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ON SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY A GENERALISED OPERATOR

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ABSTRACT

In this article new subclass for harmonic univalent in the unit disk U define by the constructed operator L_n^σ. Properties such as coefficient bounds, distortion bounds, extreme points, and convolution will be studied.

Key words: *Harmonic function, harmonic univalent function, coefficient inequality, extreme point, convex combination, integral operator.*

1. INTRODUCTION

Let f = u + iv be a complex valued harmonic function in a complex domain ℂ that is both u and v are real harmonic in ℂ. Let

$$f(z) = h + \bar{g} \tag{1.1}$$

where h and g are analytic in $\mathfrak{D} \subset \mathbb{C}$ and $\mathfrak{D}$ is any simply connected domain. Let $S\mathcal{H}$ be the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathcal{U}= \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = h(0) = f'(0) - 1 = 0$, h and g define as follows

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=2}^{\infty} b_n z^n, \ |b_1| < 1. \tag{1.2}$$

In 1984 Clunie and Sheil-Small [8] introduced and investigated the class SH as well as its geometric subclasses and obtained some properties of this class and this motivated many researchers to introduce some subclasses of the class SH, (see [3, 4, 6]). The importance of these functions is due to their use in the study of minimal surfaces as well as in various problems related to applied mathematics. Let D^n with ($n \in N_0 = 0, 1, 2, \dots$), be the Salagean derivative operator defined as $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ with $D^0 f(z) = f(z)$ given as

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_n z^k. \tag{1.3}$$

Let I^σ one-parameter Jung-Kim-Srivastava integral operator defined as $I^\sigma f(z) = \frac{2^\sigma}{2\Gamma\sigma} \int_0^z (\log\frac{z}{t})^{\sigma-1} f(t)dt$ given as

$$I^\sigma f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^\sigma a_k z^k. \tag{1.4}$$

The operator L_n^σ was define as follows in [1]

$$L_n^\sigma f(z) = z + \sum_{k=2}^{\infty} K^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k. \tag{1.5}$$

with $L_n^0 f(z)(z) = D^n f(z)$ and $L_0^\sigma f(z)(z) = I^\sigma f(z)$ We define the operator on f as follows

$$L_0^\sigma f(z)(z) = L_0^\sigma h(z)(z) + (-(-1)^n \overline{L_n^\sigma g(z)} \tag{1.6}$$

Where $L_n^\sigma f(z) = z + \sum_{k=2}^{\infty} K^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k$ and

$$L_n^\sigma g(z) = z + \sum_{k=2}^{\infty} K^n \left(\frac{2}{k+1}\right)^\sigma a_k z^k \text{ and}$$

also L $L_0^0 f(z) f(z) = h(z) + g(z)$. (1.7)

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The two operators have been used by researchers to generalised the concepts of starlikeness and convexity of functions in the unit disk. (see [9, 10, 11]). We define $M_n^{\sigma}(\alpha)$ be the family of harmonicfunctions of the form (1) such that

$$\text{Re}\,(M_{\sigma}^{n+1} \text{ f(z)} \frac{M_{\sigma}^{n+1}\text{ f(z)}}{M_n^{\sigma}(\alpha)} >)\ \beta \qquad . \tag{1.8}$$

Clearly the class $M_n^{\sigma}(\alpha)$) includes a variety of well-known subclasses of SH.

For example, $M_0^0(\alpha)) \equiv \text{SH}(\alpha)$ is the class of sense-preserving, harmonic univalent functions f which are starlike of order β in U and $M_0^0(\beta) \equiv$ KH is t On Subclass of Harmonic Univalent Functionsclass of sense-preserving, harmonic univalent functions f which are convex of order β in U studied by Jahangiri [2], $M_0^1(\alpha)$ is the class of Salagean-type harmonic univalent functions introduced by Jahangiri *et al.* [5, 7]. We let the subclass $\overline{M\,g\,(z)(\alpha)}$ (consist of harmonic functions $f_n = h(z) + g_n(z)$ in the class $M_n^{\sigma}(\beta)$ where h and f are of the form

$$h(z) = L_n^{\sigma} f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k,\ g(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k,\ |b_k| < 1. \tag{1.9}$$

In this work, we give the sufficient condition for functions in the class $M_{\sigma}^n(\beta)$ which is sufficient for the functions in the class) $\overline{M_{\sigma}^n}(\alpha)$. The distortion, extreme point and convolution for the functions in the class $\overline{M_{\sigma}^n}(\alpha)$) were also obtained.

