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EXISTENCE OF SOLUTIONS OF NONLOCAL IMPULSIVE FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we discuss the existence and uniqueness of solutions of initial value problem for impulsive fractional mixed integrodifferential equations using Schaefer fixed point theorem.

Keywords: *Fractional integrodifferential equations. Schaefer fixed point theorem, impulsive conditions, Caputo fractional derivative.*

2000 Mathematics subjects Classifications: *34K05, 34A12, 34A40.*

1. INTRODUCTION

In this paper, we study the existence and uniqueness of solutions for the nonlocal impulsive fractional mixed integrodifferential equations of the form

$$^{c}D^{\alpha}\, y(t) = f\left(t, y(t), \int_0^t g(t,s,y(s)ds), \int_0^a k(t,s,y(s)ds)\right) \text{ for } t \in J = [0, a] \text{ at } t \neq t_k \tag{1.1}$$

$$\Delta y\,|_{t=t_k} = I_k(y(t_k^-)) \tag{1.2}$$

$$y(0) = y_o - q(y) \tag{1.3}$$

where k= 1,2,…m. $0 < a \le 1$, $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J \times R \times R \times R \to R$, $g : J \times J \times R \to R$, $k : J \times J \times R \to R$ are given continuous functions and $I_k : R \to R$ and $y_0 \in R$, $0 = t_0 < t_1 < \ldots t_m < t_{m+1} = a$.

$\Delta y\,|_{t=t_k} = y(t_k^+) - y(t_k^-)$,

$y(t_k^+) = \lim_{h\to 0^+} y(t_k+h)$ and

$y(t_k^-) = \lim_{h\to 0^-} y(t_k+h)$.

represent the right and left limits of y(t) at $t = t_k$.

Nonlocal conditions were in initiated by Byszewski [13] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski [11, 12], the nonlocal conditions can be more useful than the standard initial condition to describe some physical phenomena. For example q(y) may given by

$$q(y) = \sum_{i=1}^{p} c_i y(\tau_i)$$

where c_i, i = 1,2,…p are given constants and $0 < \tau_1 < \tau_2 < \ldots \tau_p \le a$.

Differential equations of fractional order have proved to be valuable tools in the modelling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control; porous media, electromagnetic, etc. (see [16, 24, 25, 28, 33, 34, 38]). There has been a

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significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas et al[30], Miller and Ross[35], Samko et al[44] and the papers of Agarwal et al [1], Murugesu and Suguna [39] Babakhani and Daftardar-Gejji [2, 3, 14], Belmekki et al [6], Benchohra et al [5,7,10], Delbosco and Rodino [15], Diethelm et al [16, 18], El-Sayed [19, 21], Furati and Tatar [22,23], Kaufmann and Mboumi [29], Kilbas and Marzan [30], Mainardi [33], Momani and Hadid [36], Momani et al [37], Podlubny et al [41, 42, 43], Yu and Gao [46] and Zhang [47] and the references therein. Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain y(0), y′(0), etc., the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivative of both the Riemann-Liouville and Caputo types see [27, 41].

Impulsive differential equations (for $\alpha \in N$) have become important in recent years as mathematical models of phenomena in both the physical and social services. There has been a significant improvement in impulsive theory especially in the area of impulsive differential equations with fixed moments; see, for instance, the monographs by Bainov and Simeonov [4], Benchohra et al [9], Lakshmikantham et al [32] and Samoilenko and Perestyuk [44] and the references therein. This paper is organized as follows. In section 2 we present some preliminary results about fractional derivative and integration needed in the following sections. Section 3 will be concerned with existence and uniqueness results for the equations (1.1) – (1.3). The result is based on Schaefer fixed point theorem.

