, x_2, x_3)$ and $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of order α given by equation (7).

Taking the Sumudu transform of the above equation (26) and making use of the result given by equation (27), we have

$$S\left[u\left(x,t\right)\right] = \exp\left(-x_1 - x_2 - x_3\right) + \omega^{\alpha}\left[S\left(-\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}\right)\right]. \tag{28}$$

Applying inverse Sumudu transform to the equation (28), we obtain

$$u(x,t) = \exp\left(-x_1 - x_2 - x_3\right) + S^{-1}\left[\omega^{\alpha}\left[S\left(-\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u}{\partial x_i^2}\right)\right]\right]. \tag{29}$$

Substituting the results from equations (19) to (21) in the equation (29) and making use of the results given by the equations (23) to (24), we determine the components of the SDM solution as follows

$$u_0(x,t) = \exp\left(-x_1 - x_2 - x_3\right), \tag{30}$$

$$u_1(x,t) = S^{-1}\left[\omega^{\alpha}\left[S\left(-\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u_0}{\partial x_i^2}\right)\right]\right] = -\exp\left(-x_1 - x_2 - x_3\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{31}$$

$$u_2(x,t) = S^{-1}\left[\omega^{\alpha}\left[S\left(-\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u_1}{\partial x_i^2}\right)\right]\right] = \exp\left(-x_1 - x_2 - x_3\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{32}$$

$$u_3(x,t) = S^{-1}\left[\omega^{\alpha}\left[S\left(-\frac{1}{3}\sum_{i=1}^{3}\frac{\partial^2 u_2}{\partial x_i^2}\right)\right]\right] = -\exp\left(-x_1 - x_2 - x_3\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{33}$$

and so on. The other components can be determined in the same manner.

Thus, the series form of approximate analytical solution can be obtained as

$$\begin{aligned} u(x,t) &\cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots \\ &= \exp\left(-x_1 - x_2 - x_3\right)\left(1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots\right) \\ &= \exp\left(-x_1 - x_2 - x_3\right)\sum_{m=0}^{\infty}\frac{\left(-t^{\alpha}\right)^m}{\Gamma(n\alpha+1)} \\ &= \exp\left(-x_1 - x_2 - x_3\right)E_{\alpha}\left(-t^{\alpha}\right). \end{aligned} \tag{34}$$

The same result was obtained by Jafari *et al.* [15] using LDM, Jafari and Momani [14] using modified HPM, and Jafari and Daftardar-Gejji [13] using ADM.

If we put $\alpha = 1$, in equation (34), we have

$$u(x,t) = \exp\left(-x_1 - x_2 - x_3\right)e^{-t}. \tag{35}$$

Example 2: Consider the following linear fractional wave equation [13, 14, 15]

$$D_t^{\alpha} u(x,t) = 2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right), \quad -\infty < x_1, x_2 < \infty, \quad t > 0,\ 1 < \alpha \le 2, \tag{36}$$

with the initial conditions

$$u(x,0) = \sin x_1 \sin x_2, \frac{\partial u(x,0)}{\partial t} = 0, \tag{37}$$

where $x = (x_1, x_2)$ and $D_t^{\alpha} u(x,t)$ is the Caputo fractional derivative of order α given by equation (7).

Taking the Sumudu transform of the above equation (36) and making use of the results given by equation (37), we have

$$S\left[u\left(x,t\right)\right]=\left(\sin x_1 \sin x_2\right)+\omega^{\alpha} S\left[2\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)\right]. \tag{38}$$

Applying inverse Sumudu transform to the equation (38), we obtain

$$u(x,t)=\left(\sin x_1 \sin x_2\right)+S^{-1}\left[\omega^{\alpha} S\left[2\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right)\right]\right]. \tag{39}$$

Substituting the results from equations (19) to (21) in the equation (39) and making use of the results given by the equations (23) to (24), we determine the components of the SDM solution as follows

$$u_0(x,t)=\left(\sin x_1 \sin x_2\right), \tag{40}$$

$$u_1(x,t)=S^{-1}\left[\omega^{\alpha} S\left[2\left(\frac{\partial^2 u_0}{\partial x_1^2}+\frac{\partial^2 u_0}{\partial x_2^2}\right)\right]\right]=-\left(\sin x_1 \sin x_2\right)\frac{4t^{\alpha}}{\Gamma(\alpha+1)}, \tag{41}$$

$$u_2(x,t)=S^{-1}\left[\omega^{\alpha} S\left[2\left(\frac{\partial^2 u_1}{\partial x_1^2}+\frac{\partial^2 u_1}{\partial x_2^2}\right)\right]\right]=\left(\sin x_1 \sin x_2\right)\frac{4^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{42}$$

$$u_3(x,t)=S^{-1}\left[\omega^{\alpha} S\left[2\left(\frac{\partial^2 u_2}{\partial x_1^2}+\frac{\partial^2 u_2}{\partial x_2^2}\right)\right]\right]=-\left(\sin x_1 \sin x_2\right)\frac{4^3 t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{43}$$

and so on. The other components can be determined in the same manner.

