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# ANALYSIS OF THE PROPERTIES OF THE TOEPLITZ OPERATOR

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## *ABSTRACT*

*The purpose of this paper is to study some of the elementary algebraic properties of Toeplitz operator.*

*We will be dealing with the questions like correspondence phi going to Tphi, when toeplitz operators are hermitian, what is toeplitz matrix, relation of toeplitz operator, bilateral shift operator and unilateral shift operator, about analyticity of the toeplitz operator. We will follow the method of proof to learn about the properties.*

*There have been many findings about the toeplitz operators. Some of them regarding the necessary and sufficient condition when two toeplitz operators commute, unitary toeplitz operator, it being an isometry are discussed in this paper. Then much work has been done on the characterisation of a complex symmetric toeplitz operator on the weighted Bergman space.*

***Keywords:*** *Toeplitz operator, Shift operator, complex symmetric operator.*

Firstly, I need to define Toeplitz operator. The Toeplitz operators are those operators whose matrices with respect to the standard basis of $\mathbb{H}^2$ have constant diagonals.

We need to define another function $M_\varphi$, for $\varphi \in \mathbb{L}^\infty$

$M_\varphi : \mathbb{L}^2 \to \mathbb{L}^2$
by $M_\varphi(f) = \varphi f$

Now defining Toeplitz matrix —

A finite matrix, or a doubly infinite matrix (i.e. a matrix with entries in position $(m, n)$ for $m$ and $n$ integers) or a singly infinite matrix (i.e. a matrix with entries or a singly infinite matrix (i.e. a matrix with entries in positions $(m, n)$ for $m$ and $n$ non-negative integers) is called a Toeplitz matrix if its entries are constant along each diagonal, i.e. the matrix $(a_{m,n})$ is Toeplitz if $a_{m_1,n_1} = a_{m_2,n_2}$ whenever $m_1 - n_1 = m_2 - n_2$. If $A$ has toeplitz matrix, then

$$\langle A_{e_n, e_m} \rangle = \langle A_{e_{n+k}}, e_{m+K} \rangle$$

**Theorem:** A bounded linear operator on $\mathbb{L}^2$is multiplication by an $\mathbb{L}^\infty$ function iff its matrix with respect to the standard basis in $\mathbb{L}^2$ is a Toeplitz matrix.

**Proof:** If the linear operator is multiplication by an $\mathbb{L}^\infty$ function, the matrix of it is a Toeplitz matrix as for each pair of integers $(m, n)$,

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$$\langle M_\varphi e_n\,, e_m\rangle = \frac{1}{2\pi}\int_0^{2\pi} \left(M_\varphi e_n\right)\left(e^{i\theta}\right).\overline{e_m\left(e^{i\theta}\right)}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\varphi e_n)(e^{i\theta})\,.\,e^{\overline{im\theta}}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \phi\left(e^{i\theta}\right)e^{in\theta}\cdot e^{-im\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right)e^{-i(m-n)\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(e^{i\theta}\right)e^{\overline{i(m-n)\theta}}\,d\theta$$

$= \varphi_{m-n}$ for each integer $k$, the entry of the matrix for $M_\varphi$ in position $(m,n)$ is $\varphi_K$ whenever $m - n = k$.

Conversely, let $A$ be a bounded linear operator on $\mathbb{L}^2$ such that it has Toeplitz matrix.

**Claim:** $A = M_\varphi$ for some $\varphi \in \mathbb{L}^\infty$.

**Sufficient to Show:** $AW = WA$ where $W$ = bilateral shift [because the commutant of $w$ is $\{M\varphi\colon \varphi \in \mathbb{L}^\infty\}$]→*
For all integers $m$ and $n$,

$$\begin{aligned}\langle A\,W_{en}, e_m\rangle &= \langle A_{n+1}, e_m\rangle\\ &= \langle Ae_{n+1-1}, e_{m-1}\rangle \text{(from *)}\\ &= \langle Ae_n, e_{m-1}\rangle\\ &= \langle Ae_n, w^*e_m\rangle\\ &= \langle WAe_n, e_m\rangle \qquad (1)\end{aligned}$$