2. PRELIMINARIES

In this section, we introduce notation, definitions and preliminary facts which are used throughout this paper. By C(J, R), we denote the Banach space of all continuous functions from J into R with the norm

$$\|y\|_\infty := \sup\{\|y(t)\| \,;\, t \in J\}$$

Definition: 2.1 ([31, 41]). The fractional (arbitrary) order integral of the function $h \in L^1([a, b], R_+)$ of order $\alpha \in R_+$ defined by

$$I_a^\alpha h(t) = \int_a^t \left(\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\right) h(s)ds$$

where $\boldsymbol{\Gamma}$ is the gamma function. When a = 0, we write $I^\alpha h(t) = [h * \Phi_\alpha](t)$, where $\Phi_\alpha(t) = \left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)$ for t > 0, and $\Phi_\alpha(t) = 0$ for $t \le 0$, and $\Phi_\alpha(t) \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition: 2.2([31, 41]) For a function h given on the interval [a, b], the α-th Riemann-Liouville fractional derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \left(\frac{1}{\Gamma(n-\alpha)}\right)\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1} h(s)ds$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definitions: 2.3([30]) For a function h given on the interval [a, b], the Caputo fractional order derivative of order α of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \left(\frac{1}{\Gamma(n-\alpha)}\right)\int_a^t ((t-s)^{n-\alpha-1} h^{(n)}(s)ds$$

Where $n = [\alpha] + 1$.

3. EXISTENCE OF SOLUTIONS

Consider the set of functions

$PC(J, R) = \{y : J \to R : y \in C((t_k, t_{k+1}], R),\ k = 0,1,2,\ldots m$ and there exist $y(t_k^-)$ and $y(t_k^+)$, $k = 1,2,\ldots m$ with $y(t_k^-) = y(t_k)$. This set is a Banach space with the norm

$$\|y\|_{PC} = \sup_{t \in J} \|y(t)\|$$

Set $J' := [0, a] \setminus \{t_1, t_2, \ldots t_m\}$

Lemma: 3.1 A function $y \in PC(J, R)$ whose α –derivative exists on J′ is said to be a solution of (1.1) – (1.3) if y satisfies the equation

$${}^cD^\alpha y(t) = f(t, y(t), \int_0^t g(t,s,y(s))ds, \int_0^a k(t,s,y(s))ds)$$

on J′ and satisfy the conditions

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,2,\ldots m,$$

$$y(0) = y_0.$$

To prove the existence of solutions to (1.1) – (1.3), we need the following auxiliary lemmas.

Lemma: 3.2([47]) Let $\alpha > 0$, then the differential equation ${}^cD^\alpha h(t) = 0$ has solutions

$h(t) = c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1}$, $c_i \in R$, $i = 0,1,2,\ldots,n-1, n = [\alpha] + 1$.

Lemma: 3.3([47]) Let $\alpha > 0$, then $I^\alpha\, {}^cD^\alpha h(t) = h(t) + c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1}$ for some $c_i \in R$, $i = 0,1,2,\ldots n-1$, $n = [\alpha] + 1$. As a consequence of Lemma 3.2 and Lemma 3.3 we have the following result which is useful in what follows.

Lemma: 3.4 Let $0 < \alpha \le 1$ and let $f : J \times R \times R \times R \to R$ be continuous. A function y is a solution of the fractional integrodifferential equations

$$y(t) = \begin{cases} y_0 - q(y) + \left(\frac{1}{\Gamma(a)}\right)\int_0^t((t-s)^{\alpha-1})h(s)ds \quad \text{if } t \in [0.t_1] \\ y_0 - q(y) + \frac{1}{\Gamma(a)}\sum_{i=1}^{k}\int_{t_{i-1}}^{ti}(t_i - s)^{(\alpha-1)}h(s)ds \end{cases} \tag{1.4}$$

$$+\frac{1}{\Gamma(a)}\int_{t_k}^{t}(t-s)^{(\alpha-1)}h(s)ds + \sum_{i=1}^{k} I_i((y(t_i^-))), \text{ if } t \in (t_k, t_{k+1}]$$

where $h(s) = f(t, y(t), \int_0^t g(t,s,y(s))ds), \int_0^a k(t,s,y(s))ds)$, $k = 1, 2,\ldots, m$

if and only if y is a solution of fractional integrodifferential equations

$${}^cD^\alpha y(t) = f(t,y(t), \int_0^t g(t,s,y(s))ds), \int_0^a k(t,s,y(s))ds) \quad t \in J' \tag{1.5}$$

$$\Delta y\,|_{t=t_k} = I_k(y(t_k^-)), \; k = 1,2,\ldots m \tag{1.6}$$

$$y(0) = y_0 - q(y) \tag{1.7}$$

Proof: Assume y satisfies (1.5) - (1.7). If $t \in [0, t_1]$ then

$${}^cD^\alpha y(t) = f(t,y(t), \int_0^t g(t,s,y(s))ds, \int_0^a k(t,s,y(s))ds).$$

