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GRAHAM'S PEBBLING CONJECTURE ON PRODUCT OF THORN GRAPHS OF PATHS

C. MUTHULAKSHMI@SASIKALA*[1] AND A. ARUL STEFFI[2]

[1]Department of Mathematics,
Sri Paramakalyani College, Alwarkurichi– 627412, India.

[2]Department of Mathematics,
St. Xavier's College (Autonomous), Palayamkottai- 627002, India.

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ABSTRACT

Given a distribution of pebbles on the vertices of a connected graph G, the pebbling number of a graph G, is the least number f(G) such that no matter how these f(G) pebbles are placed on the vertices of G, we can move a pebble to any vertex by a sequence of pebbling moves, each move taking two pebbles off one vertex and placing one on an adjacent vertex. Let $p_1, p_2, \dots, p_n$ be positive integers and G be such a graph, V(G) = n. The thorn graph of the graph G, with parameters $p_1, p_2, \dots, p_n$ is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, where i = 1, 2,…, n. In this paper we discuss about the pebbling number of the thorn graph of path of length n also called as thorn path and we show that Graham's conjecture holds for thorn path and it satisfies the two pebbling property. As a corollary, Graham's conjecture holds when G and H are thorn paths with every $p_i \geq 2$, i = 1, 2,…, n.

Keywords: *Graphs, Pebbling Number, Thorn path, two pebbling property, Graham's pebbling conjecture.*

1. INTRODUCTION

Pebbling in graphs was first studied by Chung [1]. A pebbling move consists of taking two pebbles off one vertex and placing one pebble on an adjacent vertex. The pebbling number of a vertex *v* in a graph G is the smallest number f(G,*v*) such that from every placement of f(G, *v*) pebbles, it is possible to move a pebble to *v* by a sequence of pebbling moves. Then the pebbling number of a graph G, denoted by f(G), is the maximum f(G.*v*) over all the vertices *v* in G. Given a configuration of pebbles placed on G, let p(G) be the number of pebbles placed on the graph G, q be the number of vertices with atleast one pebble and let *r* be the number of vertices with an odd number of pebbles. We say that G satisfies the two pebbling property(respectively, weak or odd two-pebbling property), if it is possible to move two pebbles to any specified target vertex when the total starting number of pebbles is 2f(G) – q + 1 (respectively 2f(G) – r + 1). Note that any graph which satisfies the two pebbling property also satistifes the weak or odd two pebbling property.

Result 1.1: All cycles have the 2-pebbling property [7] and a tree satisfies the 2-pebbling property [1].

Theorem 1.1: [6] Let G be a graph with diameter, diam(G) = 2. Then G has the 2-pebbling property.

Theorem 1.2: [8] The pebbling number of star graph $K_{1,n}$ is f($K_{1,n}$) = n + 2 if n > 1.

Definition 1.1: [4] Let $p_1, p_2, \dots, p_n$ be positive integers and G be a graph with V(G) = n. The thorn graph of the graph G, with parameters $p_1, p_2, \dots, p_n$ is obtained by attaching p_i new vertices of degree 1 to the vertex v_i of the graph G, i = 1, 2,…, n.

The thorn graph of the graph G will be denoted by G^* or by $G^*(p_1, p_2, \dots, p_n)$, if the respective parameters need to be specified. In this paper, we will consider the thorn graph with every $p_i \geq 2$ (i = 1, 2,…, n).

Corresponding Author: C. Muthulakshmi@Sasikala*[1],
[1]Department of Mathematics, Sri Paramakalyani College, Alwarkurichi– 627412, India.

Definition 1.2: [3] Given a configuration of pebbles placed on G, a transmitting subgraph of G is a path $v_1, v_2, \ldots, v_n$ such that there are atleast two pebbles on v_1 and atleast one pebble on each of the other vertices in the path, possibly except v_n. Thus, we can transmit a pebble from v_1 to v_n.

Throughout this paper, G will denote a simple connected graph with vertex set V(G) and edge set E(G). The graph P_n denotes the path graph of length n. Also, for any vertex v of a graph G, p(v) refers to the number of pebbles on v.