For $x \in \mathbb{L}^2$, we have $x = \sum_{-\infty}^{\infty} c_m e_m$ and so consider
$\langle AWe_n, x\rangle = \langle AWe_n, \sum_{-\infty}^{\infty} c_m e_m\rangle$and so consider

$$\begin{aligned}\langle AWe_n, x\rangle &= \langle AWe_n, \sum_{-\infty}^{\infty} c_m e_m\rangle.\\ &= \sum_{-\infty}^{\infty}\overline{c_m}\langle AWe_n, e_m\rangle\\ &= \sum_{-\infty}^{\infty}\overline{c_m}\langle WAe_n, e_m\rangle \quad \text{(from (1))}\\ &= \langle WAe_n, \sum_{-\infty}^{\infty} c_m e_m\rangle\\ &= \langle WAe_n, x\rangle \qquad (2)\end{aligned}$$

$\Rightarrow AWe_n = WAe_n \forall\, n \in Z/n$
$\Rightarrow AWx = WAx \qquad \forall\, x \in \mathbb{L}^2$
$\Rightarrow AW = WA.$

Now we are in a position to give another definition of Toeplitz operator.

For each symbol $\varphi \in \mathbb{L}^\infty$, the Toeplitz operator with symbol $\varphi$ is the operator $T_\varphi$ defined by

$$T_\varphi f = P\varphi f, \quad T_\varphi\colon \widetilde{\mathbb{H}}^2 \longrightarrow \widetilde{\mathbb{H}}^2$$

for each $f \in \widetilde{\mathbb{H}}^2$, where $P$ is the orthogonal projection of $\mathbb{L}^2$ onto $\widetilde{\mathbb{H}}^2$

**Theorem:** The matrix of the Toeplitz operator with symbol $\varphi$ with respect to the basis $\left\{e^{in\theta}\right\}_{n=0}^{\infty}$ of $\widetilde{\mathbb{H}}^2$is

$$T_\varphi = \begin{bmatrix}\varphi_0 & \varphi_{-1} & \varphi_{-2} & \varphi_3 & \cdots\\ \varphi_1 & \varphi_0 & \varphi_{-1} & \vdots & \cdots\\ \varphi_2 & \varphi_1 & \varphi_0 & \vdots & \cdots\\ \varphi_3 & \varphi_2 & \varphi_1 & \vdots & \cdots\\ \vdots & \vdots & \vdots & \vdots & \cdots\end{bmatrix}$$

where $\varphi_\kappa$ is the $k^{th}$ Fourier coefficient of $\varphi$.

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**Proof:** For $m, n \geq 0$

$$\langle T_\varphi e_n, e_m \rangle = \langle P\varphi e_n, e_m \rangle$$
$$\Rightarrow \langle T_\varphi e_n, e_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} P\big(\varphi e^{i\theta}\big)e^{in\theta}\big)\, e^{-im\theta}\; d\theta.$$

Because $\varphi \in \mathbb{L}^\infty$, so

$$\varphi\big(e^{i\theta}\big) = \sum_{-\infty}^{\infty} \varphi_\kappa\, e^{i\kappa\theta}$$

be the fourier series of $\varphi$, where

$$\varphi_\kappa = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\, e^{-i\kappa\theta}\; d\theta$$

Therefore

$$P\big(\varphi\big(e^{i\theta}\big)e^{in\theta}\big) = P\left(\sum_{\kappa=-\infty}^{\infty} \varphi_\kappa e^{i(\kappa+n)\theta}\right)$$
$$= \sum_{\kappa=-n}^{\infty} \varphi_\kappa e^{i(\kappa+n)\theta}$$

Threfore, from (1)

$$\langle T_\varphi e_n, e_m \rangle = \frac{1}{2\pi}\int_0^{2\pi} \sum_{\kappa=-n}^{\infty} \varphi_\kappa e^{i(\kappa+n-m)\theta}\; d\theta$$
$$= \varphi_{m-n}$$

Here we are giving another characterization of Toeplitz operator:

The Toeplitz operator $T_\varphi$ is an analytic Toeplitz operator if $\varphi$ in $\widetilde{\mathbb{H}}^\infty$

**Theorem:** If $T_\varphi$ is an analytic Toeplitz operator, then the matrix of $T_\varphi$ with respect to the basis $\left\{e^{in\theta}\right\}_{n=0}$ is

$$T_\varphi \begin{bmatrix} \varphi_0 & 0 & 0 & 0 & \cdots \\ \varphi_1 & \varphi_0 & 0 & 0 & \cdots \\ \varphi_2 & \varphi_1 & \varphi_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \varphi_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

where $\varphi\big(e^{i\theta}\big) = \sum_{\kappa=0}^{\infty} \emptyset e^{i\kappa\theta}$.

