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MAGNETIC BIHARMONIC CURVES IN THE HEISENBERG GROUP $\mathbb{H}_3$

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ABSTRACT

In this paper, we study magnetic biharmonic curves with respect to closed 2-forms in the 3-dimensional Heisenberg group $\mathbb{H}_3$. For any given biharmonic curve $\gamma \in \mathbb{H}_3$, we characterize locally the closed 2-forms F for which γ is a magnetic curve and we give an example of such closed 2-forms.

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1. INTRODUCTION

The notions of harmonic and biharmonic maps between Riemannian manifolds have been introduced by J. Eells and J.H. Sampson (cf. [6]).

For a map $\phi: (M, g) \to (N, h)$ between Riemannian manifolds the energy functional E_1 is defined by

$$E_1(\phi) = \frac{1}{2}\int_M |d\phi|^2 v_g.$$

Critical points of E_1 are called harmonic maps and are then solutions of the corresponding Euler-Lagrange equation

$$\tau_1(\phi) = \text{trace}\, \nabla^\phi d\phi. \tag{1}$$

Here ∇^ϕ denotes the induced connection on the pull-back bundle $\phi^{-1}(TN)$ and $\tau_1(\phi)$ is called the tension field of ϕ. Biharmonic maps are the critical points of the bienergy functional

$$E_2(\phi) = \frac{1}{2}\int_M |\tau_1(\phi)|^2 v_g,$$

whose Euler-Lagrange equation is given by the vanishing of the bitension field (cf. [10]) defined by

$$\tau_2(\phi) = -\Delta^\phi \tau_1(\phi) - \text{trace}\, \mathrm{R}^N\big(d\phi, \tau_1(\phi)\big)d\phi, \tag{2}$$

where $\Delta^\phi = -\text{ trace }_g\big(\nabla^\phi \nabla^\phi - \nabla^\phi_\nabla\big)$ is the Laplacian on the sections of $\phi^{-1}(TN)$, and R^N is the Riemannian curvature operator of (N, h). Note that

$$\tau_2(\phi) = J_\phi\big(\tau_1(\phi)\big), \tag{3}$$

where J_ϕ is the Jacobi operator along ϕ defined by

$$J_\phi(X) = -\Delta^\phi X - \text{trace}\, \mathrm{R}^N(d\phi, X)\, d\phi, \quad \forall X \in \phi^{-1}(TN). \tag{4}$$

Harmonic maps are obviously biharmonic and are absolute minimum of the bienergy. Nonminimal biharmonic submanifolds of the pseudo-euclidean spaces and of the spheres have been studied in [4] and [2].

Biharmonic curves have been investgated on many special Riemannian manifolds like Heisenberg groups [3], [7], invariant surfaces [11], Damek-Ricci spaces [5], Sasakian manifolds [8], etc.

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When a solution ϕ of the equation (1) is a curve, one says that it is a geodesic. Magnetic curves generalize geodesics. In Physics, such a curve represents a trajectory of a charged particle moving on the manifold under the action of a magnetic field. Let (M,g) be an n-dimensional Riemannian manifold. A magnetic field is a closed 2-form F on M and the Lorentz force of a magnetic field F on (M,g) is a (1,1)-tensor field Φ given by

$$g(\Phi(X),Y) = F(X,Y), \quad \forall X,Y \in TM, \tag{5}$$

where TM is the tangent bundle of M.

The magnetic trajectories of F are curves γ on M that satisfy the Lorentz equation (sometimes called the Newton equation)

$$\nabla_{\gamma'}\gamma' = \Phi(\gamma'), \tag{6}$$

where ∇ is the Levi-Civita connection of (M,g).

The Lorentz equation generalizes the equation satisfied by the geodesics of (M,g),

$$\nabla_{\gamma'}\gamma' = 0. \tag{7}$$

Therefore, from the point of view of the dynamical systems, a geodesic corresponds to a trajectory of a particle without an action of a magnetic field, while a magnetic trajectory is a flow line of the dynamical system, associated to the magnetic field. In contrast to geodesics, magnetic curves are not reversible and they cannot be rescaled, that is the trajectories depend on the energy $|\gamma'|$. The Lorentz force is skew symmetric and therefore the magnetic curves have constant speed (and hence energy)

$$v(t) = |\gamma'| = v_0. \tag{8}$$

When they are parametrized by arc lenght ($v_0 = 1$), we use to call them normal magnetic curves. See for example, [1], [9] or the original papers of Novikov, e.g. [12]. Then it is interesting to know if a non-geodesic biharmonic curve could be a magnetic curve.

