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CERTAIN SUBCLASS OF MULTIVALENT α - SPIRALLIKE FUNCTIONS

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ABSTRACT

The aim of this paper is to study certain subclasses of multivalent α- spiral starlike functions and α- spiral convex functions defined by subordination. We obtain an upper bound estimate for the second Hankel determinant of functions belonging to these classes using Toeplitz determinants. Also, the bounds rendered in this paper generalize some previous results.

Keywords and Phrases: *Analytic function, convex $\alpha -$ spiral function, multivalent function,α- spiral starlike function, Hankel determinant, Toeplitz determinants.*

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1. INTRODUCTION

For a fixed integer $p \geq 1$, let A_p denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z: |z| < 1\}$ with $p \in N = \{1,2,3,\cdots\}$.

Let S be the subclass of $A_1 =: A$, consisting of univalent functions.

Let Ω be the class of Schwarzian functions

$$w(z) = \sum_{n=p+1}^{\infty} d_n z^n,$$

which are analytic in the open unit disc $\mathbb{U} = \{z: |z| < 1\}$ and satisfies the conditions $w(0) = 0$ and $|w(z)| < 1$.

Let f and g be analytic functions in $\mathbb{U}$, we say that f is subordinate to g, written as $f \prec g$ if there exist a Schwarz function $w \in \Omega$, such that $f(z) = g(w(z)), (z \in \mathbb{U})$ [2].

In 1976, Noonan and Thomas [18] defined the q^{th} Hankel determinant of f given by (1.1) for integers $n \geq 1$ and $q \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has been investigated by several authors in the literature [1, 18]. It is interesting to note that the Hankel determinants $H_2(1) = |a_3 - a_2^2|$ and $H_2(2) = |a_2 a_4 - a_3^2|$ are well known as Fekete-Szegö functional and second Hankel determinant respectively.

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The Fekete-Szegö problem for the well known classes

$$ST := \left\{ f \in A : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathbb{U} \right\}$$

and

$$CV := \left\{ f \in A : Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in \mathbb{U} \right\}$$

was investigated by Keogh and Merkes [12]. Recently, many authors have discussed upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [3, 4, 22] and references therein.

Janteng et al. discussed the Hankel determinant problem for the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points in [9] and for the functions whose derivative has a positive real part in [10]. In their work, they have shown that if $f \in RT$ then $|a_2a_4 - a_3^2| \le \frac{4}{9}$. In [11], the authors also obtained the second Hankel determinant and sharp bounds for the familiar subclasses of S, namely starlike and convex functions denoted by ST and CV and showed that $|a_2a_4 - a_3^2| \le 1$ and $|a_2a_4 - a_3^2| \le \frac{1}{8}$ respectively. Mishra and Gochchayat [16] have obtained the sharp bound to the non-linear functional $|a_2a_4 - a_3^2|$ for the class of analytic functions denoted by $R_\lambda(\alpha,\rho), \left(0 \le \rho \le p, |\alpha| < \frac{\pi}{2p}\right)$.

Analogous to the Hankel determinant of univalent functions, we consider the Hankel determinant in the case $q = 2$ and $n = p$, known as second Hankel determinant for multivalent functions given by

$$H_2(p) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix}.$$

Estimate on the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the classes of p-valent starlike and p-valent convex functions were obtained by Krishna and Ramreddy [21]. However, for any real number μ, the sharp estimate on the functional $|a_{p+2} - \mu a_{p+1}^2|$ for the classes of p-valent starlike and convex funtions of order α were obtained by Hayami and Owa [7].

Inspired by the earlier works obtained by different researchers in this direction, we in the present paper, obtain an upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the functions f belonging to multivalent α-spiral starlike and α-spiral convex functions which are defined as follows.

Definition 1.1: *For $-1 \le B < A \le 1$, a function $f \in A_p$, given by (1.1), is said to be p-valently α −spiral starlike if it satisfies the inequality*

$$\frac{e^{-i\alpha} zf'(z)}{pf(z)} \prec \cos\,\alpha\left(\frac{1+Az}{1+Bz}\right) - is\,in\ \alpha, \quad for \quad all \quad z \in \mathbb{U}, |\alpha| \le \frac{\pi}{2p}. \tag{1.2}$$

We denote this class of functions by $SP_{p,\alpha}(A,B)$.

By specializing on the values of A, B, α and p, we obtain subclasses of analytic functions that were studied earlier in literature.

1. $SP_{p,\alpha}(1,-1) = SP_p(\alpha)$, the class of p-valently $\alpha-$ spiral functions.
2. $SP_{p,0}(1-2\alpha,-1) = ST_p(\alpha)$, the class of p-valent starlike functions of order α was studied by Hayami and Owa[7] and Vamshee Krishna *et al.* [21]
3. $SP_{1,0}(A,B) = ST^*(A,B)$, the subclass of starlike functions was studied by Goel and Mehrok [5] and G.Singh *et al.* [4].
4. $SP_{1,\alpha}(1,-1) = SP(\alpha)$, the class of α −spiral functions introduced by Spacek [20].
5. $SP_{1,0}(1,-1) = ST$, the class of starlike functions, studied by Janteng *et al.* [11].