2. MAIN RESULTS

Theorem 2.1: Let $f(z) = h(z) + \bar{g}(z)$ where h(z) and g(z) were given by (2)

$$\sum_{n=2}^{\infty} \frac{(|n-k|-\alpha\gamma)C_{nk}}{(1-\alpha)} |a_n| + \sum_{n=1}^{\infty} \frac{(|n-k|+\alpha\gamma)C_{nk}}{(1-\alpha)} |a_n| \le 1$$

(σ, k ∈ N_0, $0 \le \alpha < 1$, n ∈ N), then f(z) is harmonic univalent and sense-preserving in U and $f(z) \in A_H^k(\alpha,)$.

Proof: Firstly, to show that f(z) is harmonic univalent in U, suppose that $z_1, z_2 \in U$ for $|z_1| \le |z_2| < 1$, we have by inequality so that $z_1 \neq z_2$, then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \ge 1- \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$= 1- \left|\frac{\sum_{n=1}^{\infty} b_n (z_1^n - z_2^n)}{(z_1 - z_2) - \sum_{n=2}^{\infty} a_n (z_1^n - z_2^n)}\right|$$

$$1- \left|\frac{\sum_{n=1}^{\infty} |b_n|\, n}{1-\sum_{n=2}^{\infty} |a_n| n}\right|$$

$$\ge 1- \frac{\sum_{n=2}^{\infty} \frac{(|n-k|-\alpha)C_{nk}}{(1-\alpha)} |b_n|}{1-\sum_{n=2}^{\infty} \frac{(|n-k|-\alpha)C_{nk}}{(1-\alpha)} |a_n|} \ge 0$$

Thus f is a univalent function in U.

Note that f is sense-preserving in U. This is because

$$|h'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n|\,|z|^{n-1} > \sum_{n=2}^{\infty} n|a_n| \ge 1 - \sum_{n=2}^{\infty} \frac{(|n-k|-\alpha)C_{nk}}{(1-\alpha)} |a_n| \ge$$

$$\ge \sum_{n=1}^{\infty} \frac{(|n-k|+\alpha)C_{nk}}{(1-\alpha)} |b_n|$$

$$\ge \sum_{n=1}^{\infty} n|b_n|$$

$$\ge \sum_{n=1}^{\infty} n|a_n|\,|z|^{n-1} \ge |g'(z)|$$

According to the condition of Equation (5), we only need to show that if Equation (6) holds, then

$$\text{Re}\left\{\frac{F^{k+1} f(z)}{(1-\gamma)z + \gamma F^K f(z)}\right\} > \alpha$$

where $z = re^{i\theta}$, $0 \le \theta \le 2\pi$, $0 \le r < 1$ and $0 \le \alpha < 1$.

Note that $A(z) = F^{k+1} f(z)$ and $B(z) = F^k f(z)$.

Using the fact that Re(w) > α if and only if $|w - (1 + \alpha)| \le |w + (1 - \alpha)|$, it suffices to show that

$$|A(z) - (1+\alpha)B(z)| - |A(z) + (1-\alpha)B(z)| \le 0 \tag{7}$$

Substituting for A(z) and B(z) in $|A(z) - (1 + \alpha)B(z)|$, we obtain

$$|A(z) - (1+\alpha)B(z)| = 1\ F^{k+1} f(z) - (1+\alpha) F^k f(z)\ 1$$

$$= \left\|z + \sum_{n=2}^{\infty} c_{n(k+1)} a_n z^n + (-1)^{(k+1)} \sum_{n=1}^{\infty} C_{n(k+1)} \overline{b_n\ z^n} - (1+\alpha)[z +\right.$$

n=2∞Cnk anzn+*(−1)kn=1∞Cn(k+1)*bn zn

$$\le \alpha|z| \sum_{n=2}^{\infty} |((1+\alpha)) - |n-k||C_{nk}|a_n|$$

$$|z|^n + \sum_{n=1}^{\infty} |((1+\alpha) + |n-k|C_{nk}\, a_n \overline{|z^n|}|$$

Now, substituting for A(z) and B(z) in

$$|A(z) + (1-\alpha)B(z)|,$$

We obtain $|A(z) + (1 - \alpha)B(z)| = 1F^{k+1} f(z) + (1 - \alpha) F^{k} f(z)$

$= \left|z + \sum_{n=2}^{\infty} c_{n(k+1)} a_n z^n + (-1)^{(k+1)} \sum_{n=1}^{\infty} C_{n(k+1)} \overline{b_n\ z^n} \right. - (1-\alpha)z + \sum_{n=2}^{\infty} C_{nk}\ a_n z^n +$
$(-1)kn=1\infty Cn(k+1)$bn zn

