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A NEW NECESSARY CONDITION OF OPTIMALITY FOR SINGULAR CONTROL PROBLEMS

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1. INTRODUCTION

Necessary condition of optimality for nonsingular, unconstrained, control problems are well known. When control and state variable constraints are present, the situation is more complex, but recent research [1] indicates that many of the subtleties of this class of problems are now uncovered. In the classical calculus of variations literature, little space is devoted to the analysis of singular variational problems. Recently, interest has been aroused in singular optimal control problems, owing to the appearance of such problems in, for example, the aerospace field and the chemical industry. Kelley discovered, and Robbins, Tait and Kelley *et al*. generalized, a new necessary condition of optimality for singular arcs. The condition, known as the generalized Legendre-Clebsch condition, has, in a number of cases, proved useful, in eliminating some stationary arcs from the class of candidate arcs for minimizing solutions. The generalized Legendre-Clebsch condition is proved using special control variations. In this paper, by the use of a different special control variation, an additional necessary condition of optimality is derived. The differential dynamic programming approach is used to calculate the expression for the change in cost produced by the introduction of the special variation. The new necessary condition is deduced from this expression. Control problems without terminal constraints are considered first. For this class of problems, the special control variation is a rectangular pulse. With terminal constraints present, the rectangular pulse is followed b a control variation which is designed to keep the terminal constraints satisfied to first order.

2. PRELIMINARIES

Consider the class of control problems where the dynamical system is described by the differential equations:

$$\dot{x} = f\left(x,u,t\right), \quad x\left(t_0\right) = x_0 \ , \tag{1}$$

where

$$f\left(x,u,t\right) \equiv f_1\left(x,t\right) + f_u\left(x,t\right)u \ . \tag{2}$$

The performance of the system is measured b the cost functional

$$V\left(x_0,t_0\right) = \int_{t_0}^{t_f} L\left(x,t\right)dt + F\left(x\left(t_f\right),t_f\right) \tag{3}$$

and the terminal state must satisfy

$$\psi\left(x\left(t_f\right),t_f\right) = 0 \ . \tag{4}$$

The control u is required to satisfy the constraint

$$\left|u\left(t\right)\right| \le 1, \qquad t \in \left[t_0,t_f\right]. \tag{5}$$

Here, x is an n-dimensional state vector, and u is a scalar control. f_1 and f_u are n-dimensional vector functions of x at time t, and L and F are scalar functions. ψ is an s-dimensional column vector function of $x\left(t_f\right)$ at t_f. The final time is assumed to be given explicitly. The functions f, L and F are assumed tobe three times continuously differentiable in each argument.

The control problem is: determine the control function $u\left(\cdot\right)$ to satisfy (5) and (4) and minimize the cost $V\left(x_0,t_0\right)$.

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3. NECESSARY CONDITIONS OF OPTIMALITY

It can e shown, for the case where terminal constraints are absent, that the following necessary conditions of optimality hold:

$$-V_x = H_x\left(\bar{x},\bar{u},\bar{V}_x,t\right),\ \ \bar{V}_x\left(t_f\right) = F_x\left(\bar{x}\left(t_f\right),t_f\right)\ , \tag{6}$$

where

$$\bar{u} = \arg\min_{\substack{u \\ \{|u|\le t\}}} H\left(\bar{x},u,\bar{V}_x,t\right) \tag{7a}$$

and

$$H\left(x,u,V_x,t\right) = L\left(x,t\right) + \left\langle V_x,\ f\left(x,u,t\right)\right\rangle. \tag{7b}$$

Here, $\bar{x}\left(\cdot\right),\bar{u}\left(\cdot\right)$ denote the candidate state and control functions. The derivative $\bar{V}_x\left(\bar{x},t\right)$ is the partial derivative of $\bar{V}$ —the cost produced by the control function $\bar{u}\left(\cdot\right)$. Here, $\bar{V}_x$ can be identified with Pontryagin's adjoint variable. Note that $\bar{V}_x$ may not be equal to the first partial derivative V_x^0 of the optimal cost V^0 which is obtained when optimal feedback control is used.

In general the optimal control function (for the class of problems formulated in (2) will consist of bang-bang subarcs and singular subarcs. A bang-bang arc is one along which strict equality holds in (5), except at a finite number of 'switch times' where the control $\bar{u}$ changes sign. A singular arc is one along which

$$H_u\left(\bar{x},\bar{V}_x,t\right) = 0 \tag{8}$$

for a finite time interval. Note that this implies that, on a singular arc, *H* is independent of the control *u*. along a singular arc, Kelley *et al*., Robbins and Tait prove that an additional necessary condition of optimality is as follows:

$$\left(-1\right)^p \frac{\partial}{\partial u}\left[\frac{d^{2p}}{dt^{2p}} H_u\left(\bar{x},\bar{V}_x,t\right)\right] \ge 0\ , \tag{9}$$

pth and 2pth time derivative of H_u is the first to contain explicitly the control *u*. Inequality (9) is known as the generalized Legendre-Clebsch condition.

