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APPROXIMATE FIXED POINT RESULT FOR GENERALIZED α –CONTRACTIVE MAPPING

PANKAJ KUMAR JHADE*
Department of Mathematics,
Government PG College, Multai, India-460661.

SHAKTI BHATTA
Department of Mathematics,
Madhyanchal Professional University, Ratibad, Bhopal, India-462026.

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ABSTRACT

In this paper, we introduce the generalized α −contractive mapping and obtain a result about approximate fixed point using implicit relation in α −complete metric space. Our result extends and generalizes some classical results existing in the literature.

Keywords: *ε −fixed point, α −admissible mapping, α −complete metric space, generalized α −contractive mapping.*

MSC 2010: *47H10, 54H25.*

INTRODUCTION

Fixed point theory is an important tool for solving various problems in nonlinear functional analysis since it has many useful for proving the existence solutions for nonlinear differential and integral equations. However, in several practical situations, the conditions in the fixed point theorems are too strong and so the existence of a fixed point is not guaranteed. In this situation, we can consider nearly fixed points what we call as approximate fixed points. For self mapping$T$ on a nonempty set X, the study of approximate fixed point $x \in X$ of T, we mean in a sense that Tx is "near to" x. The study of approximate fixed point theorems is equally interesting to that of fixed point theorems.

In 2006, inspired and motivated by the work of Tijs *et al.* [14], Berinde [2] studied and gave some fundamental approximate fixed point theorems in metric space. In 2013, Dey and Saha [5] established the existence of approximate fixed point for the Reich operator [11] which is turn generalizes approximate fixed point theorems of Berinde [2]. Miandaragh *et al.* [10] introduced the concept of generalized α −contractive mapping and gave two results about approximate fixed points and fixed points of the mapping on metric spaces. There have appeared many works on approximate fixed point results (see for example [4, 6, 9] and references therein).

The aim of this paper is to introduce the generalized α −contractive mapping and obtain a result about approximate fixed point using implicit relation in α −complete metric space. Our result extends and generalizes some classical results existing in the literature.

PRELIMINARIES

Throughout this paper $\mathbb{N}$ be the set of all positive integers and $\mathbb{R}$ the set of real numbers.

Definition 2.1 ([12]): Let (X, d) be a metric space, $T: X \to X$ be a mapping and $\varepsilon > 0$ be a given real number. A point $x_0 \in X$ is said to be an ε −*fixed point (approximate fixed point)* of T if

$$d(x_0, Tx_0) < \varepsilon.$$

Corresponding Author: Pankaj Kumar Jhade*
Department of Mathematics, Government PG College, Multai, India-460661.

Remark 2.2: We observe that fixed point is ε −fixed point for all $\varepsilon > 0$. However, the converse is not true.

For a metric space (X, d) and a given $\varepsilon > 0$, the set of all ε −fixed points of $T: X \to X$ is denoted by
$$F_\varepsilon(T) = \{x \in X: d(x, Tx) < \varepsilon\}.$$

Definition 2.3 ([8]): Let (X, d) be a metric space and $T: X \to X$ be a mapping. We say that T has the *approximate fixed point property* if for all $\varepsilon > 0$, there exists an ε −fixed point of T , that is,
$$\forall \varepsilon > 0, \qquad F_\varepsilon(T) \neq \phi$$
or, equivalently,
$$\inf_{x \in X} d(x, Tx) = 0.$$

In 1966, Browder and Petryshyn [3] defined the following notion.

Definition 2.4 ([3]): A self mapping T on a metric space (X, d) is said to be *asymptotically regular* at a point $x \in X$ if $d(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$, where $T^n x$ denotes the n-*th*iterate of T at x.

Lemma 2.5 ([9]): Let (X, d) be a metric space and $T: X \to X$ be an asymptotically regular mapping at a point $z \in X$, then T has the approximate fixed-point property.
In 2012, Samet *et al.* [13] introduced the concept of is α −admissible as follows:

Definition 2.6 ([13]): Let T be a self mapping on a nonempty set X and $\alpha: X \times X \to [0, \infty)$ be a mapping. We say that T is α −*admissible* if
$$x, y \in X, \qquad \alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 2.7 ([7]): Let (X, d) be a metric space and $\alpha: X \times X \to [0, \infty)$ be a mapping. The metric space X is said to be α-*complete* if and only if every Cauchy sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$, converges in X.

Remark 2.8: If X is complete metric space, then X is α −complete metric space, but the converse is not true.

Definition 2.9 ([7]): Let (X, d) be a metric space, $\alpha: X \times X \to [0, \infty)$ and $T: X \to X$ be two mappings. We say that T is α −*continuous mapping* on (X, d) if for each sequence $\{x_n\}$ in X with $x_n \to x$ as $n \to \infty$ for some $x \in X$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \Rightarrow Tx_n \to Tx$ as $n \to \infty$.

Remark 2.10: If T is continuous mapping, then T is α −continuous mapping, where $\alpha: X \times X \to [0, \infty)$ is an arbitrary mapping.

