International Journal of Mathematical Archive-14(1), 2023, 6-9
Available online through www.ijma.info ISSN 2229 – 5046

ALGEBRAIC DESCRIPTION OF TRIADIC LOGIC

NABILA M. BENNOUR*

University of Benghazi, Faculty of Science, Department of Mathematics.

(Received On: 20-11-22; Revised & Accepted On: 04-01-23)

ABSTRACT

Algebraic description of triadic logic in terms of Boolean algebras of ternary relations is presented.

INTRODUCTION

Algebraization of logic was originated with G. Boole who introduced Boolean Algebra to represent classical propositional logic.

There are mainly two approaches to algebraize first order logic. They are done by Tarski [4] and Halmos [2]. Algebraic descriptions of monadic and dyadic logics are given in terms of Boolean algebras of unary and binary relations [1].

In this paper the algebraic description of triadic logic is presented in terms of Boolean algebra of ternary relations. The quantifiers are described by operators in terms of infimum and supremum.

1. TRIADIC LOGIC

Consider the complete Boolean algebra$(B,\wedge,\vee,',0,1)$. Assume that X is a nonempty set. A ternary relation on X is a subset of X^3. The set of all ternary relations is $R_3(X) = \{\alpha : \alpha \subseteq X^3\}$. Then $R_3(X) \equiv B^{X^3} = \{\alpha | \alpha: X^3 \to B\}$. Define the operations $\wedge,\vee,',0,1$ on $R_3(X)$ pointwise. The order on $R_3(X)$ is also defined pointwise by $\alpha \leq \beta$ if $\alpha(x,y,z) \leq \beta(x,y,z)$ for any $(x,y,z) \in X^3$, where $\alpha,\beta \in R_3(X)$.

Theorem 1[3]: $(R_3(X),\wedge,\vee,',0,1)$ is a complete Boolean algebra.
The quantifier operators $\forall$ and $\exists$ are defined on $R_3(X)$ in terms of infimum and supremum. They exist by theorem 1.

The triple (nested) operators on $R_3(X)$ are defined by iteration on three variables x, y and z. For example $(\forall x)(\exists y)(\forall z)$ is given by $(\forall x)(\exists y)(\forall z)\alpha(x,y,z) \equiv (\forall x)\left[(\exists y)[(\forall z)\alpha(x,y,z)]\right] \equiv inf_x \left[sup_y [inf_z \alpha(x,y,z)]\right]$.
By [1, theorem 3.2] we have

Theorem 2:
i) $\forall_i \forall_j = \forall_j \forall_i$ and $\exists_i \exists_j = \exists_j \exists_i$
ii) $\exists_i \forall_j \leq \forall_j \exists_i$.

Corollary 3: For quantifiers $Q_1, Q_2, \ldots, Q_n$ we get
$Q_1\, Q_2 \ldots Q_n \leq \forall_1 \forall_2 \ldots\ \forall_m \exists_1 \exists_2 \ldots\ \exists_r$.

There are $8 \times 3! = 48$ closed formulas of the form $Q_1\, Q_2 Q_3 \alpha(x,y,z)$ of which one may form 18 equivalences and 12 implications.

Corresponding Author: Nabila M. Bennour*
University of Benghazi, Faculty of Science, Department of Mathematics.

2. EQUIVALENCES

By theorem 2 the equivalences are organized as follows
E-I) $(\exists x)(\exists y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\exists z)(\exists x)(\exists y)\alpha(x,y,z) \Leftrightarrow (\exists y)(\exists z)(\exists x)\alpha(x,y,z) \Leftrightarrow (\exists x)(\exists z)(\exists y)\alpha(x,y,z) \Leftrightarrow (\exists z)(\exists y)(\exists x)\alpha(x,y,z) \Leftrightarrow (\exists y)(\exists x)(\exists z)\alpha(x,y,z)$