**Proof:** Because every analytic Toeplitz operator is a Toeplitz, so we can write

$$T_\varphi = [\varphi_{m-n}]_{m,n>0}$$

But, $\varphi \in \widetilde{\mathbb{H}}^\infty$ implies $\varphi \in \widetilde{\mathbb{H}}^2$ and $\varphi$ has fourier series $\varphi\big(e^{i\theta}\big) = \sum_{\kappa=0}^{\infty} \varphi_\kappa e^{i\kappa\theta}$.

Threfore, all fourier coefficients of $\varphi$ with negative indices are $O_1$, i.e.,

$$\varphi_1 = \varphi_2 = \cdots = 0.$$
$$\Rightarrow T_\varphi = \begin{bmatrix} \varphi_0 & 0 & 0 & \cdots \\ \varphi_1 & \varphi_0 & 0 & \cdots \\ \varphi_2 & \varphi_1 & \vdots & \cdots \\ \vdots & \vdots & \varphi_0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

**Theorem:** The communtant of the unilateral shift acting on $\widetilde{\mathbb{H}}^2$ is

$$\left\{T_\varphi\ \varphi \in \widetilde{\mathbb{H}}^\infty\right\}.$$

**Proof:** We first prove that any analytic Toeplitz operator $T_\varphi, \varphi \in \widetilde{\mathbb{H}}^2$ commutes with unilateral shift $\cup = M_{e^{i\theta}}$ for $f \in \widetilde{\mathbb{H}}^2$

$$\big(T_\varphi M_{e^{i\theta}}\big)f = T_\varphi\,(M_{e^{i\theta}} f)$$
$$= T_\varphi\,(e^{i\theta} f)$$
$$= \varphi\, e^{i\theta} f \qquad \text{[because } T_\varphi \text{ is Analytic, because } T_\varphi f = \varphi f\text{ ]}$$
$$= e^{i\theta}\varphi f$$
$$= \big(M_{e^{i\theta}} T_\phi\big) f$$

Therefore,

$$T_\varphi M_{e^{i\theta}} = M_{e^{i\theta}} T_\varphi$$

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Next, it remains to show that any operator $A$ that commutes with $U$ is $T_\varphi$ for some $\varphi \in \widetilde{\mathbb{H}}^\infty$.

Let $AU = UA$ and $\varphi = Ae_0$.
Then,

$$\varphi \in \widetilde{\mathbb{H}}^2 [\text{becasuse } A: \widetilde{\mathbb{H}}^2 \longrightarrow \widetilde{\mathbb{H}}^2]$$

For each $n \in \mathbb{N}$,

$$\begin{aligned} Ae_n &= AU^n_{e_0} \\ &= U^n A_{\ell_0} \\ &= U^n \varphi \\ &= M_{e^{i\theta^n}} \varphi \\ &= e^{in\theta} \varphi \\ &= e_n \varphi = \varphi e_n \end{aligned}$$

Therefore, for every polynomial $P \in \widetilde{\mathbb{H}}^2, Ap = \varphi p$[by linearity].

For an arbitrary $f \in \widetilde{\mathbb{H}}^2$, choose a sequence of trigonometric polynomials$p_n \in \widetilde{\mathbb{H}}^2$ such that $P_n \to f$ in $\widetilde{\mathbb{H}}^2$.

Because $A$ is bounded, so, $\{Ap_n\} \to Af$

$$\Rightarrow \{P_n \varphi\} \to Af \text{ in } \widetilde{\mathbb{H}}^2$$

Therfore, there exists a subsequence $\{P_{n_j}\}$ conversging a.e. to $f$
[As every sequence converging in $\mathbb{L}^2$ has a subsequence converging a.e.]

Therefore, $\left\{\varphi P_{n_j}\right\} \to \varphi f$ a.e.

$$\left( \begin{array}{ccc} because \ \lim lim & \varphi p_{n_j} = & \lim \varphi p_n \\ & || & || \\ & Af & \phi f \end{array} \right)$$

$$\left. \begin{aligned} becasue\ Af &= \varphi f \ a.e \\ &= P(\varphi f) \\ &= T_\varphi f \end{aligned} \right\} \quad (2)$$

Therefore, A$= T_\varphi$.