4. EXPRESSION FOR CHANGE IN COST WHEN CONTROL VARIATION IS PRESENT: TERMINAL STATE UNCONSTRAINED

If a control function $\bar{u}\left(\cdot\right)+\delta u\left(\cdot\right)$ is applied to the system, then a trajectory $\bar{x}\left(\cdot\right)+\delta x\left(\cdot\right)$ is produced. At time $t, V\left(\bar{x}+\delta x,t\right)$ is the cost to go, from *t* to the final time t_f, when starting in state $\bar{x}\left(t\right)+\delta x\left(t\right)$ and using controls $\bar{u}\left(\cdot\right)+\delta u\left(\cdot\right)$. Let us assume that the cost can be expanded in a Taylor series about $\bar{x},t$:

$$V\left(\bar{x}+\delta x,t\right) = V\left(\bar{x},t\right) + \left\langle V_x\left(\bar{x},t\right),\delta x\right\rangle + \frac{1}{2}\left\langle \delta x, V_{xx}\left(\bar{x},t\right)\delta x\right\rangle + \text{higher order terms.} \tag{10}$$

The partial derivatives in (10) are obtained by changing *x* but keeping the control function fixed at $\bar{u}\left(\cdot\right)+\delta u\left(\cdot\right), V\left(\bar{x},t\right)$, the cost to go from *t* to t_f when starting in state $\bar{x}\left(t\right)$ and using controls $\bar{u}\left(\cdot\right)+\delta u\left(\cdot\right)$, can be written as

$$V\left(\bar{x},t\right) = \bar{V}\left(\bar{x},t\right) + a\left(\bar{x},t\right), \tag{11}$$

where $a\left(\bar{x},t\right)$ is the change in cost, when starting at time *t* in state $\bar{x}\left(t\right)$, produced by the variation $\delta u\left(\tau\right),\ \tau \in \left[t, t_f\right]$.

Using (11) in (10):

$$V\left(\bar{x}+\delta x,t\right) = \bar{V}\left(\bar{x},t\right) + a\left(\bar{x},t\right) + \left\langle V_x\left(\bar{x},t\right),\delta x\right\rangle + \frac{1}{2}\left\langle \delta x, V_{xx}\left(\bar{x},t\right)\delta x\right\rangle$$
$$+ \text{higher order terms.} \tag{12}$$

From (3) it is clear that

$$\dot{V}\left(\bar{x}+\delta x,t\right) = -L\left(\bar{x}+\delta x,t\right), \tag{13}$$

whence,

$$-\frac{\partial V}{\partial t}\left(\overline{x}+\delta x,t\right)=L\left(\overline{x}+\delta x,t\right)+\left\langle V_x\left(\overline{x}+\delta x,t\right),\ f\left(\overline{x}+\delta x,\overline{u}+\delta u,t\right)\right\rangle. \tag{14}$$

Substituting (12) into (14) and expanding *L* and *f* in Taylor series about $\overline{x}$, we obtain

$$-\frac{\partial \overline{V}}{\partial t}-\frac{\partial a}{\partial t}-\left\langle \frac{\partial V_x}{\partial t},\delta x\right\rangle-\frac{1}{2}\left\langle \delta x,\frac{\partial V_{xx}}{\partial t}\delta x\right\rangle \text{ + higher order terms}$$

$$=H\left(\overline{x},\overline{u}+\delta u,V_x,t\right)+\left\langle H_x+V_{xx}f,\delta x\right\rangle$$

$$+\frac{1}{2}\left\langle \delta x,\left(H_{xx}+f_x^TV_{xx}+V_{xx}f_x+\frac{1}{2}f^TV_{xxx}+\frac{1}{2}V_{xxx}\right)\delta x\right\rangle\text{+ higher order terms.} \tag{15}$$

All derivatives in (15) are evaluated at $\overline{x},\overline{u}+\delta u,V_x,t$.