In the following section 2, we present some preliminaries on the 3-dimensional Heisenberg group $\mathbb{H}_3$ and on the biharmonic curves in $\mathbb{H}_3$. In the section 3, for a closed 2-form F on $\mathbb{H}_3$, we determine the corresponding Lorentz force. And for a non-geodesic biharmonic curve in $\mathbb{H}_3$, we characterize locally the closed 2-forms for which this curve is a magnetic curve. Finally, we determine, for each non-geodesic biharmonic curve in $\mathbb{H}_3$, an example of closed 2-forms for which it is a magnetic curve.

2. PRELIMINARIES

The Heisenberg group $\mathbb{H}_3$ can be seen as the Euclidean space $\mathbb{R}^3$ endowed with the Riemannian metric g given by

$$g = dx^2 + dy^2 + \left(dz + \frac{y}{2}dx - \frac{x}{2}dy\right)^2, \tag{9}$$

for all $(x,y,z) \in \mathbb{R}^3$.

The vector fields

$$e_1 = \frac{\partial}{\partial x} - \frac{1}{2}y\frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y} + \frac{1}{2}\frac{\partial}{\partial z}, \quad e_3 = \frac{\partial}{\partial z}, \tag{10}$$

for all $(x,y,z) \in \mathbb{R}^3$; define an orthonormal basis of $\mathbb{H}_3$.

Furthermore, the Levi-Civita connection of $\mathbb{H}_3$ is defined by:

$$\begin{cases} \nabla_{e_1}e_1 = 0, & \nabla_{e_1}e_2 = \frac{1}{2}e_3, & \nabla_{e_1}e_3 = -\frac{1}{2}e_2, \\ \nabla_{e_2}e_1 = -\frac{1}{2}e_3, & \nabla_{e_2}e_2 = 0, & \nabla_{e_2}e_3 = \frac{1}{2}e_1, \\ \nabla_{e_3}e_1 = -\frac{1}{2}e_2, & \nabla_{e_3}e_2 = \frac{1}{2}e_1, & \nabla_{e_3}e_3 = 0. \end{cases} \tag{11}$$

Explicit formulas for biharmonic non-geodesic curves in the 3-dimensional Heisenberg group $\mathbb{H}_3$ have been given in the following result.

Theorem 2.1: *[3] The parametric equations of all non-geodesic biharmonic curves*

$$\begin{aligned} \gamma: I &\longrightarrow \mathbb{H}_3 \\ s &\longmapsto (x(s), y(s), z(s)) \end{aligned}$$

Are

$$\begin{cases} x(s) = & \frac{1}{A}\sin\alpha_0 \sin(As + a) + b \\ y(s) = & -\frac{1}{A}\sin\alpha_0 \cos(As + a) + c \\ z(s) = & \left(\cos\alpha_0 + \frac{\sin^2\alpha_0}{2A}\right)s - \frac{b}{2A}\sin\alpha_0\cos(As + a) \\ & -\frac{c}{2A}\sin\alpha_0 \sin(As + a) + d, \end{cases} \tag{12}$$

for all $s \in I$, where I is a nonempty open interval,

$$A = \frac{\cos\alpha_0 \pm \sqrt{5\cos^2\alpha_0 - 4}}{2},$$
$$\alpha_0 \in \left(0, \arccos\left(+\frac{2\sqrt{5}}{5}\right)\right] \cup \left[\arccos\left(-\frac{2\sqrt{5}}{5}\right), \pi\right) \tag{13}$$
$$\text{and} \quad a, b, c, d \in \mathbb{R}.$$

3. MAGNETIC BIHARMONIC CURVES IN $\mathbb{H}_3$

Let F be a closed 2-form on $\mathbb{H}_3$ defined by:

$$F = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy, \tag{14}$$

where P, Q and R are C^∞ functions defined on $\mathbb{R}^3$ with values in $\mathbb{R}$, and satisfied

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0, \tag{15}$$

since F is a closed 2-form.