2. PEBBLING NUMBER OF THORN PATH P_n^*:

Definition 2.1: Let P_n be a path of length n where V(P_n) = $\{v_0, v_1, \ldots, v_n\}$ and E(P_n) =$\{e_1, e_2, \ldots, e_m\}$. Let $X_i = \{x_{i1}, x_{i2}, \ldots, x_{ip_i}\}$ where $p_i \geq 2$ and i = 0, 1,…, n. Consider the graph P_n^* obtained from P_n such that V(P_n^*) = $\{v_i \cup X_i$ / i = 0, 1,…, n$\}$ and E(P_n^*) = E(P_n) ∪ $\{v_i x_{ij}$ / i = 0, 1,…, n and j = 1, 2, …, $p_i\}$. Then P_n^* is called the thorn path of length n.

Let G_i be the graph obtained from P_n^* by the removal of the edges $\{e_1, e_2, \ldots, e_m\}$ such that V(G_i) = $v_i \cup X_i$ and E(G_i) = $\{v_i x_{ij}$ / j = 1, 2, …, $p_i\}$ for i = 0, 1, …, n.

Note 2.1: In [1] Chung determined the pebbling number of a tree as f(T, v) = $2^{a_1} + 2^{a_2} + \ldots + 2^{a_r}$ – r + 1 where a_1, a_2, …, a_r is the sequence of the path sizes in a maximum path – partition P of T_v. Though thorn path is a tree, we give an alternate approach in finding the pebbling number of the thorn path.

Note 2.2: Every star graph $K_{1,n}$ is a thorn path of length zero. i.e, $K_{1,n}$ is P_0^*.

Lemma 2.1: The pebbling number of the thorn path of length zero P_0^* is f(P_0^*) = p_0+ 2 where $p_0 \geq 2$.

Proof: We know that every thorn path of length zero is a star graph, K_{1,p_0} with v_0 as hub vertex and p_0 as the number of pendant vertices adjacent to v_0. From theorem 1.2, the pebbling number of the star graph K_{1,p_0} is p_0+ 2. Hence f(P_0^*) = p_0+ 2.

Theorem 2.1: Let P_n^* be the thorn graph of the path P_n of length n. Then f(P_n^*) = $2^{n+2} + \sum_{i=0}^{n} p_i - 2$, where $p_i \geq 2$.

Proof: Let the vertices of P_n be $v_0, v_1, \ldots, v_n$. Let x_{ij} (j = 1, 2, …, p_i) be the pendant vertices that are attached to the vertex v_i (i = 0, 1,…, n). The graph that is composed of these vertices is P_n^*. Let p(G) be the number of pebbles placed on G. Let x_{n1} be our target vertex and p(x_{n1}) = 0.

Consider the following distribution of $2^{n+2} + \sum_{i=0}^{n} p_i - 3$ pebbles on P_n^*.

i) p(v_i) = 0 for i = 0, 1,…, n
ii) p(x_{ij}) = 1 for i = 1, 2, …, n-1 and j = 1, 2, …, p_i
iii) p(x_{0j}) = 1 for j = 2, 3, …, p_0 and p(x_{nk}) = 1 for k = 2, 3, …, p_n.
iv) p(x_{01}) = 2^{n+2} – 1.

In this distribution we cannot move one pebble to x_{n1} as the length of the path (x_{01}, x_{n1}) is n + 2.

Hence f(P_n^*) ≥ $2^{n+2} + \sum_{i=0}^{n} p_i - 2$.

Now we show that f(P_n^*) ≤ $2^{n+2} + \sum_{i=0}^{n} p_i - 2$. Let us consider any distribution of $2^{n+2} + \sum_{i=0}^{n} p_i - 2$ pebbles on P_n^*. There are only two types of possible target vertices.

Case-1: Suppose that the target vertex is v_i where i = 0, 1, …, n. Without loss of generality, let us assume that our target vertex is v_k, 0 ≤ k ≤ n and p(v_k) = 0. If p(x_{kj}) ≥ 2 for some j = 1, 2, …, p_k then we can move one pebble from x_{kj} to v_k. If p(x_{kj}) < 2 for all j = 1, 2, …, p_k then three cases arise.

Subcase-1.1: If p(P_n) = 0 then all $2^{n+2} + \sum_{i=0}^{n} p_i - 2 - p_k$ pebbles are placed on the thorns of $v_0, v_1, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n$. Let $X = X_1 \cup X_2 \cup \ldots \cup X_n$. Then all $2^{n+2} + \sum_{i=0}^{n} p_i - 2 - p_k$ pebbles are placed on $X - X_k$. Clearly 2^n pebbles can be moved to P_n and hence one pebble can be moved to v_k.

