International Journal of Mathematical Archive-13(2), 2022, 6-8
Available online through www.ijma.info ISSN 2229 – 5046

A UNIQUE COMMON FIXED POINT THEOREM IN COMPLETE METRIC SPACE

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(Received On: 13-01-22; Revised & Accepted On: 03-02-22)

ABSTRACT

In this paper, we prove a generalized unique common fixed point theorem for four self- mappings for reciprocal continuous and weakly compatible mappings in complete metric space, which is a generazation some of the recent results existing in the literature.

***2000 AMS Classification**: 54H25, 47H10.*

***Keywords**: Fixed point, common fixed point, reciprocal continue, and weakly compatible.*

1. INTRODUCTION AND PRELIMINARIES

Banach fixed point theorem has been generalized and extended by many Mathematicians in many ways for e.g. [1 2, 4 ,5] . Recently A.Djoudi [3] proved some results in metric space. Our result is generelization of A.Djoudi [3].

Defination 1.1: [1] Two self maps S and T of a metric space (X, d) are said to be commute if ST=TS., Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 1.2: [2] The maps S and T of a metric space (X, d) are said to be reciprocally continuous if $\lim_{n\to\infty} STx_n = S(t)$ and $\lim_{n\to\infty} TSx_n = T(t)$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = t$ and $\lim_{n\to\infty} Tx_n = t$, for some $t \in X$.

Definition 1.3: [2] Let S, T: X → X. Then the pair (S, T) is called weakly compatible, if ST z = T Sz for all z ∈ X such that Tz = Sz.

Notation 1.1: Let R_+ be the set of non negative real numbers and let $\phi : R_+^{\,5} \to R_+$ be a function satisfying the following conditions:φ is upper semi continuous in each coordinate variable and non decreasing.

$$\phi(t) = \max\{\varphi(0,t,0,0,t), \varphi(t,0,0,t,t), \varphi(t,t,t,2t,0), \varphi(0,0,t,t,0)\} < t, \text{ for any } t > 0.$$

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2. MAIN RESULT

The following result is generalization of the result of [3].

Theorem 2.1: Let S, T, I and J are four self mappings in a complete metric space (X, d) and satisfying the following conditions

(i) $S(X) \subseteq J(X)$ and $T(X) \subseteq I(X)$
(ii) $d(Sx, Ty) \leq \phi\{d(Ix, Jy), d,(Ix, Sx), d(Jy, Ty), d(Ix, Ty), d(Jy, Sx)\}$
(iii) the pair (S,I) is reciprocally continuous and compatible.
(iv) The pair (T, J) is weakly compatible.
(v) the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots. Sx_{2n}, Tx_{2n+1}$ …..converges to $z \in X$. Then S, I, T, and J have a unique comon fixed point in X.

Proof: Let (X, d) be complete metric space, for any $x_0 \in X$ and iterated sequence $\{x_n\}$ for four self maps the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \ldots. Sx_{2n}, Tx_{2n+1}$ … convergence to some point $z \in X$.

From (v) $\quad Sx_{2n} \to z$ and $Tx_{2n+1} \to z$ as $n \to \infty$ … (1)

Since (S, I) is reciprocal continuous $SIx_{2n} \to Sz$ and $ISx_{2n} \to Iz$ as $n \to \infty$. From the compatibility of the pair (S, I) gives $\lim_{n\to\infty} d(SIx_{2n}, ISx_{2n}) = 0$. Implies $d(Sz, Iz) = 0$, that is $Sz = Iz$. Since $S(X) \subseteq J(X) \Rightarrow$ there exists $u \in X$ such that $Ju = z$. and $T(X) \subseteq I(X) \Rightarrow$ there exists $v \in X$ such that $Iv = z$. Now to prove $Sz = z$, put $x = z$ and $y = x_{2n+1}$ in (ii) we get that

$$d(Sz, Tx_{2n+1}) \leq \phi\{d(Iz, Jx_{2n+1}), d(Iz, Sz), d(Jx_{2n+1}, Tx_{2n+1}), d(Iz, Tx_{2n+1}), d(Jx_{2n+1}, Sz)\}.$$