Definition 1.2: *For $-1 \le B < A \le 1$, a function $f \in A_p$, given by (1.1), is said to be p-valently convex α −spiral if it satisfies the inequality*

$$\frac{1}{p}\left[e^{-i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \cos\alpha\left(\frac{1+Az}{1+Bz}\right) - i\sin\,\alpha, \quad for \quad all \quad z \in \mathbb{U}, |\alpha| \le \frac{\pi}{2p}. \tag{1.3}$$

We denote this class of functions by $CVSP_{p,\alpha}(A,B)$.

By specializing on the values of A, B, α and p, we obtain subclasses of analytic functions that were studied earlier in literature.

1. $CVSP_{p,\alpha}(1,-1) = CVSP_p(\alpha)$, the class of p-valently convex $\alpha-$ spiral functions.
2. $CVSP_{p,0}(1-2\alpha,-1) = CV_p(\alpha)$, the class of p-valent convex functions of order α was studied by Hayami and Owa [7] and Vamshee Krishna *et al.* [21].
3. $CVSP_{1,0}(A,B) = K(A,B)$, the subclass of convex functions was studied by Goel and Mehrok [5] and G.Singh *et al.* [4].
4. $CVSP_{1,\alpha}(1,-1) = CVSP(\alpha)$, the class of convex $\alpha-$spiral functions studied by Vamshee Krishna *et al.* [22].
5. $CVSP_{1,0}(1,-1) = CV$, the class of convex functions, studied by Janteng *et al.* [11].

In order to prove our main results, we shall need the following preliminary lemma.

Let P denote the class of functions $p(z)$ of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \tag{1.4}$$

which are analytic in the open unit disk $\mathbb{U}$ for which $Re\{p(z)\} > 0$.

Lemma 1.3: *[19] If the function $p \in P$ is given by the series (1.4), then the following sharp estimate holds :*

$$|c_k| \leq 2, \ k = 1,2,\cdots. \tag{1.5}$$

Lemma 1.4: *[13, 14] If the function $p \in P$ is given by the series (1.4), then*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$
$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y,$$

for some x, y with $|x| \leq 1, |y| \leq 1$ and $c_1 \in [0,2]$.

2. MAIN RESULTS ON α- SPIRAL STARLIKE FUNCTION

Theorem 2.1: *Let the function f given by (1.1) be in the class $SP_{p,\alpha}(A,B)$. Then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{(A-B)^2 p^2 \cos^2(\alpha)}{4}\left(-\frac{\pi}{2p} \leq \alpha \leq \frac{\pi}{2p}\right). \tag{2.1}$$

Proof: If $f(z) \in SP_{p,\alpha}(A,B)$, then there exist a Schwarz function $w(z) \in \Omega$ such that

$$\frac{e^{-i\alpha} z f'(z)}{p f(z)} = \cos\alpha \ \phi\big(w(z)\big) - i \sin \ \alpha, \quad z \in U \tag{2.2}$$

where

$$\begin{aligned}\phi(z) = \frac{1+Az}{1+Bz} &= 1 + (A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + \cdots \\ &= 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \end{aligned} \tag{2.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad z \in \mathbb{U} \tag{2.4}$$

Since $w(z)$ is a schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$.

Define the function $h(z)$ by

$$h(z) = \frac{e^{-i\alpha} z f'(z) + ip \sin \ \alpha f(z)}{p f(z) \cos\alpha}, \quad z \in \mathbb{U} \tag{2.5}$$

In view of the equations (2.2), (2.4) and (2.5), we have

$$\begin{aligned} h(z) &= \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1 z + c_2 z^2 + c_3 z^3 + \dots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots}\right) \\ &= \phi\left(\frac{1}{2}c_1 z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right) \\ &= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 \\ &\quad + \left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right] z^3 + \cdots \end{aligned} \tag{2.6}$$

From (2.5), we have

$$e^{-i\alpha} z f'(z) + ip \ \sin \ \alpha f(z) = p \ \cos \ \alpha \ \{f(z) \times h(z)\}. \tag{2.7}$$

Replacing $f(z), f'(z)$ by their equivalent p −valent expressions and also the equivalent expression for $h(z)$ in (2.6), we have

$$e^{-i\alpha} z\left(pz^{p-1} + \sum_{n=p+1}^{\infty} na_n z^{n-1}\right) + ip\ \sin\alpha\left(z^p + \sum_{n=p+1}^{\infty} a_n z^n\right)$$
$$= p\cos\ \alpha \left[\left(z^p + \sum_{n=p+1}^{\infty} a_n z^n\right) \times \left(1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 \right.\right.$$
$$\left.\left. + \left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right] z^3 + \cdots\right)\right].$$

Upon simplification, we obtain

$$e^{-i\alpha}\left(a_{p+1} z^{p+1} + 2a_{p+2} z^{p+2} + 3a_{p+3} z^{p+3} + \cdots\right)$$
$$= p\cos\ \alpha\left(z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + a_{p+3} z^{p+3} + \cdots\right)$$
$$\left(\frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots \qquad (2.8)$$