Thus, the series form of approximate analytical solution can be obtained as

$$\begin{aligned} u(x,t) &\cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\ldots \\ &=\left(\sin x_1 \sin x_2\right)\left(1-\frac{4t^{\alpha}}{\Gamma(\alpha+1)}+\frac{4^2 t^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4^3 t^{3\alpha}}{\Gamma(3\alpha+1)}+\ldots\right) \\ &=\left(\sin x_1 \sin x_2\right)\left(\sum_{m=0}^{\infty}\frac{\left(-4t^{\alpha}\right)^m}{\Gamma\left(n\alpha+1\right)}\right) \\ &=\left(\sin x_1 \sin x_2\right)E_{\alpha}\left(-4t^{\alpha}\right). \end{aligned} \tag{44}$$

The same result was obtained by Jafari *et al.* [15] using LDM, Jafari and Momani [14] using modified HPM, and Jafari and Daftardar-Gejji [13] using ADM.

If we put $\alpha=2,$ in equation (44), we have

$$u(x,t)=\left(\sin x_1 \sin x_2\right)\cos\left(2t\right). \tag{45}$$

This result was earlier achieved by Wazwaz [29] using the ADM approach.

Example 3: Consider the following non-linear fractional wave equation [13, 14, 15]

$$D_t^{\alpha} u(x,t)+au_{xx}+\beta u+\gamma u^3=0,\ 0<x<1,\ t>0,\ 1<\alpha\le 2, \tag{46}$$

with initial conditions

$$u(x,0)=B\tan\left(Kx\right),\frac{\partial u(x,0)}{\partial t}=BcK\sec^2\left(Kx\right), \tag{47}$$

where $a,c,\beta,\gamma,$ are constants and $B=\sqrt{\dfrac{\beta}{\gamma}},\ K=\sqrt{\dfrac{-\beta}{2(a+c^2)}}$ and $D_t^{\alpha}u(x,t)$ is the Caputo fractional derivative of order α given by equation (7).

Taking the Sumudu transform of the above equation (46) and making use of the results given by equation (47), we have

$$S\left[u(x,t)\right]=\left[B\tan\left(Kx\right)+\omega BcK\sec^2\left(Kx\right)\right]-\omega^{\alpha} S\left[au_{xx}+\beta u+\gamma u^3\right]. \tag{48}$$

Applying inverse Sumudu transform to the equation (48), we obtain

$$u(x,t)=\left[B\tan\left(Kx\right)+tBcK\sec^2\left(Kx\right)\right]-S^{-1}\left[\omega^{\alpha} S\left(au_{xx}+\beta u+\gamma u^3\right)\right]. \tag{49}$$

Substituting the results from equations (19) to (21) in the equation (49) and making use of the results given by the equations (23) to (24), we determine the components of the SDM solution as follows

$$u_0(x,t) = \left[B\tan(Kx) + tBcK\sec^2(Kx)\right], \tag{50}$$

$$u_{m+1}(x,t) = -a\ ^{-1}\left[\omega SS\left(\frac{\partial^2 u_m}{\partial x^2}\right)\right] - \beta S^{-1}\left[\omega^\alpha S\left(u_m\right)\right] - \gamma S^{-1}\left[\omega^\alpha S\left(A_m\right)\right], m = 0,1,..., \tag{51}$$

where A_m are Adomian polynomials defined in equation (21). We have

$$\begin{aligned} A_0 &= \beta u_0 + \gamma u_0^3, \\ &= \beta\left[B\tan\left(Kx\right) + BtcK\sec^2\left(Kx\right)\right] + \gamma\left[B^3\tan^3\left(Kx\right) + B^3t^3c^3K\sec^6\left(Kx\right)\right. \\ &\quad \left. + 3B^3c^3K^2t^2\sec^4\left(Kx\right)\tan\left(Kx\right) + 3B^3tcK\sec^2\left(Kx\right)\tan^2\left(Kx\right)\right], \end{aligned} \tag{52}$$