E-II)
$(\forall x)(\forall y)(\forall z)\alpha(x,y,z) \Leftrightarrow (\forall z)(\forall x)(\forall y)\alpha(x,y,z) \Leftrightarrow (\forall y)(\forall z)(\forall x)\alpha(x,y,z) \Leftrightarrow (\forall x)(\forall z)(\forall y)\alpha(x,y,z) \Leftrightarrow (\forall z)(\forall y)(\forall x)\alpha(x,y,z) \Leftrightarrow (\forall y)(\forall x)(\forall z)\alpha(x,y,z)$

E-III)
- i) $(\forall x)(\forall y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\forall y)(\forall x)(\exists z)\alpha(x,y,z)$
- ii) $(\forall z)(\forall x)(\exists y)\alpha(x,y,z) \Leftrightarrow (\forall x)(\forall z)(\exists y)\alpha(x,y,z)$
- iii) $(\forall y)(\forall z)(\exists x)\alpha(x,y,z) \Leftrightarrow (\forall z)(\forall y)(\exists x)\alpha(x,y,z)$

E-IV)
- i) $(\forall x)(\exists y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\forall x)(\exists z)(\exists y)\alpha(x,y,z)$
- ii) $(\forall z)(\exists x)(\exists y)\alpha(x,y,z) \Leftrightarrow (\forall z)(\exists y)(\exists x)\alpha(x,y,z)$
- iii) $(\forall y)(\exists x)(\exists z)\alpha(x,y,z) \Leftrightarrow (\forall y)(\exists x)(\exists z)\alpha(x,y,z)$

We prove the following equivalences

E-I) $(\exists x)(\exists y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\exists x)(\exists z)(\exists y)\alpha(x,y,z)$

Proof:
$\alpha(x,y,z) \le sup_y\,\alpha(x,y,z) \le sup_z sup_y\,\alpha(x,y,z) \le sup_x sup_z sup_y\,\alpha(x,y,z)$
$sup_z\,\alpha(x,y,z) \le sup_x sup_z sup_y\,\alpha(x,y,z)$ then
$sup_y sup_z\,\alpha(x,y,z) \le sup_x sup_z sup_y\,\alpha(x,y,z)$ therefore
$sup_x sup_y sup_z\,\alpha(x,y,z) \le sup_x sup_z sup_y\,\alpha(x,y,z)$

Similarly $sup_x sup_z sup_y\,\alpha(x,y,z) \le sup_x sup_y sup_z\,\alpha(x,y,z)$

$\therefore sup_x sup_z sup_y\,\alpha(x,y,z) = sup_x sup_y sup_z\,\alpha(x,y,z).$

Then $(\exists x)(\exists y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\exists x)(\exists z)(\exists y)\alpha(x,y,z)$.
E-II) Apply (E-I) to $\alpha'(x,y,z)$. Then$(\exists x)(\exists y)(\exists z)\alpha'(x,y,z) \Leftrightarrow (\exists x)(\exists z)(\exists y)\alpha'(x,y,z)$. By using DeMorgan we get: $(\forall x)(\forall y)(\forall z)\alpha(x,y,z) \Leftrightarrow (\forall x)(\forall z)(\forall y)\alpha(x,y,z)$.

By theorem 2 we have the following two equivalences.
E-III) i) $(\forall x)(\forall y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\forall y)(\forall x)(\exists z)\alpha(x,y,z)$ and
E-IV) i) $(\forall x)(\exists y)(\exists z)\alpha(x,y,z) \Leftrightarrow (\forall x)(\exists z)(\exists y)\alpha(x,y,z)$.
The rest are similar.

3. IMPLICATIONS

By corollary 3 the implications are arranged as follows:
I-I)
- i) $(\forall x)(\exists y)(\forall z)\alpha(x,y,z) \Rightarrow (\forall x)(\forall z)(\exists y)\alpha(x,y,z)$
- ii) $(\forall z)(\exists y)(\forall x)\alpha(x,y,z) \Rightarrow (\forall z)(\forall x)(\exists y)\alpha(x,y,z)$
- iii) $(\forall z)(\exists x)(\forall y)\alpha(x,y,z) \Rightarrow (\forall z)(\forall y)(\exists x)\alpha(x,y,z)$
- iv) $(\forall x)(\exists z)(\forall y)\alpha(x,y,z) \Rightarrow (\forall x)(\forall y)(\exists z)\alpha(x,y,z)$
- v) $(\forall y)(\exists z)(\forall x)\alpha(x,y,z) \Rightarrow (\forall y)(\forall x)(\exists z)\alpha(x,y,z)$
- vi) $(\forall y)(\exists x)(\forall z)\alpha(x,y,z) \Rightarrow (\forall y)(\forall z)(\exists x)\alpha(x,y,z)$.