Subcase-1.2: If p(P_n) ≥ 2^n, then one pebble can be moved to v_k as f(P_n) = 2^n [8].

Subcase-1.3: If $0 < p(P_n) < 2^n$, Let $p(P_n) = s$. Now the number of pebbles placed on $X - X_k$ is $p(X - X_k) = 2^{n+2} + \sum_{i=0}^{n} p_i - 2 - p_k - s$. Let r_k be the number of vertices in $X - X_k$ with odd pebbles, then $r_k \leq \sum_{i=0}^{n} p_i - p_k$. Now the total number of pebbles that can be brought to P_n from $X - X_k$ is atleast $\frac{2^{n+2} + \sum_{i=0}^{n} p_i - 2 - p_k - s - r_k}{2} \geq \frac{2^{n+2} - 2 - s}{2} = 2^{n+1} - 1 - \frac{s}{2}$.

Since P_n already has s pebbles, now the total number of pebbles in P_n is atleast $2^{n+1} - 1 - \frac{s}{2} + s = 2^{n+1} + \frac{s}{2} - 1 > 2^n$. Hence one pebble can be moved to v_k.

Case-2: Suppose that the target vertex is x_{ij} where i = 0, 1, 2, …, n and j = 1, 2, …, p_i. Without loss of generality let us assume that x_{k1} be our target vertex, where $0 \leq k \leq n$ and $p(x_{k1}) = 0$. If $p(v_k) \geq 2$ then one pebble can be moved to x_{k1}. If $p(v_k) = 1$ then if there exists atleast one vertex x_{kj} ($j \neq 1$) such that $p(x_{kj}) \geq 2$ then $\{x_{kj}, v_k, x_{k1}\}$ forms a transmitting subgraph. Hence one pebble can be moved to x_{k1}. If $p(x_{kj}) < 2$ for all j = 2, 3, …, p_k, then the number of pebbles placed on $P_n^* - X_k$ is atleast $2^{n+2} + \sum_{i=0}^{n} p_i - 2 - (p_k - 1) = 2^{n+2} + \sum_{i=0}^{n} p_i - p_k - 1$, then by proceeding as in subcase 1.3 of Case 1, one pebble can be moved to v_k and from v_k one pebble can be moved to x_{k1}. If $p(v_k) = 0$ then the following cases arise.

Subcase-2.1: If there exists atleast two vertices x_{kj_1}, x_{kj_2} with $p(x_{kj_1}) \geq 2$ and $p(x_{kj_2}) \geq 2$ where $j_1, j_2 \neq 1$, among the vertices $x_{k1}, x_{k2}, \dots, x_{kp_k}$ then we can move one pebble from x_{kj_1} to v_k. So $\{x_{kj_2}, v_k, x_{k1}\}$ forms a transmitting subgraph. Hence one pebble can be moved to x_{k1}.

Subcase-2.2: If $p(x_{kj_1}) \geq 4$ for only one $j_1 \neq 1$ and $p(x_{kr}) < 2$ for all $r \neq 1, j_1$ then two pebble can be moved from x_{kj_1} to v_k and hence one pebble can be moved to x_{k1}.

Subcase-2.3: If $2 \leq p(x_{kj_1}) < 4$ for only one $j_1 \neq 1$ and $p(x_{kr}) < 2$ for all $r \neq 1, j_1$, then we can move one pebble from x_{kj_1} to v_k. Now by proceeding as in subcase 1.3 of Case 1, another pebble can be moved to v_k. So v_k get two pebbles and hence one pebble can be moved from v_k to x_{k1}.

Subcase-2.4: If $p(x_{kr}) < 2$ for all $r \neq 1$, then by proceeding as in Case 1, the number of pebbles that can be moved to P_n is atleast $\frac{2^{n+2} - s - 1}{2}$. Therefore the number of pebbles in P_n will be atleast $\frac{2^{n+2} - s - 1}{2} + s = 2^{n+1} + \frac{s-1}{2} > 2^{n+1}$. Hence two pebbles can be moved to v_k and thus one pebble can be moved from v_k to x_{k1}. Thus $2^{n+2} + \sum_{i=0}^{n} p_i - 2$ pebbles are enough to place a pebble on any vertex of P_n^*. Hence $f(P_n^*) = 2^{n+2} + \sum_{i=0}^{n} p_i - 2$.