Lemma 3.3 implies

$$y(t) = y_0 - q(y) + \frac{1}{\Gamma(a)}\int_0^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau))d\tau)ds.$$

If $t \in (t_1, t_2]$, then Lemma 3.3 implies

$$\begin{aligned} y(t) &= y(t_1^+) + \frac{1}{\Gamma(a)}\int_{t_1}^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau))d\tau)ds \\ &= \Delta y\,|_{t=t_1} + y(t_1^-) + \frac{1}{\Gamma(a)}\int_{t_1}^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau)d\tau)ds \\ &= I_1(y(t_1^-)) + y_0 - q(y) + \frac{1}{\Gamma(a)}\int_0^{t_1}(t_1 - s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \\ &\quad \int_0^a k(s,\tau,y(\tau))d\tau)ds + \frac{1}{\Gamma(a)}\int_{t_1}^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau))d\tau)ds \end{aligned}$$

If $t \in (t_2, t_3]$, then by Lemma 3.3, we get

$$\begin{aligned} y(t) &= y(t_2^+) + \frac{1}{\Gamma(a)}\int_{t_2}^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau))d\tau)ds \\ &= \Delta y\,|_{t=t_2} + y(t_2^-) + \frac{1}{\Gamma(a)}\int_{t_2}^t(t-s)^{\alpha-1}f(s,y(s), \int_0^s g(s,,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau))d\tau)ds \\ &= I_2(y(t_2^-)) + I_1(y(t_1^-)) + y_0 - q(y) + \frac{1}{\Gamma(a)}\int_{t_2}^t(t_1 - s)^{\alpha-1}f(s,y(s), \int_{t_2}^s g(s,\tau,y(\tau))d\tau, \\ &\quad \int_{t_2}^a k(s,\tau,y(\tau))d\tau)ds + \frac{1}{\Gamma(a)}\int_{t_1}^{t_2}(t_2 - s)^{\alpha-1}f(s,y(s), \int_{t_1}^s g(s,\tau,y(\tau))d\tau, \int_{t_1}^a k(s,\tau,y(\tau))d\tau)ds \\ &\quad + \frac{1}{\Gamma(a)}\int_{t_2}^t(t-s)^{\alpha-1}f(s,y(s), \int_{t_2}^s g(s,\tau,y(\tau))d\tau, \int_{t_2}^a k(s,\tau,y(\tau))d\tau)ds. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, then again from Lemma 3.3, we get (1.4)

Conversely assume that y satisfies the impulsive fractional integral equations (1.4). If $t \in [0, t_1]$ then $y(0) = y_0$ and using the fact that ${}^cD^\alpha$ is the left inverse of I^α, we get

$${}^cD^\alpha y(t) = f(t, y(t), \int_0^t g(t,s,y(s))ds, \int_0^a k(t,s,y(s))ds) \text{ for each } t \in [0, t_1].$$

If $t \in [t_k, t_{k+1})$, $k = 1,2,\ldots,m$ and using the fact that ${}^cD^\alpha C = 0$ where C is a constant, we get

$${}^cD^\alpha y(t) == f(t,y(t), \int_0^t g(t,s,y(s))ds, \int_0^\alpha k(t,s,y(s))ds) \text{ for each } t \in [t_k, t_{k+1}).$$

Also we can easily show that

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)),\ k = 1,2,\ldots m.$$

Schaefer's Theorem: Let E be a normed linear space. Let $F : E \to E$ be a completely continuous operator, that is, it is continuous and the image of any bounded set is contained in a compact set and let $\zeta(F) = \{ x \in E;\ x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$. Then either $\zeta(F)$ is unbounded or F has a fixed point.