I-II)
- i) $(\exists x)(\forall y)(\exists z)\alpha(x,y,z) \Rightarrow (\forall y)(\exists x)(\exists z)\alpha(x,y,z)$
- ii) $(\exists z)(\forall y)(\exists x)\alpha(x,y,z) \Rightarrow (\forall y)(\exists z)(\exists x)\alpha(x,y,z)$
- iii) $(\exists x)(\forall z)(\exists y)\alpha(x,y,z) \Rightarrow (\forall z)(\exists x)(\exists y)\alpha(x,y,z)$
- iv) $(\exists y)(\forall z)(\exists x)\alpha(x,y,z) \Rightarrow (\forall z)(\exists y)(\exists x)\alpha(x,y,z)$
- v) $(\exists y)(\forall x)(\exists z)\alpha(x,y,z) \Rightarrow (\forall x)(\exists y)(\exists z)\alpha(x,y,z)$
- vi) $(\exists z)(\forall x)(\exists y)\alpha(x,y,z) \Rightarrow (\forall x)(\exists z)(\exists y)\alpha(x,y,z)$

We prove the following implications
I-I) i) $(\forall x)(\exists y)(\forall z)\alpha(x,y,z) \Longrightarrow (\forall x)(\forall z)(\exists y)\alpha(x,y,z)$

Proof:
$\alpha(x,y,z) \leq sup_y\,\alpha(x,y,z)$ then $inf_z\alpha(x,y,z) \leq inf_z sup_y\,\alpha(x,y,z)$ then $sup_y inf_z\alpha(x,y,z) \leq inf_z sup_y\,\alpha(x,y,z)$.

Therefore $inf_x sup_y inf_z\alpha(x,y,z) \leq inf_x inf_z sup_y\,\alpha(x,y,z)$. Then
$(\forall x)(\exists y)(\forall z)\alpha(x,y,z) \Longrightarrow (\forall x)(\forall z)(\exists y)\alpha(x,y,z)$.
I-II) i) $(\exists x)(\forall y)(\exists z)\alpha(x,y,z) \Longrightarrow (\forall y)(\exists x)(\exists z)\alpha(x,y,z)$

Proof:
$\alpha(x,y,z) \leq sup_z\alpha(x,y,z)$ then $sup_z\alpha(x,y,z) \leq sup_z\alpha(x,y,z)$ then $sup_z\alpha(x,y,z) \leq sup_x sup_z\alpha(x,y,z)$
then $inf_y sup_z\alpha(x,y,z) \leq inf_y sup_x sup_z\alpha(x,y,z)$. Thus $sup_x inf_y sup_z\alpha(x,y,z) \leq inf_y sup_x sup_z\alpha(x,y,z)$.
Then $(\exists x)(\forall y)(\exists z)\alpha(x,y,z) \Longrightarrow (\forall y)(\exists x)(\exists z)\alpha(x,y,z)$.
The rest are similar.

4. COUNTER EXAMPLES

The following examples show that the reverse of the implications do not hold in general.
I-I) i) $(\forall x)(\exists y)(\forall z)\alpha(x,y,z) \Longrightarrow (\forall x)(\forall z)(\exists y)\alpha(x,y,z)$