Then according to (5) and (10) the Lorentz force Φ of the magnetic field F on $\mathbb{H}_3$ is given by:

$$\begin{aligned} \Phi(e_1) &= \left(R + \tfrac{1}{2}yP - \tfrac{1}{2}xQ\right)e_2 - Qe_3, \\ \Phi(e_2) &= \left(-R - \tfrac{1}{2}yP + \tfrac{1}{2}xQ\right)e_1 + Pe_3, \\ \Phi(e_3) &= Qe_1 - Pe_2, \end{aligned} \tag{16}$$

for all $(x, y, z) \in \mathbb{R}^3$.

For a non-geodesic biharmonic curve $\gamma = (x, y, z)$ defined in (12) we have

$$\gamma' = A(-y(s) + c)e_1 + A(x(s) - b)e_2 + (\cos\alpha_0)e_3, \tag{17}$$

$$\begin{aligned} \Phi(\gamma') = \; & \left[A(x(s) - b)\left(-R - \tfrac{1}{2}y(s)P + \tfrac{1}{2}x(s)Q\right) + (\cos\alpha_0)Q\right]e_1 \\ & + \left[A(-y(s) + c)\left(R + \tfrac{1}{2}y(s)P - \tfrac{1}{2}x(s)Q\right) - (\cos\alpha_0)P\right]e_2 \\ & + [-A(-y(s) + c)Q + A(x(s) - b)P]e_3 \end{aligned} \tag{18}$$

and

$$\tau_1(\gamma) = -A(x(s) - b)(A - \cos\alpha_0)e_1 + A(-y(s) + c)(A - \cos\alpha_0)e_2, \tag{19}$$

for all $s \in I$.

Thus we obtain the following result.

Proposition 3.1: *A non-geodesic biharmonic curve γ defined by (12) is a magnetic curve with respect to a closed 2 -form F defined by (14) if and only if*

$$\begin{aligned} \frac{1}{2}A(x(s) - b)y(s)P + \left[-\frac{1}{2}A(x(s) - b)x(s) - \cos\alpha_0\right]Q \\ +A(x(s) - b)R = A(x(s) - b)(A - \cos\alpha_0) \end{aligned} \tag{20}$$

$$\begin{aligned} \left[-\frac{1}{2}A(-y(s) + c)y(s) + \cos\alpha_0\right]P + \frac{1}{2}A(-y(s) + c)x(s)Q \\ -A(-y(s) + c)R = -A(-y(s) + c)(A - \cos\alpha_0) \end{aligned} \tag{21}$$

and

$$(x(s) - b)P - (-y(s) + c)Q = 0 \tag{22}$$

for all $s \in I$; where the functions P, Q and R are evaluated at $(x(s), y(s), z(s))$.

Now for a non-geodesic biharmonic curve γ in $\mathbb{H}_3$, we characterize locally the closed 2-forms F for which γ is a magnetic curve. In fact, we have the following Theorem.

Theorem 3.2: Let γ be a non-geodesic biharmonic curve defined by (12) such as there exists $s_0 \in I$ and $x(s_0) \neq b$. Then there exists a neighborhood $J \subset I$ of s_0 such that $x(s) \neq b, \forall s \in J$ and then the curve

$$\begin{aligned} \gamma: \quad & J \longrightarrow \mathbb{H}_3 \\ & s \longmapsto (x(s), y(s), z(s)) \end{aligned}$$

is a magnetic curve with respect to a closed 2-form F defined by (14) if and only if

$$P = \left(\frac{-y(s) + c}{x(s) - b}\right)Q \text{ and} \tag{23}$$

$$\begin{aligned} R = \frac{1}{A(x(s) - b)}\left[\frac{1}{2}A\left(x^2(s) + y^2(s)\right) - \frac{1}{2}Abx(s) - \frac{1}{2}Acy(s) + \cos\alpha_0\right]Q \\ +(A - \cos\alpha_0), \end{aligned} \tag{24}$$

for all $s \in J$, where the functions P, Q and R are evaluated at $(x(s), y(s), z(s))$.

Proof: Let γ be a non-geodesic biharmonic curve defined in (12) such as there exists $s_0 \in I$ and $x(s_0) \neq b$. Then because of the continuity of x at s_0 there exists a neighborhood J of s_0 such as $J \subset I$ and $x(s) \neq b, \forall s \in J$. Furthermore, according to Proposition 3.1 and for $s \in J$, we can consider (P, Q, R) like a solution of the system of equations (20), (21) and (22) whose determinant is null. So, with the condition $x(s) \neq b$, the equation (22) gives the solution (23). Then by considering the equation (20) we obtain the solution (24) and by direct computations we can check easily that the equation (21) is satisfied by the solutions obtained in (23) and (24).