Theorem: 3.5 Assume that

(H_1) The function $f: J \times R \times R \times R \to R$, $g : J \times J \times R \to R$, $k : J \times J \times R \to R$ are continuous functions

(H_2) There exists a constant $M^* > 0$ such that $\|f(t, x_1, x_2, x_3)\| \le M^*$ for each $t \in J = [0, a]$ and each $x_i \in X$, for $i = 1,2,3$.

(H_3) There exists a constant $M_1 > 0$ such that

$\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \le M_1(\|x_1 - y_1\| + \|x_2 - y_2\| + \|x_3 - y_3\|)$

(H_4) There exist constants $M_2 > 0$, $M_3 > 0$ such that

$\|g(t, s, x_1) - g(t, s, y_1)\| \le M_2(\|x_1 - y_1\|)$ and $\|k(t, s, x_1) - k(t, s, y_1)\| \le M_3(\|x_1 - y_1\|)$

(H_5) The function $I_k : R \to R$ is continuous and there exists a constant $M^{**} > 0$ such that $\|I_k(x)\| \le M^{**}$ for each $x \in R$, $k = 1,2,\ldots,m$.

(H_6) There exists a constant $M > 0$ such that $\|y_0 - q(y)\| \le M$. Then (1.1) − (1.3) has atleast one solution on J.

Proof: We shall use Schaefer's fixed point theorem to prove that F has a fixed point. The proof will be given to several steps.

Step 1: F is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in PC (J, R). Then for each $t \in J$

$$\begin{aligned}
\|F(y_n)(t) - F(y)(t)\| \le{}& \|y_0 - q(y) - y_0 + q(y)\| + \frac{1}{\Gamma(a)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} \\
& \|f(s, y_n(s) \int_0^s g(s,\tau,y_n(\tau))d\tau, \int_0^a k(s,\tau,y_n(\tau))d\tau)ds \\
& - f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \int_0^a k(s,\tau,y(\tau)d\tau)ds\| \\
& + \frac{1}{\Gamma(a)}\int_{t_k}^t (t-s)^{\alpha-1}\|f(s, y_n(s), \int_0^s g(s,\tau,y_n(\tau))d\tau, \\
& \int_0^a k(s,\tau,y_n(\tau))d\tau)ds - f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau, \\
& \int_0^a k(s,\tau,y(\tau)d\tau))ds\| + \sum_{0<t_k<t}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|
\end{aligned}$$

$$\begin{aligned}
\le{}& \frac{1}{\Gamma(a)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}M_1[\|y_n(s) - y(s)\| \\
& + \|\int_0^s g(s,\tau,y_n(\tau))d\tau - \int_0^s g(s,\tau,y(\tau))d\tau\| + \|\int_0^a k(s,\tau,y_n(\tau))d\tau - \int_0^a k(s,\tau,y(\tau))d\tau\|]ds \\
& + \frac{1}{\Gamma(a)}\int_{t_k}^t (t-s)^{\alpha-1}M_1[\|y_n(s) - y(s)\| + \|\int_0^s g(s,\tau,y_n(\tau))d\tau - \int_0^s g(s,\tau,y(\tau))d\tau\| \\
& + \|\int_0^a k(s,\tau,y_n(\tau))d\tau - \int_0^a k(s,\tau,y(\tau))d\tau\|]ds + \sum_{0<t_k<t}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|
\end{aligned}$$

$$\begin{aligned}
\le{}& \frac{1}{\Gamma(a)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}M_1[\|y_n(s)-y(s)\| + a.M_2\|y_n(\tau) - y(\tau)\| + a.M_3\|y_n(\tau) \\
& - y(\tau)\|]ds + \frac{1}{\Gamma(a)}\int_{t_k}^t (t-s)^{\alpha-1}M_1[\|y_n(s) - y(s)\| + a.M_2\|y_n(\tau) - y(\tau)\| + a.M_3\|y_n(\tau) - y(\tau)\|]ds \\
& + \sum_{0<t_k<t}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|
\end{aligned}$$