Claim: $\varphi \in \widetilde{\mathbb{H}}^\infty$

Sufficient to show that : $\phi \in \mathbb{L}^\infty$

If $A = 0$, then $\varphi = 0$, a.e., so, $\phi \in \mathbb{L}^\infty$

So, assume that $\|A\| \neq 0$.

Define the measurable function $\psi$ by

$$\psi = \frac{\phi}{\|A\|}$$

Note that $\psi \in \widetilde{\mathbb{H}}^2 (\because \varphi \in \widetilde{\mathbb{H}}^2)$
Then,

$$\psi f = \frac{\varphi f}{\|A\|} = \frac{Af}{\|A\|} \to \text{(from 2)} \ \forall \ f \in \widetilde{\mathbb{H}}^2$$

Therefore, $\|\psi\| = \left\|\frac{Af}{A}\right\|$

$$\begin{aligned} 1 &= \frac{1}{\|A\|} \|Af\| \\ &\leq \frac{1}{\|A\|} \|A\| \|f\| \end{aligned}$$

Therefore,

$$\|\psi f\| \leq \|f\|, \qquad \forall \ f \in \widetilde{\mathbb{H}}^2 \quad (3)$$

Take $f = 1$, then $\|\psi\| \leq 1$ (from 3)
Then,

$$\|\psi^n\| \leq \|\psi\| \cdot \|\psi\| \dots \|\psi\|$$
$$\leq 1 \ \forall \ n \in \mathbb{N}$$

Take

$$E = \{e^{i\theta} : |\psi(e^{i\theta})| \geq 1 + \epsilon\} \text{ then } \exists \ \epsilon > 0 \text{ such that } m(E) > 0.$$

and

$$1 \geq \|\psi^n\| = \frac{1}{2\pi}\int_0^{2\pi} |\psi^n(e^{i\theta})| \, d\theta$$
$$\geq \frac{1}{2\pi}\int |\psi(e^{i\theta})|^n \, d\theta$$
$$\geq \frac{1}{2\pi}\int (1+\epsilon)^n \, d\theta$$
$$= (1+\epsilon)^n \frac{1}{2\pi}\int_E d\theta$$
$$= (1+\epsilon)^n . m(E).$$
$$\Rightarrow m(E) \leq \frac{1}{(1+\epsilon)^n} \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Therefore, $m(E) = 0$

$$\Rightarrow |\psi(e^{i\theta})| < 1 \quad a.e$$
$$\Rightarrow \left|\frac{\varphi(e^{i\theta})}{\|A\|}\right| < 1$$
$$\Rightarrow |\varphi(e^{i\theta})| < \|A\| \ a.e$$
$$\Rightarrow \varphi \in \mathbb{L}^\infty$$

**Theorem:** The operator $T$ is a Toeplitz operator iff $U^*TU = T$, where $U$ is unilateral shift.

**Proof:** for non-negative integers $m, n$, we have

$$\langle U^*TUe_n, e_m\rangle = \langle TUe_n, Ue_m\rangle$$
$$= \langle Te_{n+1}, e_{m+1}\rangle \longrightarrow 1 \qquad [\because U = M_{e^{i\theta}}]$$

Let $T$ be a Toeplitz operator, then

$$\langle Te_{n+1}, e_{m+1}\rangle = \langle Te_n, e_m\rangle.$$

Therefore, $\langle U^*TUe_n, e_m\rangle = \langle Te_n, e_m\rangle$ and so

$$U^*TU = T \tag{2}$$

Conversely, let $U^*TU = T$, then

$$\langle Te_n, e_m\rangle = \langle U^*TUe_n, e_m\rangle$$
$$= \langle Te_{n+1}, e_{m+1}\rangle$$

From induction,

$$\langle Te_n, e_m\rangle = \langle Te_{n+\kappa}, e_{m+\kappa}\rangle \ \ \forall \kappa \in \mathbb{N}.$$

Therefore, the matrix of $T$ has constant diagonals i.e $T$has a Toeplitz matrix.

Therefore, $T$ is a Toeplitz operator.

Now coming to one more definition

The operator $T_\varphi$ is said to be coanalytic if $T_\varphi^*$ is analytic.

**Theorem:** For $\psi$ and $\phi$ in $\mathbb{L}^\infty$, $T_\psi T_\varphi$ is a Toeplitz operator iff either $T_\psi$ is coanalytic or $T_\varphi$ is analytic. In both cases, $T_\psi T_\varphi = T_{\psi\varphi}$.