Equating the coefficients of like powers of z^{p+1}, z^{p+2} and z^{p+3} respectively in (2.8), we have

$$a_{p+1} e^{-i\alpha} = \frac{B_1 c_1}{2} p\cos\ \alpha,$$
$$2a_{p+2} e^{-i\alpha} = a_{p+1} \frac{B_1 c_1}{2} p\cos\ \alpha + \left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right) p\cos\ \alpha,$$
$$3a_{p+3} e^{-i\alpha} = a_{p+2} \frac{B_1 c_1}{2} p\cos\ \alpha + a_{p+1}\left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right) p\cos\ \alpha$$
$$+ \left[\frac{B_1}{2}\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{B_2 c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3 c_1^3}{8}\right] p\cos\ \alpha.$$

After simplifying using (2.3), we obtain

$$a_{p+1} = e^{i\alpha} \frac{(A-B)c_1}{2} p\cos\ \alpha,$$
$$a_{p+2} = \frac{e^{i\alpha}(A-B)}{8}[e^{i\alpha}(A-B)c_1^2 p\cos\ \alpha + 2c_2 - c_1^2 - Bc_1^2] p\cos\ \alpha,$$
$$a_{p+3} = \frac{e^{i\alpha}}{48}(A-B)[8c_3 + e^{2i\alpha}(A-B)^2 c_1^3 p^2 \cos^2(\alpha) - 3(A-B)e^{i\alpha} p\cos\ \alpha c_1^3$$
$$-3B(A-B)e^{i\alpha} c_1^3 + (2 + 4B + 2B^2)c_1^3 + (6e^{i\alpha} p\cos\ \alpha(A-B) - 8 - 8B)c_1 c_2] p\cos\ \alpha.$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} in the second Hankel functional, we have

$$|a_{p+1} a_{p+3} - a_{p+2}^2| = \left|\frac{e^{2i\alpha}(A-B)^2 p^2 \cos^2\alpha}{192}[16c_3 c_1 - e^{2i\alpha}(A-B)^2 c_1^4 p^2 \cos^2\alpha\right.$$
$$\left. -12c_2^2 + (1+B)^2 c_1^4 - 4(1+B)c_2 c_1^2]\right|.$$

By using the facts $|xp + yq| \leq x|p| + y|q|$, where x, y, p and q are real numbers and $|e^{ni\alpha}| = 1$, upon simplification , we obtain

$$|a_{p+1} a_{p+3} - a_{p+2}^2| \leq \frac{(A-B)^2 p^2 \cos^2\alpha}{192} |[16c_3 c_1 - (A-B)^2 c_1^4 p^2 \cos^2\alpha$$
$$-12c_2^2 + (1+B)^2 c_1^4 - 4(1+B)c_2 c_1^2]|. \qquad (2.9)$$

Substituting the values of c_2 and c_3 from Lemma 1.4, we have

$$|a_{p+1} a_{p+3} - a_{p+2}^2| \leq \frac{(A-B)^2 p^2 \cos^2\alpha}{192} |4c_1[c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2$$
$$+2(4 - c_1^2)(1 - |x|^2)y]\ - (A-B)^2 c_1^4 p^2 \cos^2\alpha + (1+B)^2 c_1^4$$
$$-3[c_1^4 + (4 - c_1^2)^2 x + 2c_1^2(4 - c_1^2)x] - 2(1+B)c_1^2[c_1^2 + (4 - c_1^2)x]|.$$

Assume that $c_1 = c$ and $c \in [0,2]$, using triangular inequality and $|y| \leq 1$, we have

$$|a_{p+1} a_{p+3} - a_{p+2}^2| \leq \frac{(A-B)^2 p^2 \cos^2\alpha}{192}(|10 + 4B + B^2|c^4 + (A-B)^2 p^2 \cos^2\alpha c^4$$
$$+|(16 + 2B)c^2(4 - c^2)|\delta + |(12 - 7c^2 - 8c)(4 - c^2)|\delta^2|)$$
$$= \frac{(A-B)^2 p^2 \cos^2\alpha}{192} F(c,\delta), \quad where \quad \delta = |x| \leq 1 \quad and$$

$$F(c,\delta) = |10 + 4B + B^2|c^4 + (A-B)^2 p^2 \cos^2\alpha\ c^4 + |[(16 + 2B)c^2(4 - c^2)]|\delta + |[(12 - 7c^2 - 8c)(4 - c^2)]|\delta^2. \qquad (2.10)$$

Now the function $F(c,\delta)$ is maximized on the closed square $[0,2] \times [0,1]$. Differentiating $F(c,\delta)$ in (2.10), partially with respect to δ, we get

$$\frac{\partial F}{\partial \delta} = |[(16 + 2B)c^2(4 - c^2)]| + [(12 - 7c^2 - 8c)(4 - c^2)]|(2\delta). \qquad (2.11)$$

For $0 < \delta < 1,$ and for fixed c with $0 < c < 2,$ from (2.11) we observe that $\frac{\partial F}{\partial \delta} > 0.$

Consequently, $F(c, \delta)$ is an increasing function of δ and hence cannot have maximum value at any point in the interior of the closed square [0,2] $\times$ [0,1]. Moreover, for fixed $c \in [0,2],$ we have

$$\max_{0 \le \delta \le 1} F(c, \delta) = F(c, 1) = G(c). \tag{2.12}$$

Upon simplifying the relation (2.10) and (2.12), we obtain

$$G(c) = (A-B)^2 p^2 cos^2\alpha\ c^4 + (1+B)^2 c^4 + 8c^3 + |(24+8B)|c^2 - 32c + 48. \tag{2.13}$$

Differentiation yields :

$$G'(c) = (A-B)^2 p^2 cos^2\alpha(4c^3) + (1+B)^2(4c^3) + 24c^2 + 2|(24+8B)|c - 32. \tag{2.14}$$

From the expression (2.14), we observe that $G'(c) \le 0$ from all values of c in the interval $0 \le c \le 2$ and for a fixed valued of α with $\left(\frac{-\pi}{2p} \le \alpha \le \frac{\pi}{2p}\right)$.