Suppose that $\alpha(x,y,z)=0$ if $y \neq z, \alpha(x,y,z)=1$ if $y=z$.
$(\forall x)(\forall z)(\exists y)\alpha(x,y,z) = (\forall z)(\exists y)\alpha(a,y,z)\wedge(\forall z)(\exists y)\alpha(b,y,z)\wedge(\forall z)(\exists y)\alpha(c,y,z)$
$= [(\exists y)\alpha(a,y,a)\wedge(\exists y)\alpha(a,y,b)\wedge(\exists y)\alpha(a,y,c)]\wedge$
$[(\exists y)\alpha(b,y,a)\wedge(\exists y)\alpha(b,y,b)\wedge(\exists y)\alpha(b,y,c)]\wedge[(\exists y)\alpha(c,y,a)\wedge(\exists y)\alpha(c,y,b)\wedge(\exists y)\alpha(c,y,c)]$
$= [\big(\alpha(a,a,a)\vee\alpha(a,b,a)\vee\alpha(a,c,a)\big)\wedge\big(\alpha(a,a,b)\vee\alpha(a,b,b)\vee\alpha(a,c,b)\big)\wedge$
$\big(\alpha(a,a,c)\vee\alpha(a,b,c)\vee\alpha(a,c,c)\big)]\wedge[\big(\alpha(b,a,a)\vee\alpha(b,b,a)\vee\alpha(b,c,a)\big)\wedge$
$\big(\alpha(b,a,b)\vee\alpha(b,b,b)\vee\alpha(b,c,b)\big)\wedge\big(\alpha(b,a,c)\vee\alpha(b,b,c)\vee\alpha(b,c,c)\big)]\wedge$
$[\big(\alpha(c,a,a)\vee\alpha(c,b,a)\vee\alpha(c,c,a)\big)\wedge\big(\alpha(c,a,b)\vee\alpha(c,b,b)\vee\alpha(c,c,b)\big)\wedge$
$\big(\alpha(c,a,c)\vee\alpha(c,b,c)\vee\alpha(c,c,c)\big)] = 1$
$(\forall x)(\exists y)(\forall z)\alpha(x,y,z) = (\exists y)(\forall z)\alpha(a,y,z)\wedge(\exists y)(\forall z)\alpha(b,y,z)\wedge(\exists y)(\forall z)\alpha(c,y,z)$
$= [(\forall z)\alpha(a,a,z)\vee(\forall z)\alpha(a,b,z)\vee(\forall z)\alpha(a,c,z)]\wedge[(\forall z)\alpha(b,a,z)\vee(\forall z)\alpha(b,b,z)\vee$
$(\forall z)\alpha(b,c,z)]\wedge[(\forall z)\alpha(c,a,z)\vee(\forall z)\alpha(c,b,z)\vee(\forall z)\alpha(c,c,z)] = [(\alpha(a,a,a)\wedge$
$\alpha(a,a,b)\wedge\alpha(a,a,c))\vee(\alpha(a,b,a)\wedge\alpha(a,b,b)\wedge\alpha(a,b,c))\vee(\alpha(a,c,a)\wedge\alpha(a,c,b)\wedge$
$\alpha(a,c,c))]\wedge[(\alpha(b,a,a)\wedge\alpha(b,a,b)\wedge\alpha(b,a,c))\vee(\alpha(b,b,a)\wedge\alpha(b,b,b)\wedge\alpha(b,b,c))\vee$
$(\alpha(b,c,a)\wedge\alpha(b,c,b)\wedge\alpha(b,c,c))]\wedge[(\alpha(c,a,a)\wedge\alpha(c,a,b)\wedge\alpha(c,a,c))\vee(\alpha(c,b,a)\wedge$
$\alpha(c,b,b)\wedge\alpha(c,b,c))\vee(\alpha(c,c,a)\wedge\alpha(c,c,b)\wedge\alpha(c,c,c))] = 0$.