Since equality holds for all δx, we equate coefficients to obtain

$$-\frac{\partial \overline{V}}{\partial t}-\frac{\partial a}{\partial t}=H\left(\overline{x},\overline{u}+\delta u,V_x,t\right),$$

$$-\frac{\partial V_x}{\partial t}=H_x\left(\overline{x},\overline{u}+\delta u,V_x,t\right)+V_{xx}f\left(\overline{x},\overline{u}+\delta u,t\right),$$

$$-\frac{\partial V_{xx}}{\partial t}=H_{xx}\left(\overline{x},\overline{u}+\delta u,V_x,t\right)+f_x^T\left(\overline{x},\overline{u}+\delta u,t\right)V_{xx}$$

$$+V_{xx}f_x\left(\overline{x},\overline{u}+\delta u,t\right)+\frac{1}{2}V_{xxx}f\left(\overline{x},\overline{u}+\delta u,t\right)+\frac{1}{2}f^T\left(\overline{x},\overline{u}+\delta u,t\right)V_{xxx}. \tag{16}$$

The higher order equations are not presented.

Now,

$$\frac{d}{dt}\left(\overline{V}+a\right)=\frac{d}{dt}V=\frac{\partial V}{\partial t}+\left\langle V_x,\ f\left(\overline{x},\overline{u},t\right)\right\rangle.$$

Therefore,

$$\frac{d}{dt}\left(\overline{V}+a\right)=\frac{\partial}{\partial t}\left(\overline{V}+a\right)+\left\langle V_x,\ f\left(\overline{x},\overline{u},t\right)\right\rangle \tag{17a}$$

and

$$\dot{V}_x=\frac{\partial V_x}{\partial t}+V_{xx}f\left(\overline{x},\overline{u},t\right),$$

$$\dot{V}_{xx}=\frac{\partial V_{xx}}{\partial t}+\frac{1}{2}V_{xxx}f\left(\overline{x},\overline{u},t\right)+\frac{1}{2}f^{T\left(\overline{x},\overline{u},t\right)V_{xxx}}.$$

Using (17) in (16), the following equations result:

$$-\dot{a}=H-H\left(\overline{x},\overline{u},V_x,t\right),$$

$$-\dot{V}_x=H_x+V_{xx}\left(f-f\left(\overline{x},\overline{u},t\right)\right),$$

$$-\dot{V}_{xx}=H_{xx}+f_x^TV_{xx}+V_{xx}f_x+\frac{1}{2}V_{xxx}\left(f-f\left(\overline{x},\overline{u},t\right)\right)+\frac{1}{2}\left(f-f\left(\overline{x},\overline{u},t\right)\right)^TV_{xxx}, \tag{18}$$

where, unless otherwise specified, all quantities are evaluated at $\overline{x},\overline{u}+\delta u,V_x,t$. Using the special structure of *f*, equation (2), equations (18) become:

$$-\dot{a}=H_u\delta u,$$

$$-\dot{V}_x=H_x+\left(H_{xu}+V_{xx}f_u\right)\delta u,$$

$$-\dot{V}_{xx}=H_{xx}+f_x^TV_{xx}+V_{xx}f_x+\left(H_{xxu}+f_{xu}^TV_{xx}+V_{xx}f_{xu}+\frac{1}{2}V_{xxx}f_u+\frac{1}{2}f_u^TV_{xxx}\right)\delta u. \tag{19}$$

In (19), all quantities are now evaluated at $\overline{x}, \overline{u}, V_x, t$. Boundary conditions for (19) are, clearly,

$$a\left(t_f\right) = 0,$$
$$V_x\left(t_f\right) = F_x\left(\overline{x}, t_f\right),$$
$$V_{xx}\left(t_f\right) = F_{xx}\left(\overline{x}, t_f\right). \tag{20}$$

The change in cost owing to the presence of a control variation $\delta u(\tau); \tau \in [t_1, t_2],\ t_2 > t_1$, is given by

$$a(t_1) = a(t_2) + \int_{t_2}^{t_1} \dot{a}(t)\, d \ . \tag{21}$$

5. NEW NECESSARY CONDITION: UNCONSTRAINED TERMINAL STATE

A singular are is assumed to lie in an interval $[t_a, t_b]$. A control variation in the form of a rectangular pulse of height η and duration *T* is introduced in an interval $[t_1, t_2]$
where

$$t_a < t_i < t_b, \qquad i = 1, 2, \quad t_2 > t_1. \tag{22}$$

See Fig. 1.