Therefore, G(c) is a monotonically decreasing function of c in the interval [0,2]. So, that its maximum value occurs at c = 0. From (2.13), we get

$$\max_{0 \le c \le 2} G(0) = 48. \tag{2.15}$$

After simplifying the expressions (2.9) and (2.15) we obtain

$$|16c_3c_1 - (A-B)^2 c_1^4 p^2 cos^2\alpha - 12c_2^2 + (1+B)^2 c_1^4 - 4(1+B)c_2c_1^2| \le 48. \tag{2.16}$$

Upon simplifying the expressions (2.9) and (2.16), we get

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \frac{(A-B)^2 p^2 cos^2(\alpha)}{4}. \tag{2.17}$$

Choosing $c_1 = c = 0$ and selecting $x = -1$ in Lemma 1.3, we find that $c_2 = -2$ and $c_3 = 0.$ Substituting these values in (2.16), it is observed that equality is attained which shows that our result is sharp. This completes the proof.

Choosing $A = 1, B = -1,$ Theorem 2.1 gives the following result.

Corollary 2.2: If $f(z) \in SP_p(\alpha)\left(\frac{-\pi}{2p} \le \alpha \le \frac{\pi}{2p}\right)$, then $|a_{p+1}a_{p+3} - a_{p+2}^2| \le p^2 cos^2(\alpha).$

Choosing $A = 1 - 2\beta\ \ (0 \le \beta < 1), B = -1, \alpha = 0$ Theorem 2.1 gives the following result.

Corollary 2.3: If $f(z) \in ST_p(\beta)\left(0 \le \beta \le \left(p - \frac{1}{2}\right)\right)$, then $|a_{p+1}a_{p+3} - a_{p+2}^2| \le (p-\beta)^2.$

Choosing $p = 1, \alpha = 0$ Theorem 2.1 gives the following result.

Corollary 2.4: If $f(z) \in ST^*(A, B)(-1 \le B < A \le 1)$, then $|a_2a_4 - a_3^2| \le \frac{(A-B)^2}{4}.$

Choosing $A = 1, B = -1,$ and $p = 1$ Theorem 2.1 gives the following result.

Corollary 2.5 *If $f(z) \in SP(\alpha)$ and for $\left(\frac{-\pi}{2} \le \alpha \le \frac{\pi}{2}\right)$ then $|a_2a_4 - a_3^2| \le cos^2(\alpha).$*

Choosing $A = 1, B = -1, p = 1$ and $\alpha = 0$ Theorem 2.1 gives the following result.

Corollary 2.6: *If $f(z) \in ST,$ then $|a_2a_4 - a_3^2| \le 1.$*

This inequality is sharp and coincides with that of Janteng, Halim and Darus [11].

3. MAIN RESULTS ON α- SPIRAL CONVEX FUNCTION

Theorem 3.1: *Let the function f given by (1.1) be in the class $CVSP_{p,\alpha}(A, B)$. Then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \frac{p^4(A-B)^2 cos^2(\alpha)[6-(1+B)(p^2+4p+7)+3(A-B)pcos\ \ \alpha]^2 + 48(p+1)(p+3)\Delta(A,B)}{12(p+1)(p+2)^2(p+3)\Delta(A,B)} \tag{3.1}$$

where

$$\Delta(A,B) = 2(p^2+4p+7)+(A-B)^2(p^2+4p+1)p^2cos^2(\alpha)+6(A-B)pcos(\alpha)(1+B)$$
$$-4(1+B)(p^2+4p+1)-(p^2+4p+7)(B^2+4B+3).$$

Proof: If $f(z) \in CVSP_{p,\alpha}(A,B)$, then there exist a Schwarz function $w(z) \in \Omega$ such that

$$\frac{1}{p}\left[e^{-i\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)\right] = cos\alpha\ \phi(w(z)) - isin\ \alpha, \quad z \in \mathbb{U} \tag{3.2}$$

where

$$\phi(z) = \frac{1+Az}{1+Bz} = 1+(A-B)z - B(A-B)z^2 + B^2(A-B)z^3 + \cdots$$
$$= 1 + B_1z + B_2z^2 + B_3z^3 + B_4z^4 + \cdots. \tag{3.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots, \quad z \in \mathbb{U} \tag{3.4}$$

Since $w(z)$ is a schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1.$ Define the function $h(z)$ by

$$h(z) = \frac{e^{-i\alpha}[f'(z)+zf''(z)]+ipsin\ \alpha f'(z)}{pcos\ \alpha f'(z)}, \quad z \in \mathbb{U} \tag{3.5}$$

In view of the equations (3.2), (3.4) and (3.5), we have

$$h(z) = \phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = \phi\left(\frac{c_1z+c_2z^2+c_3z^3+\cdots}{2+c_1z+c_2z^2+c_3z^3+\cdots}\right)$$
$$= \phi\left(\frac{1}{2}c_1z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right)$$
$$= 1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2$$
$$+\left[\frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}\right]z^3 + \cdots. \tag{3.6}$$