I-II) i) $(\exists x)(\forall y)(\exists z)\alpha(x,y,z) \Longrightarrow (\forall y)(\exists x)(\exists z)\alpha(x,y,z)$

Suppose that $\alpha(x,y,z)=1$ if $x=y=z, \alpha(x,y,z)=0$ otherwise.
$(\forall y)(\exists x)(\exists z)\alpha(x,y,z) = (\exists x)(\exists z)\alpha(x,a,z)\wedge(\exists x)(\exists z)\alpha(x,b,z)\wedge(\exists x)(\exists z)\alpha(x,c,z)$
$= [(\exists z)\alpha(a,a,z)\vee(\exists z)\alpha(b,a,z)\vee(\exists z)\alpha(c,a,z)]\wedge[(\exists z)\alpha(a,b,z)\vee(\exists z)\alpha(b,b,z)$
$\vee(\exists z)\alpha(c,b,z)]\wedge[(\exists z)\alpha(a,c,z)\vee(\exists z)\alpha(b,c,z)\vee(\exists z)\alpha(c,c,z)]$
$= [(\alpha(a,a,a)\vee\alpha(a,a,b)\vee\alpha(a,a,c))\vee(\alpha(b,a,a)\vee\alpha(b,a,b)\vee\alpha(b,a,c))\vee$
$\alpha(c,a,a)\vee\alpha(c,a,b)\vee\alpha(c,a,c))]\wedge[(\alpha(a,b,a)\vee\alpha(a,b,b)\vee\alpha(a,b,c))\vee(\alpha(b,b,a)\vee$
$\alpha(b,b,b)\vee\alpha(b,b,c))\vee(\alpha(c,b,a)\vee\alpha(c,b,b)\vee\alpha(c,b,c))]\wedge[(\alpha(a,c,a)\vee\alpha(a,c,b)\vee$
$\alpha(a,c,c))\vee(\alpha(b,c,a)\vee\alpha(b,c,b)\vee\alpha(b,c,c))\vee(\alpha(c,c,a)\vee\alpha(c,c,b)\vee\alpha(c,c,c))] = 1$
$(\exists x)(\forall y)(\exists z)\alpha(x,y,z) = (\forall y)(\exists z)\alpha(a,y,z)\vee(\forall y)(\exists z)\alpha(b,y,z)\vee(\forall y)(\exists z)\alpha(c,y,z)$
$= [(\exists z)\alpha(a,a,z)\wedge(\exists z)\alpha(a,b,z)\wedge(\exists z)\alpha(a,c,z)]\vee[(\exists z)\alpha(b,a,z)\wedge(\exists z)\alpha(b,b,z)\wedge$
$(\exists z)\alpha(b,c,z)]\vee[(\exists z)\alpha(c,a,z)\wedge(\exists z)\alpha(c,b,z)\wedge(\exists z)\alpha(c,c,z)] = [(\alpha(a,a,a)\vee$
$\alpha(a,a,b)\vee\alpha(a,a,c))\wedge(\alpha(a,b,a)\vee\alpha(a,b,b)\vee\alpha(a,b,c))\wedge(\alpha(a,c,a)\vee\alpha(a,c,b)\vee$
$\alpha(a,c,c))]\vee[(\alpha(b,a,a)\wedge\alpha(b,a,b)\wedge\alpha(b,a,c))\vee(\alpha(b,b,a)\wedge\alpha(b,b,b)\wedge\alpha(b,b,c))\vee$
$(\alpha(b,c,a)\wedge\alpha(b,c,b)\wedge\alpha(b,c,c))]\vee[(\alpha(c,a,a)\wedge\alpha(c,a,b)\wedge\alpha(c,a,c))\vee$
$(\alpha(c,b,a)\wedge\alpha(c,b,b)\wedge\alpha(c,b,c))]\vee[(\alpha(c,c,a)\wedge\alpha(c,c,b)\wedge\alpha(c,c,c))] = 0$.

REFERENCES

1. Kahtan H. Alzubaidy, Algebraic Description of Monadic and Dyadic Logics, International Journal of Mathematics And its Applications, 10(2)(2022),5-8.
2. P.R. Halmos, Algebraic Logic, Chelsea, New York, (1962).
3. P.R. Halmos, Lectures on Boolean Algebras, Princeton, New Jersey, (1963).
4. L. Henkin, J. D. Monk and A. Tarski, Cylindric Algebras, Part I, North-Holland, Amsterdam, (1971).

Source of support: Nil, Conflict of interest: None Declared.