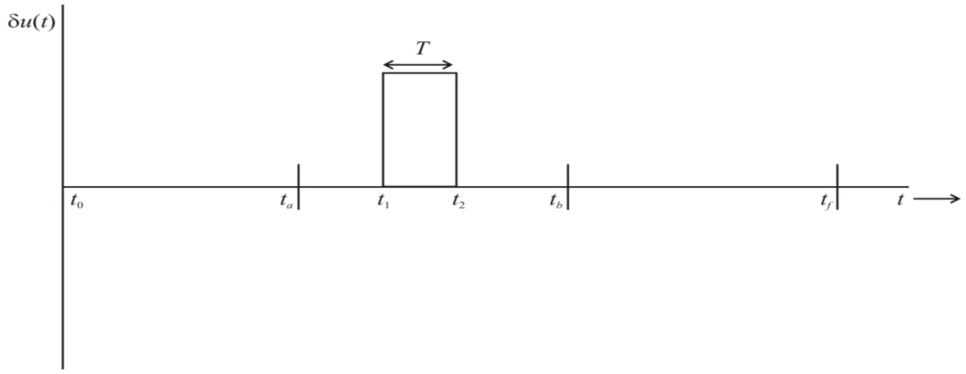

Figure-1

The change in cost produced by this variation is given by

$$a(t_1) = \int_{t_2}^{t_1} \dot{a}dt + a(t_2) = \int_{t_2}^{t_1} -H_u \delta u dt + a(t_2), \tag{23}$$

where H_u is evaluated at $\overline{x}, V_x, t$. Expanding the integral in a Taylor series in *T*, the expression for the change in cost becomes

$$a(t_1) = H_u\, \delta u\big|_{t_2} T - \frac{1}{2}\frac{d}{dt}\left[H_u \delta u\right]\Big|_{t_2} T^2 + \ldots + a(t_2).$$

At time t_2, one has

$$a(t_2) = 0,$$
$$V_x(t_2) = \overline{V}_x(t_2),$$
$$V_{xx}(t_2) = \overline{V}_{xx}(t_2), \tag{25}$$

where $\overline{V}_x(t_2)$ and $\overline{V}_{xx}(t_2)$ are computed using (19) and (20) with $\delta u(t) = 0, t \in (t_2, t_f]$.

Since $\overline{x}(t_2)$ is on the singular arc, $H_u\left(\overline{x}, \overline{V}_x, t_2 = 0\right)$. Thus, the first nonzero term in expansion (24) is the T^2 one. We have that

$$\frac{d}{dt}\left[H_u \delta u\right]\Big|_{t_2} = \dot{H}_u \delta u\big|_{t_2} + H_u \delta \dot{u}\big|_{t_2} = \dot{H}_u\left(\overline{x}, \overline{V}_x, t\right)\Big|_{t_2} \eta. \tag{26}$$

From (19), (20),

$$\dot{H}_u\left(\bar{x},V_x,t\right)\Big|_{t_2}=\left\{f_u^TV_x+f_u^T\left[-H_x-\left(H_{xu}+V_{xx}f_u\right)\eta\right]\right\}\Big|_{t_2}. \tag{27}$$

The first two terms in (27) sum to zero. Using (27) and (26) in (24), the change in cost is

$$a\left(t_1\right)=\frac{1}{2}f_u^T\left(\bar{x},t_2\right)\left[H_{xu}\left(\bar{x},\bar{V}_x,t_2\right)+\bar{V}_{xx}f_u\left(\bar{x},t_2\right)\right]\eta^2T^2+\text{higher order terms}. \tag{28}$$

For the singular arc to be a candidate as a minimizing arc, it is necessary that the change in cost, owing to the presence of the control variation, be nonnegative. From (28) this implies that

$$f_u^T\left(\bar{x},t\right)\left[H_{xx}\left(\bar{x},\bar{u},\bar{V}_x,t\right)+\bar{V}_{xx}f_u\left(\bar{x},t\right)\right]\geq 0, \tag{29}$$

where

$$-\dot{\bar{V}}_x=H_x\left(\bar{x},\bar{u},\bar{V}_x,t\right),$$

$$-\dot{\bar{V}}_{xx}=H_{xx}\left(\bar{x},\bar{u},\bar{V}_x,t\right)+f_x^T\left(\bar{x},\bar{u},t\right)\bar{V}_{xx}+\bar{V}_{xx}f_x\left(\bar{x},\bar{u},t\right) \tag{30}$$

and

$$\bar{V}_x\left(t_f\right)=F_x\left(\bar{x}\left(t_f\right),t_f\right),$$

$$\bar{V}_{xx}\left(t_f\right)=F_{xx}\left(\bar{x}\left(t_f\right),t_f\right). \tag{31}$$

Inequality (29) is the new necessary condition of optimality for singular control problems with unconstrained terminal states.

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