Remark 3.3: *For a non-geodesic biharmonic curve γ defined by (12), a parameter $s_0 \in I$ such as $x(s_0) \neq b$ exists necessarily since the function $s \longmapsto x(s)$ can not be constant on the nonempty open interval I.*
Then we establish the following corollary.

Corollary3. 4:
1. Let γ be a non-geodesic biharmonic curve defined by (12). Let F be a closed 2-form defined by (14) such as
$$P = \left(\frac{-y+c}{x-b}\right) Q \tag{25}$$
$$R = \frac{1}{A(x-b)}\left[\frac{1}{2}A(x^2+y^2) - \frac{1}{2}Abx - \frac{1}{2}Acy + \cos\alpha_0\right] Q + (A - \cos\alpha_0) \tag{26}$$
for all $(x, y, z) \in S = \{(x, y, z) \in \mathbb{R}^3, x \neq b\}$.
Then
$$(x-b)(-y+c)\frac{\partial Q}{\partial x} + (x-b)^2\frac{\partial Q}{\partial y} + \frac{(x-b)}{A}\left[\frac{1}{2}A(x^2+y^2) - \frac{1}{2}Abx - \frac{1}{2}Acy + \cos\alpha_0\right]\frac{\partial Q}{\partial z} = 0 \tag{27}$$
and γ is a magnetic curve with respect to the closed 2-form F.

2. Any non-geodesic biharmonic curve γ in $\mathbb{H}_3$ defined by (12) is a magnetic curve with respect to the closed 2-form F defined by:
$$P(x, y, z) = (-y+c)\left[\frac{1}{2}x^2 - bx + \frac{1}{2}y^2 - cy + k\right] \tag{28}$$
$$Q(x, y, z) = (x-b)\left[\frac{1}{2}x^2 - bx + \frac{1}{2}y^2 - cy + k\right] \text{ and} \tag{29}$$
$$R(x, y, z) = \frac{1}{A}\left[\frac{1}{2}A(x^2+y^2) - \frac{1}{2}Abx - \frac{1}{2}Acy + \cos\alpha_0\right]\left[\frac{1}{2}x^2 - bx + \frac{1}{2}y^2 - cy + k\right] + (A - \cos\alpha_0), \tag{30}$$
for all $(x, y, z) \in \mathbb{R}^3$, where k is a constant real number.

Proof:
1. If we extend the formulas (23) and (24) to the set $S = \{(x, y, z) \in \mathbb{R}^3, x \neq b\}$ then we obtain closed 2-forms defined by (25), (26) and (27), and for wich a given non-geodesic biharmonic curve in $\mathbb{H}_3$ is a magnetic curve.
2. According to the last point 1., if we suppose that $\frac{\partial Q}{\partial z} = 0$ then the partial differential equation (27) reduces to
$$(x-b)(-y+c)\frac{\partial Q}{\partial x} + (x-b)^2\frac{\partial Q}{\partial y} + (y-c)Q = 0, \tag{31}$$
for all $(x, y, z) \in S$.
In order to obtain some continuous extensions of the formulas (25) and (26) to $x = b$, we suppose that
$$Q(x, y, z) = (x-b)K(x, y), \tag{32}$$
for all $(x, y, z) \in \mathbb{R}^3$, where K is a C^∞ function on $\mathbb{R}^2$, then since Q is a solution of the partial differential equation (31), the function K satisfies
$$(-y+c)\frac{\partial K}{\partial x} + (x-b)\frac{\partial K}{\partial y} = 0, \tag{33}$$
for all $(x, y) \in \mathbb{R}^2, x \neq b$.

So if we take
$$\frac{\partial K}{\partial x}(x, y) = x - b \text{ and} \tag{34}$$
$$\frac{\partial K}{\partial y}(x, y) = y - c \tag{35}$$
for all $(x, y) \in \mathbb{R}^2$, we obtain
$$K(x, y) = \frac{1}{2}x^2 - bx + \frac{1}{2}y^2 - cy + k, \#(36)$$
for all $(x, y) \in \mathbb{R}^2$, where k is a constant real number. Finally, by using the formulas (25) and (26) we obtain the formulas in (28), (29) and (30) as stated.

Remark 3.5: *Any geodesic in the 3-dimensional Heisenberg group $\mathbb{H}_3$, can be seen as a trivial magnetic curve with respect to the null 2-form on $\mathbb{H}_3$.*

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