$\geq (2-\alpha)|z| - \sum_{n=2}^{\infty}(\alpha - 1) - |n-k|\, C_{nk}\, |a_n|\,|z|^n -$
$\sum_{n=1}^{\infty}|n-k| - ((1-\alpha)C_{nk}\,|a_n|\bar{z}^n - |a_n|\bar{z}^n +$ (9)

Substituting for Equations (8) and (9) in the inequality we obtain

$|A(z) - (1 + \alpha)B(z)| - |A(z) + (1 - \alpha)B(z)|$
$\leq \alpha|z| + \sum_{n=1}^{\infty} \left|((1+\alpha)) - |n-k|\right| C_{nk}\ |a_n||z|^n + \sum_{n=1}^{\infty}((1+\alpha)|n-k|\, C_{nk}\ b_n||\overline{z^n|z|}^{\,n} + (\alpha -$
$2)\overline{|z|} \sum_{n=1}^{\infty}((1+\alpha)|n-k|\, C_{nk}\,|a_n|\ z|^n + \sum_{n=1}^{\infty}((1+\alpha)|n-k|\, C_{nk}\ b_n||z|^n$
$= 2\sum_{n=2}^{\infty}|n-k| - \alpha)C_{nk}|a_n| + 2\sum_{n=1}^{\infty}|n-k| + \alpha)C_{nk}|b_n| - 2(1-\alpha)$
≤ 0. (by hypothesis).

Therefore, we have

$\sum_{n=2}^{\infty}|n-k| - \alpha)C_{nk}|a_n| + \sum_{n=1}^{\infty} (|n-k| + \alpha)C_{nk}|bn| \leq (1-\alpha)$.

$f(z) = z + \sum_{n=2}^{\infty} \frac{1}{\frac{(|n-k\ |-\alpha)}{(1-\alpha)}} \mathcal{X}_n z^n \quad + \sum_{n=1}^{\infty} \frac{1}{\frac{(|n-k\ |+\alpha)}{(1-\alpha)}} \overline{z^n \mathcal{Y}_n}$ (10)

where $k \in N_0$ and $\sum_{n=2}^{\infty}|\mathcal{X}_n| + \sum_{n=1}^{\infty}|\mathcal{Y}_n| = 1$, shows that the coefficient bound given by Equation (6) is sharp. Since

$\sum_{n=2}^{\infty} \frac{(|n-k\ |-\alpha)C_{nk}}{(1-\alpha)} \frac{1}{\frac{(|n-k\ |-\gamma)}{(1-\alpha)}} |\mathcal{X}_n| \ + \sum_{n=1}^{\infty} \frac{(|n-k\ |+\alpha\gamma)C_{nk}}{(1-\alpha)} \frac{1}{\frac{(|n-k\ |+\gamma)}{(1-\alpha)}} |\mathcal{Y}_n|$

$\sum_{n=2}^{\infty}|\mathcal{X}_n| + \sum_{n=1}^{\infty}|\mathcal{Y}_n| = 1$

Now, we show that the condition of Equation (6) is also necessary for functions $f_k = h + \overline{g_k}$, where h and g_n are given by Equation (6).

Theorem 2.2: Let $f_k = h + \overline{g_k}$ be given by Equation (6). Then $f_k(z) \in A_{\bar{H}}^{k}(\alpha,)(\alpha,)$ if and only if the coefficient in condition of Equation (6) holds.

Proof: We only need to prove the "only if" part of the theorem because of $A_{\bar{H}}(k, \alpha, \gamma) \subset AH(k, \alpha, \gamma)$. Then by Equation (5), we have

$\mathrm{Re}\left\{ \frac{F^{k+1}f(z)}{F^{K}f(z)} \right\} > \alpha$

$\mathrm{Re}\left\{ \frac{|z + \sum_{n=2}^{\infty} c_{n(k+1)} a_n z^n + (-1)^{(k+1)} \sum_{n=1}^{\infty} C_{n(k+1)} \overline{b_n\ z^n} \ -(1-\alpha)[\gamma z + \gamma \sum_{n=2}^{\infty} C_{nk}\ a_n z^n + \gamma(-1)^k \sum_{n=1}^{\infty} C_{n(k+1)} \overline{b_n\ z^n}|}{(1-\alpha)[\gamma z + \gamma \sum_{n=2}^{\infty} C_{nk}\ a_n z^n + \gamma(-1)^k \sum_{n=1}^{\infty} C_{n(k)} \overline{b_n\ z^n}} \right\} > \alpha$