MAIN RESULT

Definition 3.1: (Implicit Relation). Let Φ be the family of all real valued continuous functions $\phi: \mathbb{R}_+^3 \to \mathbb{R}_+$ non-decreasing in the first argument for three variables. For some $\mu \in [0,1)$ we consider the following conditions:
(i) For any $x, y \in \mathbb{R}_+$ if $x \leq \phi\left(x, \frac{x+y}{2}, \frac{x+y}{2}\right)$ then $x \leq \mu y$.
(ii) For any $x, y \in \mathbb{R}_+$ if $x \leq \phi(x, 0, x)$ then $y = 0$ since $\mu \in [0,1)$.

Example 3.2: Let $\phi(r, s, t) = r - \alpha \min(s, t) + (2 + \alpha)t$, where $\alpha > 0$.

Example 3.3: Let $\phi(r, s, t) = r^2 - ar\, max(s, t) - bs$, where $a > 0, b > 0$.

Example 3.4: Let $\phi(r, s, t) = r + c\, max\,(s, t)$, where $c \geq 0$.

Definition 3.5: (Generalized α −contractive mapping). Let (X, d) be a metric space, $\alpha: X \times X \to [0, \infty)$ and $T: X \to X$ be two mappings. Then we say that the self map T of X is a generalized α −contractive mapping whenever there exists $\phi \in \Phi$ such that
$$\alpha(x, y)\alpha(Tx, Ty) \leq \phi\left(d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right)$$
for all $x, y \in X$.

Theorem 3.6: Let (X, d) be a metric space and T a generalized α −contractive and α −admissible self mapping on X such that $\alpha(x_0, Tx_0) \geq 1$ for some $x_0 \in X$. Then T has an approximate fixed point.

Moreover , T has a fixed point whenever T is α −continuous and (X, d) is α −complete metric space.

Proof: Fix $1 > r > \mu$and $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Define the sequence $\{x_n\}$ by $x_{n+1} = T^{n+1}x_0$ for all $n \geq 0$.

If $x_n = x_{n+1}$ for some n, then we have nothing to prove.

So, we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α −admissible, it is easy to check that $\alpha(x_n, x_{n+1}) \geq 1$ for all n.
Now,

$$d(x_1, x_2) = d(Tx_0, Tx_1) \leq \alpha(x_0, x_1) d(Tx_0, Tx_1)$$
$$\leq \phi\left(d(x_0, x_1), \frac{d(x_0, Tx_0) + d(x_1, Tx_1)}{2}, \frac{d(x_0, Tx_1) + d(x_1, Tx_0)}{2}\right)$$
$$\leq \phi\left(d(x_0, x_1), \frac{d(x_0, x_1) + d(x_1, x_2)}{2}, \frac{d(x_0, x_1) + d(x_1, x_2)}{2}\right)$$

By using Definition 3.1, we obtain

$$d(x_1, x_2) \leq \mu d(x_0, x_1) < r d(x_0, x_1)$$

Similarly,

$$d(x_2, x_3) = d(Tx_1, Tx_2) \leq \alpha(x_1, x_2) d(Tx_1, Tx_2)$$
$$\leq \phi\left(d(x_1, x_2), \frac{d(x_1, Tx_1) + d(x_2, Tx_2)}{2}, \frac{d(x_1, Tx_2) + d(x_2, Tx_1)}{2}\right)$$
$$\leq \phi\left(d(x_1, x_2), \frac{d(x_1, x_2) + d(x_2, x_3)}{2}, \frac{d(x_1, x_2) + d(x_2, x_3)}{2}\right)$$

Again, by using Definition 3.1, we obtain

$$d(x_2, x_3) \leq \mu d(x_1, x_2) < r d(x_1, x_2) < r^2 d(x_0, x_1)$$

By continuing in this way, we obtain

$$d(x_n, x_{n+1}) < r^n d(x_0, x_1) \text{ for all } n.$$

Hence $d(T^{n+1}x_0, T^n x_0) \to 0$ as $n \to \infty$. Therefore, T is an asymptotically regular at a point $x_0 \in X$. By using Lemma 2.5, we conclude that T has the approximate fixed point property.

Next, we show that T has a fixed point provided that T is α −continuous and (X, d) is an α −complete metric space.

Firstly, we claim that $\{x_n\}$ is a Cauchy sequence in X.

Let $m, n \in \mathbb{N}$ such that $n > m$, then it is easy to see that

$$d(x_m, x_n) \leq (r^m + r^{m+1} + \cdots + r^{n-1}) d(x_0, x_1)$$
$$< \frac{r^m}{1 - r} d(x_0, x_1)$$

Thus $\{x_n\}$ is a Cauchy sequence in X.

Since $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $\{x_n\}$ is a Cauchy sequence in X by using α −completeness of X there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since T is α −continuous, $Tx_n \to Tx^*$ as $n \to \infty$. Therefore, $x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*$ and thus T has a fixed point.

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