Corollary 2.1: The pebbling number of the thorn rod of length n, P_n^* (whose end vertices only has thorns) is $2^{n+2} + p_0 + p_n - 2$.

Proof: The corollary follows from Theorem 2.1.

3. TWO PEBBLING PROPERTY

Definition 3.1: [7] We say a graph G satisfies the 2- pebbling property if two pebbles can be moved to any specified vertex when the total starting number of pebbles is 2f(G) – q +1, where q is the number of vertices with atleast one pebble.

Theorem 3.1: Let P_n^* be the thorn graph of the path P_n of length n. Then P_n^* satisfies the two pebbling property.

Proof: Let p be the number of pebbles on the thorn path P_n^* and q be the number of vertices with atleast one pebble and P+ q= $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1$. We consider the following two types of possible target vertices.

Case-1: Suppose the target vertex is v_k, $0 \leq k \leq n$. If $p(v_k) = 1$, then the number of pebbles on all the vertices except v_k is $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 1 > 2^{n+2} + \sum_{i=0}^{n} p_i - 2$, since $q \leq n + 1 + \sum_{i=0}^{n} p_i$.

Since $f(P_n^*) = 2^{n+2} + \sum_{i=0}^{n} p_i - 2$, we can put one more pebble on v_k using the $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 1$ pebbles.

If $p(v_k) = 0$, then we consider the following cases.

Subcase-1.1: Suppose that $p(x_{kj}) \geq 2$ for some x_{kj} (j = 1, 2, …, p_k). Then we can move one pebble from x_{kj} to v_k. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 2$ pebbles, we can move another pebble to v_k.

Subcase-1.2: Suppose that p(x_{kj}) < 2 for all x_{kj} (j = 1, 2, …, p_k). Since q ≤ n + $\sum_{i=0}^{n} p_i$ as p(v_k) = 0, we have p ≥ $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - (n + \sum_{i=0}^{n} p_i)$ = $2^{n+3} + \sum_{i=0}^{n} p_i$ − (n+3). Since p(x_{kj}) < 2 for all j = 1, 2, …, p_k, we have p($P_n^* - X_k$) ≥ $2^{n+3} + \sum_{i=0}^{n} p_i$ − (n+3) − p_k. If p(P_n) = 0, then all the $2^{n+3} + \sum_{i=0}^{n} p_i$ − (n+3) − p_k pebbles are placed on $X - X_k$, then 2^{n+1} pebbles can be moved to P_n and hence two pebbles can be moved to v_k. If p(P_n) ≥ 2^{n+1}, then two pebbles can be moved to v_k. If 0< p(P_n)<2^n then let us assume that p(P_n) = s. Now the number of pebbles placed on $X - X_k$ is p($X - X_k$) ≥ $2^{n+3} + \sum_{i=0}^{n} p_i$ − (n+3) − p_k − s. Let r_k be the number of vertices in $X - X_k$ with odd pebbles, then $r_k \leq \sum_{i=0}^{n} p_i - p_k$. Now the total number of pebbles that can be brought to P_n from $X - X_k$ is atleast $\frac{2^{n+3} + \sum_{i=0}^{n} p_i - (n+3) - p_k - s - r_k}{2} \geq \frac{2^{n+3} - (n+3) - s}{2}$. Then the total number of pebbles on P_n will be atleast $\frac{2^{n+3} - (n+3) - s}{2} + s > 2^{n+1}$. Hence with these 2^{n+1} pebbles we can place two pebbles on v_k.

Case-2: Suppose that the target vertex is x_{kj} where j = 1, 2, …, p_k . Without loss of generality, let us assume that the target vertex is x_{k1}. If p(x_{k1}) = 1, then the number of pebbles on all the vertices except x_{k1} is $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 1 > 2^{n+2} + \sum_{i=0}^{n} p_i - 2$, as q ≤ n+1+ $\sum_{i=0}^{n} p_i$. Since f(P_n^*) = $2^{n+2} + \sum_{i=0}^{n} p_i - 2$, we can put one more pebble on x_{k1}. If p(x_{k1}) = 0, then we consider the following cases.