We observe that the above-required condition of Equation (11) must behold for all values of z in U. If we choose z to be real and $z \to 1^-$, we get

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty}(|n-k| - \alpha C_{nk}\,|a_n|}{[\sum_{n=2}^{\infty} C_{nk}\ |a_n|z^{n-1} + \gamma \sum_{n=2}^{\infty} C_{nk}\ |b_n|z^{-n-1}} \geq 0$$

(12) If the condition (6) does not hold, then the numerator in Equation (12) is negative for r sufficiently closed to 1.

Hence there exist z0 = r0 in (0, 1) for which the quotient in Equation (12) is negative, therefore there is a contradicts the required condition for $f_k \in A_H^k(\alpha,)(\ \alpha,)$.

Extreme Points Here, we determine the extreme points of the closed convex hull of $A_{\bar{H}}$ (k, α, γ), denoted by clco$A_H^k(\alpha,)(\ \alpha,)$.

Theorem 2.3: Let f_k given by (1.2). Then $f_k \in A_H^k(\alpha,)(\ \alpha,)$ i f and only i f

$f_k(z) = \sum_{n=1}^{\infty} \mathcal{X}_n h_n + \mathcal{Y}_n g_{km}$ where $h_1(z) = z$, $h_n(z) = z - \frac{1}{\frac{(|n-k\ |-\alpha)}{(1-\alpha)}} z^n$ n = 2, 3, . . . ,

$g_{kn}(z) = z + \frac{1}{\frac{(|n-k\ |+\alpha)}{(1-\alpha)}} z^n$ n = 1, 2, . . . ,

and $X_n \geq 0$, $Y_n \geq 0$, $X_1 = 1 - \sum_{n=2}^{\infty}(\mathcal{X}_n + \mathcal{Y}_n) \geq 0$ In particular the extreme points of $A_H^k(\alpha,)(\ \alpha,)$ are $\{h_n\}$ and $\{g_{kn}\}$.

Theorem 2.4: Let the functions $f_{k,i}(z)$, defined by Equation (13) be in the class $A_{\bar{H}}^k(, \alpha)$, for every i = 1, 2, . . . , m. Then the functions $c_i(z)$ defined by $c_i(z) = \sum_{i=1}^{\infty} t_i\ f_{k,i}(z)$ $0 \leq t_i \leq 1$ are also in the class $A_{\bar{H}}$ (α,) where $\sum_{i=1}^{\infty} t_i = 1$. 2.4. Convolution (Hadamard Product) Property

Here, we show that the class $A_{\bar{H}}^k(\alpha,)$ is closed under convolution. The convolution of two harmonic functions

$z - \sum_{n=2}^{\infty} |a_n|z^n + (-1)^n \sum_{k=1}^{\infty} |b_n|\, z^{-n}$ (14) and

$Q_n(z) = z - \sum_{n=2}^{\infty} |L_n|z^n + (-1)^n \sum_{k=1}^{\infty} |M_n|\, z^{-n}$ (15) is defined as

$(f_n * Q_n)(z) = f_n(z) * Q_n(z) = z - z - \sum_{n=2}^{\infty} |a_n L_n|z^n + (-1)^n \sum_{k=1}^{\infty} |b_n M_n|\, z^{-n}$ (16)

Using Equations (12)–(14), we prove the following theorem.

Theorem 2.5: For $0 \le \mu \le \alpha < 1$, $k \in N_0$, let $f_n \in A_{\bar{H}}^{k}$ (α,) and $Q_n \in A_{\bar{H}}^{k}$(μ,) Then $f_n * Q^n \in A_{\bar{H}}^{k}$ (α) ⊂ $A_{\bar{H}}^{k}$ (μ,).

3. INTEGRAL OPERATOR

Here, we examine the closure property of the class $A_{\bar{H}}^{k}$ (α,) under the generalized Bernardi-Libera-Livingston integral operator (see References [10,11]) $L_u(f)$ which is defined by

$$L_u(f) = \frac{u+1}{z^u} \int_o^z t^{u-1} f(t)dt, \; u > -1. \tag{17}$$

Theorem 3.1: Let $f_k(z) \in A_{\bar{H}}^{k}$ (k, α, γ). Then $L_u(f_k(z)) \in A_{\bar{H}}^{k}$ (α)

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