From (3.5), we have

$$e^{-i\alpha}[f'(z) + zf''(z)] + ipsin\ \alpha f'(z) = pcos\ \alpha\{f'(z) \times h(z)\}.$$

Replacing $f'(z)$ *and* $f''(z)$ by their equivalent p-valent expressions and also the equivalent expression for $h(z)$ in (3.6), we have

$$e^{-i\alpha}\left(pz^{p-1} + \sum_{n=p+1}^{\infty} na_nz^{n-1}\right) + z\left(p(p-1)z^{p-2} + \sum_{n=p+1}^{\infty} n(n-1)a_nz^{n-2}\right)$$
$$+ipsin\alpha\left(pz^{p-1} + \sum_{n=p+1}^{\infty} na_nz^{n-1}\right)$$
$$= pcos\ \alpha\left[\left(pz^{p-1} + \sum_{n=p+1}^{\infty} na_nz^{n-1}\right) \times \left\{1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2\right.\right.$$
$$\left.\left.+\left[\frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}\right]z^3 + \ldots\right\}\right]. \tag{3.7}$$

Equating the coefficients of like powers of z^p, z^{p+1} and z^{p+2} respectively in (3.7), we have

$$(p+1)a_{p+1}e^{-i\alpha} = p\frac{B_1c_1}{2}pcos\ \alpha,$$
$$2(p+2)a_{p+2}e^{-i\alpha} = (p+1)a_{p+1}\frac{B_1c_1}{2}pcos\ \alpha + p\left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right)pcos\ \alpha,$$
$$3(p+3)a_{p+3}e^{-i\alpha} = (p+2)a_{p+2}\frac{B_1c_1}{2}pcos\ \alpha + (p+1)a_{p+1}\left(\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right)pcos\ \alpha$$
$$+p\left[\frac{B_1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right) + \frac{B_2c_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_3c_1^3}{8}\right]pcos\ \alpha.$$

After simplifying using (3.3), we obtain

$$a_{p+1} = e^{i\alpha}\frac{(A-B)c_1}{2(p+1)}p^2cos\ \alpha,$$
$$a_{p+2} = \frac{e^{i\alpha}(A-B)}{8(p+2)}[e^{i\alpha}(A-B)c_1^2pcos\ \alpha + 2c_2 - c_1^2(1+B)]p^2cos\ \alpha,$$
$$a_{p+3} = \frac{e^{i\alpha}}{48(p+3)}(A-B)[8c_3 + e^{2i\alpha}(A-B)^2c_1^3p^2cos^2(\alpha)$$
$$-3(A-B)e^{i\alpha}pcos\ \alpha c_1^3(1+B) + 2(1+B)^2c_1^3 + 6e^{i\alpha}pcos\ \alpha(A-B)c_1c_2 - 8c_1c_2(1$$
$$+B)]p^2cos\ \alpha.$$

Substituting the values of a_{p+1}, a_{p+2} and a_{p+3} in the second Hankel functional and applying the same procedure as described in Theorem 2.1, upon simplification, we have

$$\begin{aligned}|a_{p+1}a_{p+3}-a_{p+2}^2| \leq \frac{p^4cos^2(\alpha)(A-B)^2}{12(p+1)(p+2)^2(p+3)}\,&|[(p+2)^2c_1c_3 - c_1^2c_2(1+B)(p+2)^2\\ &+\frac{c_1^4}{4}(p+2)^2(1+B)^2+\frac{3}{4}(A-B)c_1^2c_2(p+2)^2pcos\ \alpha\\ &-\frac{3}{8}(A-B)c_1^4pcos\ \alpha(1+B)(p+2)^2\\ &+\frac{(A-B)^2}{8}c_1^4p^2cos^2(\alpha)(p+2)^2-3(p+1)(p+3)\frac{c_2^2}{4}\\ &-\frac{3}{16}(p+1)(p+3)c_1^4(1+B)^2-\frac{3}{16}(A-B)^2(p+1)(p+3)c_1^4p^2cos^2(\alpha)\\ &+\frac{3}{4}c_1^2c_2(1+B)(p+1)(p+3)+\frac{3}{8}(A-B)c_1^4(1+B)pcos\ \alpha(p+1)(p+3)\\ &-\tfrac{3}{4}(p+1)(p+3)(A-B)c_1^2c_2pcos\ \alpha]|. \qquad (3.8)\end{aligned}$$

The above expression is equivalent to

$$|a_{p+1}a_{p+3}-a_{p+2}^2| \leq \frac{p^4cos^2(\alpha)(A-B)^2}{12(p+1)(p+2)^2(p+3)} \times |d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| \qquad (3.9)$$

where

$$\begin{aligned}d_1 &= (p+2)^2\\ d_2 &= -(1+B)(p+2)^2+\frac{3}{4}(1+B)(p^2+4p+3)+\frac{3}{4}(A-B)pcos\ \alpha\\ d_3 &= -\frac{3}{4}(p^2+4p+3)\\ d_4 &= \frac{1}{16}(1+B)^2(p^2+4p+7)-\frac{3}{8}(A-B)(1+B)pcos\ \alpha-\frac{1}{16}(A-B)^2p^2cos^2\alpha(p^2+4p+1)\end{aligned}$$