Subcase-2.1: If p(v_k) ≥ 2, then we can move one pebble from v_k to x_{k1}. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 2$ pebbles, we can move another pebble to x_{k1}.

Subcase-2.2: Consider p(v_k) = 1. If there is atleast one vertex $x_{kj_1}(j_1 \neq 1)$ with p(x_{kj_1}) ≥ 2 then $\{x_{kj_1}, v_k, x_{k1}\}$ forms a transmitting subgraph. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 3$ pebbles, we can move another pebble to x_{k1}. If p(x_{kr}) < 2 for all r ≠ 1 and if p(P_n) = 0 or p(P_n) ≥ 3(2^n), then three pebbles can be moved to v_k.Let us assume that p(P_n) = s. If p(x_{kr}) < 2 for all r ≠ 1 and if 0 < p(P_n)<3(2^n)then the number of pebbles placed on $X - X_k$ is p($X - X_k$) ≥ $2^{n+3} + \sum_{i=0}^{n} p_i$ − (n+3) − p_k − s. Let r_k be the number of vertices in $X - X_k$ with odd pebbles. Hence the number of pebbles that can be placed on P_n is atleast $\frac{2^{n+3} + \sum_{i=0}^{n} p_i - (n+3) - p_k - s - r_k}{2} \geq 2^{n+2} - \frac{s+n+3}{2}$. Now P_n has atleast $2^{n+2} - \frac{s+n+3}{2} + s > 2^{n+1} + 2^n$pebbles. Hence we can move three pebbles to v_k and two pebbles can be moved to x_{k1}.

Subcase-2.3: If p(v_k) = 0 and if there exists atleast two vertices $x_{kj_1}, x_{kj_2} (j_1, j_2 \neq 1)$ with p(x_{kj_1}) ≥ 2, p(x_{kj_2}) ≥2, then we can move one pebble each from x_{kj_1} *and* x_{kj_2} to v_k. Thus v_k get two pebbles and one pebble can be moved to x_{k1}. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 4$ pebbles, we can move another pebble to x_{k1} as q ≤ $n - 1 + \sum_{i=0}^{n} p_i$. If there is only one vertex $x_{kj_1}(j_1 \neq 1)$ with p(x_{kj_1}) ≥ 4 and p(x_{kr}) < 2 for all r ≠ 1, j_1 then we can move two pebbles from x_{kj_1} to v_k. So $\{v_k, x_{k1}\}$ forms a transmitting subgraph. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 4 - (p_k - 1)$ pebbles, we can move another pebble to x_{k1}. If there is only one vertex $x_{kj_1}(j_1 \neq 1)$ with 2 ≤ p(x_{kj_1}) ≤ 3 and p(x_{kr}) < 2 for all r ≠ 1, j_1, we can move one pebble from x_{kj_1} to v_k. Using the remaining $2(2^{n+2} + \sum_{i=0}^{n} p_i - 2) + 1 - q - 3 - (p_k - 1)$ pebbles, by subcase 2.2 of Case 2, we can move three pebbles to v_k.

Hence two pebbles can be moved to x_{k1}. If p(x_{kr}) < 2 for all r (r ≠ 1) and if p(P_n) = 0 or p(P_n) ≥ 2^{n+2}, then four pebbles can be moved to v_k and hence one pebble can be moved to x_{k1}. If p(x_{kr}) < 2 for all r (r ≠ 1) and if 0 < p(P_n)<2^{n+2} then let us assume that p(P_n) = s. Now the number of pebbles placed on $X - X_k$ is p($X - X_k$) ≥ 2^{n+3} + $\sum_{i=0}^{n} p_i$ − (n+2) − (p_k − 1) − s as q ≤ $n - 1 + \sum_{i=0}^{n} p_i$. Let r_k be the number of vertices in $X - X_k$ with odd pebbles. Then the total pebbles that can be moved to P_n is atleast $\frac{2^{n+3} + \sum_{i=0}^{n} p_i - (n+2) - (p_k - 1) - s - r_k}{2}$ where $r_k \leq \sum_{i=0}^{n} p_i - p_k$. Now P_n has atleast $\frac{2^{n+3} - (n+1) - s}{2}$ + s pebbles. Hence four pebbles can be moved to v_k and two pebbles can be moved to x_{k1}.