**Proof:** Given that $\psi$ and $\phi \in \mathbb{L}^\infty$.

If $T_\varphi$ is analytic, then $\phi \in \widetilde{\mathbb{H}}^\infty$, therefore

$$T_\psi T_\varphi = T_{\psi\varphi} \text{ and } T_\psi T_\varphi \text{ is Toeplitz.}$$

If $T_\psi$ is co-analytic, then $T_{\tilde{\psi}}$ is analytic $\Rightarrow \tilde{\psi} \in \widetilde{\mathbb{H}}^\infty$

$$](T_\psi T_\varphi)^* = T_\varphi^* T_\psi^* = T_{\bar{\varphi}} T_{\bar{\psi}} \Rightarrow (T_\psi T_\varphi)^* = T_{\bar{\varphi}\bar{\psi}}$$
$$= T_{\bar{\varphi}\bar{\psi}}$$

Taking adjoint both sides,

$$T_\psi T_\varphi = (T_{\bar{\varphi}\bar{\psi}})^* = T_{\psi\varphi}.$$

Thus, $T_\psi T_\varphi$ is a Toeplitz operators.

Conversely, let $T_\psi T_\varphi$ is a Toeplitz operator. We know

$$U^* T_\psi T_\varphi U - T_\psi T_\varphi = P(e^{-i\theta}\psi) \otimes P(e^{-i\theta}\bar{\varphi})$$
$$(\because T_\psi T_\varphi \text{ is Toeplitz}),$$
$\Rightarrow$ either $P(e^{-i\theta}\psi) = 0$ or $P(e^{-i\theta}\bar{\varphi}) = 0$.

of $P(e^{-i\theta}\psi) = 0$, then consider

$$e^{-i\theta}\psi(e^{i\theta}) = e^{-i\theta}\sum_{-\infty}^{\infty} a_n e^{in\theta}$$
$$= \sum_{-\infty}^{\infty} a_n e^{i(n-1)\theta.}$$

Therefore, $P(e^{-i\theta}\psi) = 0$ implies $P(\sum_{-\infty}^{\infty} a_n e^{i(n-1)\theta}) = 0$

$\Rightarrow \sum_{n=1}^{\infty} a_n e^{i\theta}\psi) = 0$ implies $P(\sum_{-\infty}^{\infty} a_n e^{i(n-1)\theta}) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} a_n e^{i(n-1)\theta} = 0$$
$$\Rightarrow a_n = 0 \qquad \forall\ n \geq 1$$

Therefore, $\tilde{\psi} \in \widetilde{\mathbb{H}}^2$

As $\psi \in \mathbb{L}^\infty \Rightarrow \psi \in \mathbb{L}^\infty$, thus $\psi \in \widetilde{\mathbb{H}}^2 \cap \widetilde{\mathbb{L}}^\infty$

$\Rightarrow \tilde{\psi} \in \widetilde{\mathbb{H}}^\infty$

$\Rightarrow T_{\tilde{\psi}}$ is analytic.

$\Rightarrow T_\psi$ is co-anlytic.

If $P(e^{-i\theta}\bar{\varphi}) = 0$, then

$P(e^{-i\theta}\bar{\varphi}) = 0$, then

$$P\left(e^{-\theta}\overline{\sum_{-\infty}^{\infty} a_n e^{in\theta}}\right) = 0$$
$$\Rightarrow P\left(\sum_{-\infty}^{\infty} a_n e^{-i(n+1)\theta}\right) = 0$$
$$\Rightarrow \sum_{n=-\infty}^{-1} \bar{a}_n e^{-i(n+1)\theta} = 0$$
$$\Rightarrow \bar{a}_n = 0 \qquad \forall\ \ n \leq -1.$$
$$\Rightarrow a_n = 0 \qquad \forall\ \ n \leq -1$$

Therefore, $\varphi(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$

$\Rightarrow \phi \in \widetilde{\mathbb{H}}^2$

Therefore $T_\varphi$ is analytic.

**Theorem:** A Toeplitz operator is self adjoint iff its symbol is real valued a.e.

**Proof:** Let, $T_\varphi$ be a Toeplitz operator. Then,$T_\varphi$ is self adjoint $(=) T_\varphi = T_{\varphi^*}$

$(=)\ T_\varphi = T_{\bar{\varphi}}$

$(=)\ \varphi = \bar{\varphi}$

$(=)\ \varphi$ real valued $a.e.$

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