Substituting the values of c_2 and c_3 from Lemma (1.4) in the right hand side of (3.9), we have

$$\begin{aligned}&|d_1c_1c_3+d_2c_1^3c_2+d_3c_2^2+d_4c_1^4| =\\ &|d_1c_1\times\tfrac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)y\}+d_2c_1^2 \qquad (3.10)\\ &\times\tfrac{1}{2}\{c_1^2+x(4-c_1^2\}+d_3\times\tfrac{1}{4}\{c_1^2+x(4-c_1^2)\}^2+d_4c_1^4|.\end{aligned}$$

After simplifying, we get

$$\begin{aligned}&4|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4| =\\ &|(d_1+2d_2+d_3+4d_4)c_1^4+2d_1c_1(4-c_1^2)y \qquad (3.11)\\ &\quad+2(d_1+d_2+d_3)c_1^2(4-c_1^2)|x|-\{(d_1+d_3)c_1^2+2d_1c_1-4d_3\}(4-c_1^2)|x|^2y|.\end{aligned}$$

Substituting the values of d_1, d_2, d_3 and d_4, we obtain

$$\begin{aligned}d_1+2d_2+d_3+4d_4 &= \frac{1}{4}(p^2+4p+7)+(1+B)(p^2+4p+1)+\frac{1}{4}(1+B)^2(p^2+4p+7)\\ &-\tfrac{3}{2}B(A-B)pcos\ \alpha-\tfrac{1}{4}(A-B)^2(p^2+4p+1)p^2cos^2\alpha, \qquad (3.12)\end{aligned}$$

$$d_1+d_2+d_3 = \tfrac{1}{4}(p^2+4p+7)-\tfrac{1}{4}(1+B)(p^2+4p+7)+\tfrac{3}{4}(A-B)pcos\ \alpha, \qquad (3.13)$$

$$(d_1+d_3)c_1^2+2d_1c_1-4d_3 = \tfrac{1}{4}(p^2+4p+7)c_1^2+2(p^2+4p+4)c_1+3(p^2+4p+3). \qquad (3.14)$$

Consider

$$\begin{aligned}&(p^2+4p+7)c_1^2+8(p^2+4p+4)c_1+12(p^2+4p+3)\\ &\quad=(p^2+4p+7)\times\left[c_1^2+\frac{8(p^2+4p+4)}{p^2+4p+7}c_1+\frac{12(p^2+4p+3)}{p^2+4p+7}\right]\\ &\quad=(p^2+4p+7)\times\left[\left\{c_1+\frac{4(p^2+4p+4)}{(p^2+4p+7)}\right\}^2-\frac{16(p^2+4p+4)}{(p^2+4p+7)^2}+\frac{12(p^2+4p+3)}{(p^2+4p+7)}\right].\end{aligned}$$

Upon simplification, the above expression can also be expressed as

$$(p^2+4p+7)c_1^2+8(p^2+4p+4)c_1+12(p^2+4p+3)$$
$$=(p^2+4p+7)\times\left[\left\{c_1+\frac{4(p^2+4p+4)}{(p^2+4p+7)}\right\}^2-\left\{\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}^2\right].$$
$$(p^2+4p+7)c_1^2+8(p^2+4p+4)c_1+12(p^2+4p+3)$$
$$=(p^2+4p+7)\times\left[c_1+\left\{\frac{4(p^2+4p+4)}{(p^2+4p+7)}+\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right]$$
$$\times\left[c_1+\left\{\frac{4(p^2+4p+4)}{(p^2+4p+7)}-\frac{2\sqrt{p^4+8p^3+18p^2+8p}+1}{(p^2+4p+7)}\right\}\right].\text{(3.15)}$$

Since $c_1\in[0,2]$, using the result $(c_1+a)(c_1+b)\geq(c_1-a)(c_1-b)$, where $a,b>0$ in the right-hand side of (3.15), upon simplification, we obtain

$$(p^2+4p+7)c_1^2+8(p^2+4p+4)c_1+12(p^2+4p+3)\geq$$
$$(p^2+4p+7)c_1^2-8(p^2+4p+4)c_1+12(p^2+4p+3). \tag{3.16}$$

From the relations (3.14) and (3.16), we obtain

$$-4(d_1+d_3)c_1^2+2d_1c_1-4d_3\leq-\{(p^2+4p+7)c_1^2-8(p^2+4p+4)c_1+12(p^2+4p+3)\}. \tag{3.17}$$

Substituting the calculated values from (3.13) and (3.17) in the right-hand side of the relation (3.12), we get

$$\begin{aligned}&16|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\\&\quad\leq|\{(p^2+4p+7)+4(1+B)(p^2+4p+1)+(1+B)^2(p^2+4p+7)\\&\quad-6B(A-B)pcos\ \alpha-(A-B)^2(p^2+4p+1)p^2cos^2\alpha\}c_1^4+8(p^2+4p+4)c_1(4-c_1^2)y\\&\quad+2[(p^2+4p+7)-(1+B)(p^2+4p+7)+3(A-B)pcos\ \alpha]c_1^2(4-c_1^2)|x|\\&\quad+[(p^2+4p+7)c_1^2-8(p^2+4p+4)c_1+12(p^2+4p+3)](4-c_1^2)|x|^2y|.\end{aligned} \tag{3.18}$$