4. PEBBLING ON $P_n^* \times P_m^*$

Definition 4.1: [9] Let G and H be two graphs, the Cartesian product of G and H , denoted by G×H, is the graph whose vertex set is the Cartesian product V(G×H) = V(G) × V(H) = $\{(x, y) : x \in V(G), y \in V(H)\}$ and two vertices $(x, y)$and (x', y') are adjacent iff $x = x'$ and $\{y, y'\} \in E(H)$ or $\{x, x'\} \in E(G)$ and $y = y'$.

Conjecture (Graham): The pebbling number of G×H satisfies f(G×H) ≤ f(G) f(H).

Lemma 4.1: [2] Let $\{x_i, x_j\}$ be an edge in G. Suppose that in G×H, we have p_i pebbles on $\{x_i\} \times$ H and r_i of these vertices have an odd number of pebbles. If $r_i \leq k \leq p_i$, and if k and p_i have the same parity, then k pebbles can be retained on $\{x_i\} \times$ H, while transferring $\frac{p_i - k}{2}$ pebbles on to $\{x_j\}$ ×H. If k and p_i have opposite parity, we must leave $k + 1$ pebbles on $\{x_i\} \times$ H, so we can only transfer $\frac{p_i-(k+1)}{2}$ pebbles onto $\{x_j\}$ ×H.

In particular, we can always transfer $\frac{p_i - r_i}{2}$ pebbles onto $\{x_j\} \times$ H, since p_i and r_i have the same parity. In all these cases, the number of vertices of $\{x_i\}$ ×H with an odd number of pebbles is unchanged by these transfers.

Lemma 4.2: [5] If T is a tree, and G satisfies the odd two pebbling property, then f((T,G), (x, y)) ≤ f(T, x) f(G) for every vertex v in G.

Theorem 4.1: If G satisfies the two pebbling property, then f($P_n^* \times$ G) ≤ f(P_n^*) f(G).

Proof: Label the vertices of P_n by $\{v_0, v_1, \ldots, v_n\}$ and let the new vertices that attaches to the vertex v_i of the graph be x_{ij} where i = 0, 1,…, n and j = 1, 2, …, p_i. The graph which is composed of these vertices is P_n^*. Let G_{ij} denote the subgraph$\{x_{ij}\}$ ×G $\subsetneq P_n^* \times$ G and H_i denote the subgraph $\{v_i\}$ ×G $\subsetneq P_n^* \times$ G.

Let a_{ij} denote the number of pebbles on the vertices of G_{ij} and r_i denote the number of pebbles on the vertices of H_i

Let b_{ij} denote the number of vertices in G_{ij} which have an odd number of pebbles and $t_i$denote the number of vertices in H_i which have an odd number of pebbles.

Take any arrangement of $(2^{n+2} + \sum_{i=0}^{n} p_i - 2)$f(G) pebbles on the vertices of $P_n^* \times$ G. First we assume that the target vertex is (v_i, y) for some y, where i = 0, 1,…, n. Without loss of generality, we may assume that the vertex is (v_o, y).

Let $P_n^* - \{x_{01}, \ldots, x_{0p_0}, x_{11}, \ldots, x_{1p_1}, \ldots, , x_{n1}, \ldots, x_{np_n}\} = P_n$. From [7], we know that f((P_n ×G), (v_0, y)) ≤ f(P_n × G)≤ 2^n f(G). Since $b_{ij} \leq$ |V(G)| ≤f(G), $\sum_{i=0}^{n}\sum_{j=1}^{p_i} a_{ij} \leq (2^{n+2} + \sum_{i=0}^{n} p_i - 2)$ f(G), then

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=1}^{p_i}(a_{ij} + b_{ij}) &= \sum_{i=0}^{n}\sum_{j=1}^{p_i} a_{ij} + \sum_{i=0}^{n}\sum_{j=1}^{p_i} b_{ij} \\ &\leq (2^{n+2} + \textstyle\sum_{i=0}^{n} p_i - 2)\text{f(G)} + \sum_{i=0}^{n} p_i \text{ f(G)} \\ &= (2^{n+2} + 2\textstyle\sum_{i=0}^{n} p_i - 2)\text{f(G)}\end{aligned}$$