$$\begin{aligned}
\le{}& \frac{1}{\alpha\Gamma(a)}\sum_{0<t_k<t}(t_k - t_{k-1})^\alpha M_1[\|y_n(s) - y(s)\| + a.M_2\|y_n(\tau) - y(\tau)\| + a.M_3\|y_n(\tau) - y(\tau)\|] \\
& + \frac{1}{\alpha\Gamma(a)}(t - t_k)^\alpha M_1[\|y_n(s) - y(s)\| + a.M_2\|y_n(\tau) - y(\tau)\| + a.M_3\|y_n(\tau) - y(\tau)\|] \\
& + \sum_{0<t_k<t}\|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|
\end{aligned}$$

$$= \frac{1}{\Gamma(a+1)}\sum_{0<t_k<t} [(t_k - t_{k-1})^\alpha + (t - t_k)^\alpha] M_1[\|y_n(s) - y(s)\| + a.M_2\|y_n(\tau) - y(\tau)\| + a.M_3\|y_n(\tau) - y(\tau)\|] + \sum_{0<t_k<t} \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|$$

$$= \frac{1}{\Gamma(a+1)}\sum_{0<t_k<t} [(t_k - t_{k-1})^\alpha + (t - t_k)^\alpha] M_1[1 + a(M_2 + M_3)] \|y_n(\tau) - y(\tau)\|_\infty + \sum_{0<t_k<t} \|I_k(y_n(t_k^-)) - I_k(y(t_k^-))\|$$

Since f, g, k and I_k, k = 1, 2… m are continuous functions, we have

$$\|F(y_n)(t) - F(y)(t)\|_\infty \to 0 \text{ as } n \to \infty.$$

Step 2: F maps bounded sets into bounded sets in PC(J,R). Indeed it is enough to show that for any η* > 0, there exists a positive constant 'ι' such that for each $y \in B\eta^* = \{y \in PC(J, R): \|y\|_\infty \le \eta^*\}$ we have $\|F(y)\|_\infty \le ι$. By (H_1) - (H_6) we have for each $t \in J$.

$$\|F(y)(t)\| \le \|y_0 - q(y)\| + \frac{1}{\Gamma(a)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1}\|f(s, y(s), \int_0^s g(s, \tau, y(\tau))d\tau, \int_0^a k(s, \tau, y(\tau))d\tau)\|ds + \frac{1}{\Gamma(a)}\int_{t_k}^{t} (t_k - s)^{\alpha-1}\|f(s, y(s), \int_0^s g(s, \tau, y(\tau)d\tau), \int_0^a k(s, \tau, y(\tau)d\tau))\|ds + \sum_{0<t_k<t} \|I_k(y_n(t_k^-))\|$$

$$\le \|y_0 - q(y)\| + \frac{1}{\Gamma(a)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha-1} + M^* ds + \frac{1}{\Gamma(a)}\int_{t_k}^{t} (t - s)^{\alpha-1} M^* ds + \sum_{0<t_k<t} \|I_k(y_n(t_k^-))\|$$

$$\le \|y_0 - q(y)\| + \sum_{0<t_k<t} \frac{1}{\Gamma(a+1)}(t_k - t_{k-1})^\alpha M^* + \frac{1}{\Gamma(a+1)}(t - t_k)^\alpha M^* + mM^{**}$$

$$\le M + \frac{1}{\Gamma(a+1)}mT^\alpha M^* + \frac{1}{\Gamma(a+1)}T^\alpha M^* + mM^{**}$$

$$= M + \frac{1}{\Gamma(a+1)}T^\alpha M^*(m+1) + mM^{**}$$

$$= M + \frac{T^\alpha M^*}{\Gamma(a+1)}(1 + m) + mM^{**}$$

Thus $\|F(y)(t)\| \le M + \frac{T^\alpha M^*}{\Gamma(a+1)}(1 + m) + mM^{**} := ι$