Choosing $c_1=c\in[0,2]$, applying triangle inequality and using $|y|\leq1$, and also replacing $|x|$ by δ in the right hand side of (3.18), it reduces to

$$\begin{aligned}&16|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\\&\quad\leq[\{(p^2+4p+7)+4(1+B)(p^2+4p+1)+(1+B)^2(p^2+4p+7)\\&\quad-6B(A-B)pcos\ \alpha-(A-B)^2(p^2+4p+1)p^2cos^2\alpha\}c_1^4+8(p^2+4p+4)c_1(4-c_1^2)\\&\quad+2[(p^2+4p+7)-(1+B)(p^2+4p+7)+3(A-B)pcos\ \alpha]c_1^2(4-c_1^2)\delta\\&\quad+[(p^2+4p+7)c_1^2-8(p^2+4p+4)c_1+12(p^2+4p+3)](4-c_1^2)\delta^2]\\&\quad=F(c,\delta),\quad for\quad 0\leq\delta=|x|\leq1.\end{aligned} \tag{3.19}$$

We assume that the upper bound for (3.19) occurs at an interior point of the set $\{\delta,c):\delta\in[0,1]$ and $c\in[0,2]\}$.

Differentiating $F(c,\delta)$ in (3.19) partially with respect to δ, we get

$$\begin{aligned}\frac{\partial F}{\partial\delta}&=\{2(p^2+4p+7)-(1+B)(p^2+4p+7)+3(A-B)pcos\ \alpha\}c^2(4-c^2)\\&\quad+\{2(p^2+4p+7)c^2-8(p^2+4p+4)c+12(p^2+4p+3)\}(4-c^2)\delta.\end{aligned} \tag{3.20}$$

For $0\leq\delta\leq1$, and for fixed c with $0\leq c\leq2$ and $\left(\frac{-\pi}{2p}\leq\alpha\leq\frac{\pi}{2p}\right)$,from (3.20) we observe that $\frac{\partial F}{\partial\delta}>0$.

Consequently, $F(c,\delta)$ is an increasing function of δ and hence cannot have maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$, we have

$$\max_{0\leq\delta\leq1}F(c,\delta)=F(c,1)=G(c). \tag{3.21}$$

Upon simplifying the relation (3.19) and (3.21), we obtain

$$\begin{aligned}G(c)&=[\{-2(p^2+4p+7)+4(1+B)(p^2+4p+1)+(p^2+4p+7)(B^2+4B+3)\\&\quad-6(A-B)pcos\ \alpha(1+B)-(A-B)^2(p^2+4p+1)p^2cos^2\alpha\}]c^4\\&\quad+[48-8(1+B)(p^2+4p+7)+24(A-B)pcos\ \alpha]c^2+48(p+1)(p+3)\end{aligned} \tag{3.22}$$

Differentiation yields:

$$\begin{aligned} G'(c) = {} & [\{-2(p^2+4p+7)+4(1+B)(p^2+4p+1)+(p^2+4p+7)(B^2+4B+3) \\ & -6(A-B)p\cos\ \alpha(1+B)-(A-B)^2(p^2+4p+1)p^2\cos^2\alpha\}](4c^3) \\ & +[48-8(1+B)(p^2+4p+7)+24(A-B)p\cos\ \alpha](2c). \end{aligned} \tag{3.23}$$

$$\begin{aligned} G''(c) = {} & [\{-2(p^2+4p+7)+4(1+B)(p^2+4p+1)+12(p^2+4p+7)(B^2+4B+3) \\ & -(A-B)p\cos\ \alpha(1+B)-(A-B)^2(p^2+4p+1)p^2\cos^2\alpha\}](12c^2) \\ & +[48-8(1+B)(p^2+4p+7)+24(A-B)p\cos\ \alpha](2). \end{aligned} \tag{3.24}$$

The maximum of minimum value of G(c) is obtained for the values of $G'(c)=0$. From the expression (3.23), we get

$$\begin{aligned} G'(c) = {} & [\{-2(p^2+4p+7)+4(1+B)(p^2+4p+1)+(p^2+4p+7)(B^2+4B+3) \\ & -6(A-B)p\cos\ \alpha(1+B)-(A-B)^2(p^2+4p+1)p^2\cos^2\alpha\}](4c^3) \\ & +[48-8(1+B)(p^2+4p+7)+24(A-B)p\cos\ \alpha](2c)=0. \end{aligned} \tag{3.25}$$

We now discuss the following cases.

Case-1: If $c=0$, then from (3.24), we obtain

$$G''(c)=96-16(1+B)(p^2+4p+7)+48(A-B)p\cos\ \alpha>0 \quad because \quad |\alpha|\le\frac{\pi}{2p}.$$

Therefore, by the second derivative test, $G(c)$ has a minimum value at $c=0$, which is ruled out.

