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COMMON FIXED-POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS USING COMMON PROPERTY (E.A) IN FUZZY METRIC SPACE

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ABSTRACT

The aim of present paper is to prove common fixed-point theorems for four self-mappings in fuzzy metric spaces using the common property (E. A.) satisfying an implicit relation.

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Keywords: *Fuzzy metric spaces, the property (E. A.), the common property (E. A.).*

INTRODUCTION

The foundation of fuzzy mathematics is laid by Zadeh [16] with the introduction of fuzzy sets in 1965. This foundation represents a vagueness in everyday life. Subsequently several authors have applied various form general topology of fuzzy sets and developed the concept of fuzzy space. Kramosil and Michalek [9] introduced concept of fuzzy metric spaces. Grabiec [6] extended fixed point theorem of Banach and Eldestien to fuzzy metric spaces in the sense of Kramosil and Michalek [8]. George *et al.* [5] modified the notion of fuzzy metric spaces with the help of continuous t-norms. A number of fixed-point theorem have been obtained by various authors in metric spacesand fuzzy metric spaces by using the concept of compatible, implicit relations, weakly compatible, R weakly compatible maps. (See, [2–15]). Saini and Gupta [11, 12] proved some fixed points theorems on expansion type maps and common coincidence points of R-Weakly commuting fuzzy maps in Fuzzy Metric Space. In this paper, the concept of implicit relation has been used for establishing common fixed-point results in a fuzzy metric space. This concept plays a vital role in the proof of the main results.

2. BASIC DEFINITIONS AND PRELIMINARIES

Definition 2.1: [13] A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* * satisfies the following conditions:

i. * is continuous,
ii. * is commutative and associative,
iii. a * 1 =a for all a ∈ [0, 1],
iv. a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0,1].

Examples of *t-norm* - a * b = ab and a * b = min{a, b}.

Definition 2.2: [9] A 3- tuple $(X, M,*)$ is called fuzzy metric space if X is an arbitrary non empty set, * is a continuous *t-norm*, and M, is fuzzy sets on $X^2 \times [0, \infty]$ satisfying the following conditions:

For each x, y, z, ∈X and t, s > 0

1. $M(x, y, t) > 0$
2. $M(x, y, t) = 1 \; iff \; x = y$
3. $M(x, y, t) = M(y, x, t)$
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$
5. $M(x, y, .): [0, \infty) \to [0,1]$ is left continuous,

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Example 2.3: [5] Let (X, d) be a metric space. Define a * b = ab for all a, b ∈ [0, 1] and let M be fuzzy seton $X^2 \times (0, \infty) \to [0,1]$ defined as follows:

$$M(x,y,t)=\frac{t}{t+d(x,y)} \text{ for all x, y} \in X \text{ and all } t > 0.$$

This fuzzy metric induced by a metric d is called the standard fuzzy metric and $(X, M,*)$ is called fuzzy metric space.

Lemma 2.1: [6]. For all, $x, y \in X$, $M(x,y,t)$ is non-decreasing.

Lemma 2.2: [10] Let $(X, M, *)$ be a fuzzy metric space, if there exists k ∈ (0, 1) such that for all x, y ∈ X,

$M(x,y,kt) \geq M(x,y,t)$for all t > 0, then x = y.

Definition 2.4: [1] A pair (*A*, *S*) of self-mappings of a fuzzy metric space$(X, M,*)$is said to satisfy the (*E.A.*) property if there exists a sequence$\{x_n\}$ in *X* such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \text{ for some } z \in X..$$

Definition 2.5: [3] Two pairs (*A*, *S*) and (*B*, *T*) of self-mappings of a fuzzy metric space $(X, M, *)$ are said to satisfy the common (*E.A*) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that for all t > 0

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z \text{ for some } z \in X$$

Definition 2.6: [8] A pair (*f*, *g*) of self-mappings of a metric space (*X*, *d*) is said to be weakly compatible mappings if the mappings commute at all of their coincidence points, i.e.,

$$fx = gx \text{ for some } x \in X \text{implies } fgx = gfx.$$

IMPLICIT RELATION

Let M_5 denotes the set of all real valued continuous function $\phi: [0,1]^5 \to R$ which are non- decreasing and satisfying the following conditions:

(A) $\phi(u,1,u,1,u) \geq 0$ *implies* $u \geq 1$
(B) $\phi(u,1,1,u,u) \geq 0$ *implies* $u \geq 1$
(C) $\phi(u,u,1,1,u) \geq 0$ *implies* $u \geq 1$

Example2.7: [10] Define $\phi: [0,1]^5 \to R$ as

$$\phi(t_1,t_2,t_3,t_4,t_5) = 11t_1 - 12t_2 + 6t_3 - 8t_4 + 3t_5$$

Clearly ϕ satisfies all condition (A), (B), (C). Therefore $\phi \in M_5$.