Step 3: F maps bounded sets into equicontinuous sets of PC (J, R). Let $\tau_1, \tau_2 \in J$, $\tau_1 < \tau_2$, $B\eta^*$ be a bounded set of PC(J, R) as in step 2, and let $y \in B\eta^*$. Then

$$\|F(y)(\tau_2) - F(y)(\tau_1)\| = \frac{1}{\Gamma(a)}\int_0^{\tau_1} \|(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\| \|f(s, y(s), \int_0^s g(s, \tau, y(\tau))d\tau, \int_0^a k(s, \tau, y(\tau))d\tau\|ds + \frac{1}{\Gamma(a)}\int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}\|f(s, y(s), \int_0^s g(s, \tau, y(\tau))d\tau, \int_0^a k(s, \tau, y(\tau))d\tau\|ds + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

$$\le M^*\frac{1}{\Gamma(a)}\int_0^{\tau_1} \|(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}\|ds + M^*\frac{1}{\Gamma(a)}\int_{\tau_1}^{\tau_2} \|(\tau_2 - s)^{\alpha-1}\|ds + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

$$\le -M^*\frac{1}{a\Gamma(a)}\{\|[(\tau_2 - \tau_1)^\alpha - (\tau_2)^\alpha] + [(\tau_1 - \tau_1)^\alpha - (\tau_1)^\alpha]\|\} - (-M^*)\frac{1}{a\Gamma(a)}\{\|[(\tau_2 - \tau_2)^\alpha - (\tau_2 - \tau_1)^\alpha]\|\} + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

$$\le M^*\frac{1}{a\Gamma(a)}\|[(\tau_2)^\alpha - (\tau_2 - \tau_1)^\alpha - (\tau_1)^\alpha]\| + M^*\frac{1}{a\Gamma(a)}\|(\tau_2 - \tau_1)^\alpha]\| + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

$$\le M^*\frac{1}{\Gamma(a+1)}2(\tau_2 - \tau_1)^\alpha + M^*\frac{1}{\Gamma(a+1)}((\tau_2)^\alpha - (\tau_1)^\alpha) + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

$$\|F(y)(\tau_2) - F(y)(\tau_1)\| \le M^*\frac{1}{\Gamma(a+1)}[2(\tau_2 - \tau_1)^\alpha + ((\tau_2)^\alpha - (\tau_1)^\alpha)] + \sum_{0<t_k<\tau_2-\tau_1} \|I_k(y(t_k^-))\|$$

As $\tau_1 \to \tau_2$, the right hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzela-Ascoli theorem. We can conclude that F : PC(J, R) → PC(J, R) is completely continuous.

Step 4: A priori bounds. Now it remains to show that the set $\in = \{y \in PC(J, R) : y = \lambda F(y)$, for some $0 < \lambda < 1\}$ is bounded. Let $y \in \in$, then $y = \lambda F(y)$, for some $0 < \lambda < 1$. Thus for each $t \in J$ we have

$$y(t) = \lambda[y_0 - q(y)] + \frac{\lambda}{\Gamma(\alpha)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} (t_k - s)^{\alpha - 1} f(s, y(s), \int_0^s g(s, \tau, y(\tau))d\tau, \int_0^a k(s,\tau, y(\tau))d\tau)ds + \frac{\lambda}{\Gamma(\alpha)}\int_{t_k}^{t} (t_k - s)^{\alpha - 1} f(s, y(s), \int_0^s g(s, \tau, y(\tau))\, dr, \int_0^a k(s,\tau, y(\tau))d\tau)ds + \lambda \sum_{0<t_k<t} \|I_k(y(t_k^-))\|$$

This implies by $(H_1) – (H_6)$ (as in step 2) that for each $t \in J$ we have

$$\|y(t)\| = \|y_0 - q(y)\| + \frac{T^\alpha M^*}{\Gamma(\alpha+1)}(1 + m) + mM^{**} := \iota$$

$$= M + \frac{T^\alpha M^*}{\Gamma(\alpha+1)}(1 + m) + mM^{**} := \iota$$

Thus for every $t \in J$, we have

$$\|y(t)\|_\infty \leq M + \frac{T^\alpha M^*}{\Gamma(\alpha+1)}(1 + m) + mM^{**} := \iota$$

This shows that the set '$\in$' is bounded. As a consequence of Schafer's fixed point theorem, we deduce that F has a fixed point which is a solution of the problem (1.1) – (1.3).

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