$$\begin{aligned} u_1(x,t) &= -a\left[\frac{2BK^2t^\alpha\sec^2(Kx)\tan\left(Kx\right)}{\Gamma(\alpha+1)} + \frac{4BcK^3t^{(\alpha+1)}\sec^4\left(Kx\right)}{\Gamma(\alpha+2)}\right. \\ &\quad \left. + \frac{8BcK^3t^{(\alpha+1)}\sec^2\left(Kx\right)\tan^2\left(Kx\right)}{\Gamma(\alpha+2)}\right] - \beta\left[\frac{Bt^\alpha\tan(Kx)}{\Gamma(\alpha+1)} + \frac{2BcKt^{(\alpha+1)}\sec^2\left(Kx\right)}{\Gamma(\alpha+2)}\right] \\ &\quad -\gamma\left[\frac{6B^3c^3K^3t^{(\alpha+3)}\sec^6\left(Kx\right)}{\Gamma(\alpha+4)} + \frac{6B^3c^2K^2t^{(\alpha+2)}\sec^4\left(Kx\right)\tan\left(Kx\right)}{\Gamma(\alpha+3)}\right. \\ &\quad \left. + \frac{6B^3cKt^{(\alpha+1)}\sec^2\left(Kx\right)\tan^2\left(Kx\right)}{\Gamma(\alpha+2)} + \frac{B^3t^\alpha\tan^3\left(Kx\right)}{\Gamma(\alpha+1)}\right], \end{aligned} \tag{53}$$

and so on. The other components can be determined in the same manner.

Thus, the series form of approximate analytical solution can be obtained as

$$\begin{aligned} u(x,t) &\cong \lim_{N\to\infty}\sum_{m=0}^{N} u_m(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t). \ . \\ &= \left[B\tan\left(Kx\right) + tBcK\sec^2\left(Kx\right)\right] - a\left[\frac{2BK^2t^\alpha\sec^2\left(Kx\right)\tan\left(Kx\right)}{\Gamma(\alpha+1)} + \frac{4BcK^3t^{(\alpha+1)}\sec^4\left(Kx\right)}{\Gamma(\alpha+2)}\right. \\ &\quad \left. + \frac{8BcK^3t^{(\alpha+1)}\sec^2\left(Kx\right)\tan^2\left(Kx\right)}{\Gamma(\alpha+2)}\right] - \beta\left[\frac{Bt^\alpha\tan\left(Kx\right)}{\Gamma(\alpha+1)} + \frac{2BcKt^{(\alpha+1)}\sec^2\left(Kx\right)}{\Gamma(\alpha+2)}\right] \\ &\quad -\gamma\left[\frac{6B^3c^3K^3t^{(\alpha+3)}\sec^6\left(Kx\right)}{\Gamma(\alpha+4)} + \frac{6B^3c^2K^2t^{(\alpha+2)}\sec^4\left(Kx\right)\tan\left(Kx\right)}{\Gamma(\alpha+3)}\right. \\ &\quad \left. + \frac{6B^3cKt^{(\alpha+1)}\sec^2\left(Kx\right)\tan^2\left(Kx\right)}{\Gamma(\alpha+2)} + \frac{B^3t^\alpha\tan^3\left(Kx\right)}{\Gamma(\alpha+1)}\right] + ... \end{aligned} \tag{54}$$

The same result was obtained by Jafari *et al.* [15] using the method of LDM and Jafari and Momani [14] using modified HPM.

If we put $\alpha = 2,$ in equation (54), we have

$$u\left(x,t\right) = B\tan\left(K\left(x+ct\right)\right). \tag{55}$$

This result was earlier achieved by Kaya and El-Sayed [17] using the ADM approach.

5. CONCLUSION

In this paper, we have determined the approximate analytical solutions of the linear and non-linear fractional diffusion and wave equations with initial conditions by using the Sumudu decomposition method (SDM). The time-fractional derivative described here in the Caputo sense. The solution was provided by the proposed algorithm in a series form that converges rapidly to the exact solution, if it exists. Based on the obtained results, it is clear that the SDM produces quite accurate solutions using only a few iterations. This study concludes that SDM can be applied to other fractional-order differential equations due to its efficiency and adaptability, as can be seen in the illustrative examples.

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