By lemma 4.1, we apply pebbling moves to all the vertices in $G_{01}, \ldots, G_{0p_0}, G_{11}, \ldots, G_{1p_1}, \ldots, , G_{n1}, \ldots, G_{np_n}$ and we can move atleast $\sum_{i=0}^{n}\sum_{j=1}^{p_i}\frac{(a_{ij} - b_{ij})}{2}$ pebbles from $G_{01}, \ldots, G_{0p_0}, G_{11}, \ldots, G_{1p_1}, \ldots, , G_{n1}, \ldots, G_{np_n}$ to the vertices of P_n ×G.

Therefore in $P_n \times$ G, we have atleast

$$\begin{aligned}(2^{n+2} + \textstyle\sum_{i=0}^{n} p_i - 2)\text{f(G)} - \sum_{i=0}^{n}\sum_{j=1}^{p_i} a_{ij} + \sum_{i=0}^{n}\sum_{j=1}^{p_i}\frac{(a_{ij} - b_{ij})}{2} &= (2^{n+2} + \textstyle\sum_{i=0}^{n} p_i - 2)\text{ f(G)} - \sum_{i=0}^{n}\sum_{j=1}^{p_i}\frac{(a_{ij} + b_{ij})}{2} \\ &\geq (2^{n+2} + \textstyle\sum_{i=0}^{n} p_i - 2)\text{f(G)} - \frac{(2^{n+2} + 2\sum_{i=0}^{n} p_i - 2)}{2}\text{ f(G)} \\ &= (2^{n+2} + \textstyle\sum_{i=0}^{n} p_i - 2 - 2^{n+1} - \sum_{i=0}^{n} p_i + 1)\text{f(G)} \\ &= (2^{n+1} - 1)\text{f(G) pebbles}\end{aligned}$$

Since f((P_n ×G), (v_0, y)) ≤ 2^n f(G), then we can move one pebble to (v_0, y).

Now let us assume that the target vertex is (x_{ij}, y) for some y, where i = 0, 1,…, n and j = 1, 2, …, p_i. Without loss of generality, we assume that the target vertex is (x_{01}, y). We know that, every thorn path P_n^* of length n is a tree.

Hence by lemma 4.2,f(($P_n^* \times$ G), (x_{01}, y)) ≤ f(P_n^*, x_{01}) f(G) = $(2^{n+2} + \sum_{i=0}^{n} p_i - 2)$ f(G). Hence one pebble can be moved to (x_{01}, y).

Corollary 4.1: Let P_n^* be the thorn path of length n and P_m be a path of length m, then f($P_n^* \times P_m$) ≤ f(P_n^*) f(P_m).

Proof: The corollary follows from Theorem 4.1 and Result 1.1.

Corollary 4.2: Let P_n^* be the thorn path of length n and C_m be a cycle with m vertices, then f($P_n^* \times C_m$) ≤ f(P_n^*)f(C_m).

Proof: The corollary follows from Theorem 4.1 and Result 1.1.

Corollary 4.3: Let P_n^* be the thorn path of length n and $K_{1,m}$ be a star graph with m > 1, then
$f(P_n^* \times K_{1,m}) \leq f(P_n^*)f(K_{1,m})$.

Proof: The corollary follows from Theorem 4.1 and Theorem 1.1.

Corollary 4.4: Let P_n^* be the thorn path of length n and W_m be a wheel graph with m≥3, then $f(P_n^* \times W_m) \leq f(P_n^*)f(W_m)$.

Proof: The corollary follows from Theorem 4.1 and Theorem 1.1.

Corollary 4.5: Let P_n^* be the thorn path of length n and P_m^* be a thorn path of length m, then $f(P_n^* \times P_m^*) \leq f(P_n^*)f(P_m^*)$.

Proof: The corollary follows from Theorem 3.1 and Theorem 4.1.

5. CONCLUSION AND OPEN PROBLEM

In this paper, we determined the pebbling number of the thorn path and also we have proved that the thorn path satisfies the 2- pebbling property and Grahams pebbling conjecture is true for the products of a thorn path by a

i) Path
ii) Cycle
iii) Star
iv) Wheel
v) Thorn path

The pebbling number of the thorn cycle is an open problem.

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