Case-2: If $c\neq 0$, then from (3.25), we obtain

$$c^2=\frac{4(6-(1+B)(p^2+4p+7)+3(A-B)p\cos\ \alpha)}{\Delta(A,B)} \tag{3.26}$$

where

$$\begin{aligned} \Delta(A,B) = {} & 2(p^2+4p+7)+(A-B)^2(p^2+4p+1)p^2\cos^2(\alpha)+6(A-B)p\cos(\alpha)(1+B) \\ & -4(1+B)(p^2+4p+1)-(p^2+4p+7)(B^2+4B+3) \end{aligned}$$

Using the value of c^2 in (3.24), after simplifying, we get

$$G''(c)=-(192-32(1+B)(p^2+4p+7)+96(A-B)p\cos\ \alpha)<0 \quad because \quad |\alpha|\le\frac{\pi}{2p}.$$

From the second derivative test, $G(c)$ has a maximum value at c, where c^2 is given by (3.26).

From the expression (3.22), we have G-maximum value at c^2, after simplifying it is given by

$$\max_{0\le c\le 2} G(c)=\frac{16[6-(1+B)(p^2+4p+7)+3(A-B)p\cos\ \alpha]^2+48(p+1)(p+3)\Delta(A,B)}{12(p+1)(p+2)^2(p+3)\Delta(A,B)} \tag{3.27}$$

where

$$\begin{aligned} \Delta(A,B) = {} & 2(p^2+4p+7)+(A-B)^2(p^2+4p+1)p^2\cos^2(\alpha)+6(A-B)p\cos(\alpha)(1+B) \\ & -4(1+B)(p^2+4p+1)-(p^2+4p+7)(B^2+4B+3) \end{aligned}$$

Considering only the maximum value of $G(c)$ at c, where c^2 is given by (3.26). From the expression (3.19) and (3.27), upon simplification , we obtain

$$|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4|\le\frac{[6-(1+B)(p^2+4p+7)+3(A-B)p\cos\ \alpha]^2+48(p+1)(p+3)\Delta(A,B)}{12(p+1)(p+2)^2(p+3)\Delta(A,B)} \tag{3.28}$$

From the expression (3.9) and (3.28), after simplifying, we get

$$\begin{aligned} & |a_{p+1}a_{p+3}-a_{p+2}^2|\le \\ & \frac{p^4(A-B)^2\cos^2(\alpha)[6-(1+B)(p^2+4p+7)+3(A-B)p\cos\ \alpha]^2+48(p+1)(p+3)\Delta(A,B)}{12(p+1)(p+2)^2(p+3)\Delta(A,B)} \end{aligned} \tag{3.29}$$

where

$$\begin{aligned} \Delta(A,B) = {} & 2(p^2+4p+7)+(A-B)^2(p^2+4p+1)p^2\cos^2(\alpha)+6(A-B)p\cos(\alpha)(1+B) \\ & -4(1+B)(p^2+4p+1)-(p^2+4p+7)(B^2+4B+3) \end{aligned}$$

Choosing $x=-1$ in Lemma 1.3, we find that $c_2=c_1^2-2$ and $c_3=c_1^3-3c_1$. The result is sharp for $c_1=c$, $c_2=c^2-2$ and $c_3=c^3-3c$ where c^2 is given by (3.26).

This completes the proof.

Choosing $A=1, B=-1$, Theorem 3.1 gives the following result.

Corollary 3.2: If $f(z) \in CVSP_p(\alpha)\left(\frac{-\pi}{2p} \leq \alpha \leq \frac{\pi}{2p}\right)$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^4(6(1+2pcos(\alpha)+p^2cos^2(\alpha)+(p+1)(p+3)(p^2+4p+7+2(p^2+4p+1)p^2cos^2(\alpha))}{(p+1)(p+2)^2(p+3)\{2(p^2+4p+1)+(p^2+4p+7)p^2sec^2(\alpha)\}}.$$

Choosing $A = 1 - 2\beta(0 \leq \beta < 1), B = -1, \alpha = 0$ Theorem 3.1 gives the following result.

Corollary 3.3: If $f(z) \in CV_p(\beta)\left(0 \leq \beta \leq \left(p - \frac{1}{2}\right)\right)$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{p^2(p-\beta)^2[6(p+1-\beta)^2+(p+1)(p+3)\{2\beta(\beta-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)\}]}{(p+1)(p+2)^2(p+3)[2\beta(\beta-2p)(p^2+4p+1)+(2p^4+8p^3+3p^2+4p+7)]}.$$

Choosing $p = 1, \alpha = 0$ Theorem 3.1 gives the following result.

Corollary 3.4: If $f(z) \in K(A,B)(-1 \leq B < A \leq 1)$, then

$$|a_2a_4 - a_3^2| \leq \frac{(A-B)^2}{576}\left[\frac{16|-A^2+2B^2+AB|-|A-5B|^2-12|A-5B|-36}{|-A^2+2B^2+AB|-|A-5B|-2}\right].$$

Choosing $A = 1, B = -1,$ and $p = 1$ Theorem 3.1 gives the following result.

Corollary 3.5: If $f(z) \in CVSP(\alpha)$ and for $\left(\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}\right)$ then $|a_2a_4 - a_3^2| \leq \frac{17(1+cos^2\alpha)+2cos\ \alpha}{144(1+sec^2\alpha)}$.

Choosing $A = 1, B = -1, p = 1$ and $\alpha = 0$ Theorem 3.1 gives the following result.

Corollary 3.6: If $f(z) \in CV$, then $|a_2a_4 - a_3^2| \leq \frac{1}{8}$.

This inequality is sharp and coincides with that of Janteng, Halim and Darus [11].

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