Letting $n\to\infty$,

$$d(Sz, z) \leq \phi\{d(Iz, z), d(Sz, Sz), d(z, z), d(Iz, z), d(z, Sz)\}.$$
$$d(Sz, z) \leq \phi\{d(Sz, z), d(Sz, z), d(z, Sz)\}.$$
$$d(Sz, z) \leq \phi\{d(Sz, z)\} < d(Sz, z), \text{ which is a contradiction. Therefore } Sz = z.$$

To prove $Tu = z$, put $x = x_{2n}$ and $y = u$ in (ii) we get that

$$d(Sx_{2n}, Tu) \leq \phi\{d(Ix_{2n}, Ju), d(Ix_{2n}, Sx_{2n}), d(Ju, Tu), d(Ix_{2n}, Tu), d(Ju, Sx_{2n})\}.$$

Letting $n\to\infty$,

$$d(z, Tu) \leq \phi\{d(z, Ju), d(z, z), d(z, Tu), d(z, Tu), d(Ju, z)\}.$$
$$d(z, Tu) \leq \phi\{d(z, z), d(z, z), d(z, Tu), d(z, Tu), d(z, z)\}.$$
$$d(z, Tu) \leq \phi\{d(z, Tu), d(z, Tu) < d(z, Tu), \text{ which is a contradiction. Therefore } Tu = z.$$

Hence $Tu = Ju = z$.

Since, (I, J) is weakly compatible $\Rightarrow TJu = Jtu \Rightarrow Tz = Jz$.

To prove $Tz = z$.

put $x = x_{2n}$ and $y = z$ in (ii) we get that

$$d(Sx_{2n}, Tz) \leq \phi\{d(Ix_{2n}, Jz), d(Ix_{2n}, Sx_{2n}), d(Jz, Tz), d(Ix_{2n}, Tz), d(Jz, Sx_{2n})\}.$$

Letting $n\to\infty$,

$$d(z, Tz) \leq \phi\{d(z, Jz), d(z, z), d(z, Tz), d(z, Tz), d(Jz, z)\}.$$
$$d(z, Tz) \leq \phi\{d(z, z), d(z, z), d(z, Tz), d(z, Tz), d(z, z)\}.$$
$$d(z, Tz) \leq \phi\{d(z, Tz), d(z, Tz) < d(z, Tz), \text{ which is a contradiction. Therefore } Tz = z.$$

Hence $Sz = Tz = z$.

To prove $Iz = z$.

put $x = Iz$ and $y = x_{2n+1}$ in (ii) we get that

$$d(SIz_{2n}, Tx_{2n+1}) \leq \phi\{d(IIz, Jx_{2n+1}), d(IIz, SIz), d(Jx_{2n+1}, Tx_{2n+1}), d(IIz, Tx_{2n+1}), d(Jx_{2n+1}, SIz)\}.$$

Letting $n\to\infty$,

$$d(Iz, z) \leq \phi\{d(Iz, z), d(Iz, Iz), d(z, z), d(Iz, z), d(z, Iz)\}.$$
$$d(Iz, z) \leq \phi\{d(Iz, z), d(Iz, z), d(z, Iz)\}.$$
$$d(Iz, z) \leq \phi\{d(Iz, z)\} < d(Iz, z), \text{ which is a contradiction. Therefore } Iz = z.$$

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To prove Jz = z. put x = z and y = Jz in (ii) we get that

$d(Sz, TJz) \leq \phi\{d(Iz, JJz), d(Iz, Sz), d(JJz, TJz), d(Iz, TJz), d(JJz, Sz)\}$.

$d(z, Jz) \leq \phi\{d(z, Jz), d(z, z), d(Jz, Jz), d(z, Jz), d(Jz, z)\}$.

$d(z, Jz) \leq \phi\{d(z, Jz), d(z, Jz), d(z, Jz)\}$.

$d(z, Jz) \leq \phi\{d(z, Jz)\} < d(z, Tz)$, which is a contradiction. Therefore Jz = z.

Therefore, Jz = Iz = z. Hence, Tz = Sz = Js = Iz = z.

Therefore, S, T, I, and J have a unique common fixed point in X. This completes the proof of the theorem.

Remark: Our theorem is generalization of the theorem of [3], which is a more general the results of [3].

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Source of support: Nil, Conflict of interest: None Declared.

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