3. MAIN RESULTS

We now establish the following results.

Theorem 3.1: Let *A*, *B*, *S and T* be self-mappings of a fuzzy metric space (X, M,*) satisfying the following conditions that:

(i) the pair (A, S) or (B, T) satisfies the property (E.A);
(ii) for any x, y ∈ X, $\phi \in M_5$ and for all t > 0, there exists $\alpha \in (0,1)$such that

$$\phi(M(Ax,By,\alpha t), M(Sx,Ty,t), M(Sx,Ax,t), M(Ty,By,{}^{\alpha}/_{2}\, t), M(Ax,Ty,t) * M(Sx,By,t)) \geq 0$$

(iii) *A*(*X*) ⊂ *T*(*X*) *orB*(*X*) ⊂*S*(*X*).

Then the pairs (A, S) and (B, T) share the common property (E.A).

Proof: Suppose that the pair (*A*, *S*) satisfies property (*E.A*), then there exists a sequence $\{x_n\}$ in *X* such that

$\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = z$ for some z ∈ X. Since *A*(*X*) ⊂*T*(*X*),

therefore, for each x_n, there exist $y_n$in *X* such that $Ax_n = Ty_n$. This gives,

$$\lim_{n\to\infty} A x_n = \lim_{n\to\infty} S x_n = \lim_{n\to\infty} Ty_n = z.$$

Now, we claim that $\lim_{n\to\infty} By_n = z$.

Applying inequality (ii), we obtain

$$\phi(M(Ax_n,By_n,\alpha t), M(Sx_n,Ty_n,t), M(Sx_n,Ax_n,t), M(Ty_n,By_n,{}^{\alpha}/_{2}\, t), M(Ax_n,Ty_n,t) * M(Sx_n,By_n,t)) \geq 0$$

Taking limit as $n \to \infty$

$$\phi(M\left(z, \lim_{n\to\infty} By_n, \alpha t\right), M(z,z,t), M(z,z,t), M\left(z, \lim_{n\to\infty} By_n, {}^{\alpha}/_{2}\, t\right), M(z,z,t) * M\left(z, \lim_{n\to\infty} By_n, t\right)) \geq 0$$

Since ϕ is non-decreasing in the first argument, we have

$$\phi(M\left(z, \lim_{n\to\infty} By_n, t\right), 1, 1, M\left(z, \lim_{n\to\infty} By_n, t\right), M\left(z, \lim_{n\to\infty} By_n, t\right)) \geq 0$$

Using (B), we get

$$M\left(z, \lim_{n\to\infty} By_n, t\right) \geq 1$$

Hence

$$M\left(z, \lim_{n\to\infty} By_n, t\right) = 1.$$

Therefore $\lim_{n\to\infty} By_n = z$.

Hence the pairs (A, S) and (B, T) share the common property (*E.A*).

Similarly, if the pair (B, T) satisfies property (*E.A*) and $B(X) \subset S(X)$, then pairs (*A*, *S*) and (*B*, *T*) share the common property (*E.A*).

Theorem 3.2: Let *A*, *B*, *S and T* be self-mappings of a fuzzy metric space (X, M,*) satisfying the following conditions that:

(i) for any x, y ∈ X, $\phi \in M_5$ and for all t > 0, there exists $\alpha \in (0,1)$ such that

$$\phi(M(Ax,By,\alpha t), M(Sx,Ty,t), M(Sx,Ax,t), M\left(Ty,By,{}^{\alpha}/_{2}\, t\right), M(Ax,Ty,t) * M(Sx,By,t)) \geq 0$$

(ii) the pairs(A, S) and(B, T) share the common property (E.A);

(iii) S(X) and T(X) are closed subsets of X.

Then each of the pairs (A, S) and (B, T) have a point of coincidence. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.

Proof: Since the pairs (A, S) and (B, T) share the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$in X such that $\lim_{n\to\infty} A\,x_n = \lim_{n\to\infty} S\,x_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some z ∈ *X*.S(X) is closed subset of X, there exists a point *u* ∈ *X* such that *z* = *Su*.

We, now claim that *Au* = *z*. By (i), we have

$$\phi(M(Au,By_n,\alpha t), M(Su,Ty_n,t), M(Su,Au,t), M\left(Ty_n,By_n,{}^{\alpha}/_{2}\, t\right),$$
$$M(Au,Ty_n,t) * M(Su,By_n,t)) \geq 0$$

Taking limit as $n \to \infty$,

$$\phi(M(Au,z,\alpha t), M(z,z,t), M(z,Au,t), M\left(z,z,{}^{\alpha}/_{2}\, t\right), M(Au,z,t) * M(z,z,t)) \geq 0$$

As ϕ is non-decreasing in the first argument, we have

$$\phi(M(Au,z,t), 1, M(z,Au,t), 1, M(Au,z,t)) \geq 0$$

Using implicit relations (*A*), we have

$$M(Au,z,t) \geq 1$$

Hence $M(Au, z, t) = 1$.

Therefore, *Au* = *z* = *Su* which shows that *u* is a coincidence point of the pair (*A*, *S*).

Since *T*(*X*) is also a closed subset of *X*, therefore, $\lim_{n\to\infty} Ty_n = z$ in *T*(*X*) and hence there exists $v \in X$ such that *Tv* = *z* = *Au* = *Su*. Now, we show that *Bv*= *z*.

By using inequality (i), we have

$$\phi(M(Au,Bv,\alpha t), M(Su,Tv,t), M(Su,Au,t), M\left(Tv,Bv,{}^{\alpha}/_{2}\, t\right), M(Au,Tv,t) * M(Su,Bv,t)) \geq 0$$

it follows

$$\phi(M(z,Bv,\alpha t), M(z,z,t), M(z,z,t), M\left(z,Bv,{}^{\alpha}/_{2}\, t\right), M(z,z,t) * M(z,Bv,t)) \geq 0$$

As $\phi$is a non-decreasing in the first argument, we have

$$\phi(M(z,Bv,t), 1, 1, M(z,Bv,t), M(z,Bv,t)) \geq 0$$

Using implicit relations (*B*), we get

$$M(z, Bv, t) \geq 1.$$

Hence $$M(z, Bv, t) = 1$$

Therefore, *Bv*= *z* = *Tv*, which shows that *v* is a coincidence point of the pair (*B*, *T*).

Moreover, since the pairs (*A*, *S*) and (*B*, *T*) are weakly compatible and *Au* = *Su*, *Bv*= *Tv*, therefore, *Az* = *ASu*= *SAu*= *Sz*, *Bz*= *BTv*= *TBv*= *Tz.*

Next, we claim that *Az* = *z* for showing the existence of a fixed point of *A*. By using inequality (i), we have

$$\phi(M(Az, Bv, \alpha t), M(Sz, Tv, t), M(Sz, Az, t), M(Tv, Bv, {}^{\alpha}/_{2}\, t), M(Az, Tv, t) * M(Sz, Bv, t)) \geq 0$$

it follows that

$$\phi(M(Az, z, \alpha t), M(Az, z, t), M(Az, Az, t), M(z, z, {}^{\alpha}/_{2}\, t), M(Az, z, t) * M(Az, z, t)) \geq 0$$

Since $\phi$is a non-decreasing in the first argument, we have

$$\phi(M(Az, z, t), M(Az, z, t), 1,1, M(Az, z, t)) \geq 0$$

On using implicit relations (*C*), we get

$$M(Az, z, t) \geq 1$$

Hence, *M*(*Az*, *z*, *t*) =1. Therefore, *Az* = *z* = *Sz*.

Similarly, we can prove that *Bz*= *Tz*= *z*. Hence, *Az* = *Bz*= *Sz*= *Tz*= *z*, which implies that *z* is a common fixed point of *A*, *B*, *S* and *T*.

Uniqueness: Let *w* be another common fixed point of *A*, *B*, *S* and *T*. Then by using (i),

$$\phi(M(Az, Bw, \alpha t), M(Sz, Tw, t), M(Sz, Az, t), M(Tw, Bw, {}^{\alpha}/_{2}\, t), M(Az, Tw, t) * M(Sz, Bw, t)) \geq 0$$

it follows that

$$\phi(M(z, w, \alpha t), M(z, w, t), M(z, z, t), M(w, w, {}^{\alpha}/_{2}\, t), M(z, w, t) * M(z, w, t)) \geq 0$$

Since $\phi$is a non-decreasing in the first argument, we have

$$\phi(M(z, w, t), M(z, w, t), 1,1, M(z, w, t)) \geq 0$$

Using implicit relations (*C*), we have

$$M(z, w, t) \geq 1.$$

Hence $M(z, w, t)$ =1.

Therefore, z = w, i.e., mappings A, B, S and T have a unique common fixed point.

Taking B = A and T = S in the Theorem 3.2. yields following corollary:

Corollary 3.1: Let A and S be self-mappings of a fuzzy metric space(X, M,*) satisfying the following conditions that

(i) the pair (A, S) share the property *(E.A);*
(ii) for any x, y ∈ X, $\phi \in M_5$ and for all t > 0, there exists $\alpha \in (0, 1)$ such that
(iii) $\phi(M(Ax, Ay, \alpha t), M(Sx, Sy, t), M(Sx, Ax, t), M(Sy, Ay, {}^{\alpha}/_{2}\, t), M(Ax, Sy, t) * M(Sx, Ay, t)) \geq 0$
(iv) S(X) is a closed subset of X.

Then A and S each have a point of coincidence. Moreover, if the pair (A, S) is weakly compatible, then A and S have a unique common fixed point.

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