

CLASS OF ANALYTIC FUNCTIONS CONSTRUCTED USING q - DERIVATIVE OPERATOR

Dr. GURMEET SINGH
H.O.D. of Deptt. Of Mathematics,
Khalsa college, Patiala; 147002, Punjab, India.

MISHA RANI*
Research Fellow, Department of Mathematics,
Punjabi University, Patiala-147002, Punjab, India.

(Received On: 02-08-21; Revised & Accepted On: 12-08-21)

ABSTRACT

In this paper, we are using a q - derivative operator of quantum calculus. By using this operator, we define Fekete – Szegő Inequality for a new class of analytic functions.

Keywords – analytic function, Fekete – Szegő Inequality, q - derivative operator, concept of subordination, starlike functions.

2010 Mathematics Subject Classification: 30C45, 30C50.

INTRODUCTION

Here, we are dealing with q – calculus and as we know, it has many applications on various branches of mathematics, so till now many researchers have worked on it. First of all, the concept of q – derivative and q – integral was developed, which was given by Jackson [9,10]. After that, Aral and Gupta proved an operator based on q - analogue [2, 3]. Recently, the concept of q - derivative operator was defined by Abdullah Alsoboh and Maslina Darus [1] which was based on q – operator and the concept of q – operator was studied by Mohammed and Darus[16].

\mathcal{A} be the family of analytic functions of the type

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

with the normalization $f(0) = 0, f'(0) = 1$.

\mathcal{S} be the family of functions of the type

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j$$

with the normalization $f(0) = 0, f'(0) = 1$ and univalent in the open disk $E = \{z \in \mathbb{C} : |z| < 1\}$.

$S^*(\phi)$ be the class of functions in $f \in \mathcal{S}$, for which

$$\frac{z f'(z)}{f(z)} < \phi(z)$$

which was introduced by Ma and Minda [15].

In this equation, “ $<$ ” denotes subordination [which states that if $f(z)$ and $g(z)$ are two analytic functions, then there exists a Schwarzian function $w(z)$ (which is analytic in E) in such a way that $|w(z)| < 1, w(0) = 0$ and $f(z) = g(w(z))$; $z \in E$, then the function $f(z)$ is subordinate to $g(z)$ and we write it as $f(z) < g(z)$].

The concept of subordination was given by Lindelof [13]. Here, $\phi(z)$ is an analytic function with positive real part on E ; which maps the unit disk E onto a region starlike with respect to 1 as well as symmetric with respect to real axis, satisfying conditions $\phi(0) = 0$ and $\phi'(0) > 0$.

Corresponding Author: Misha Rani*
Research Fellow, Department of Mathematics,
Punjabi University, Patiala-147002, Punjab, India.

Miller *et.al* [14] proved the conditions

$$|c_1| \leq 1, |c_2| \leq 1-|c_1|^2$$

for Schwarzian function, which is a function of the type $w(z) = \sum_{n=1}^{\infty} c_n z^n$, with the conditions $w(0) = 0$ and $|w(z)| < 1$. For $j \in \mathbb{N}$,

the quantum number, $|j|_q = \frac{q^j - 1}{q - 1}$; $0 < q < 1, z \neq 0$

and the quantum derivative, $D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}$; $q \neq 0, z \neq 0$.

Here $|j|_q \rightarrow j$ and $D_q f(z) \rightarrow f'(z)$, as $q \rightarrow 1^-$.

Now, a new class of q- starlike of order β for $0 \leq \beta < 1, 0 < q < 1$ and $n \in \mathbb{N}$, is defined by

$$S_{q,n}^*(\beta) = \left\{ f \in A : \operatorname{Re} \left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} \right) > \beta ; z \in E \right\}$$

And a new q - differential operator denoted by $M_q^n f(z)$, defined as

$$M_q^0 f(z) = f(z), M_q^1 f(z) = z D_q f(z) = z + \sum_{j=2}^{\infty} |j|_q a_j z^j$$

$$M_q^n f(z) = z D_q \{M_q^{n-1} f(z)\} = z + \sum_{j=2}^{\infty} |j|_q^n a_j z^j$$

The terms $S_{q,n}^*(\beta)$ and $M_q^n f(z)$ were defined by Abdullah Alsoboh and Maslina Darus [1].

It is to be noticed here that

$$S_{q,0}^*(\beta) \equiv S^*(\beta); q \rightarrow 1$$

And this concept was introduced by Seoudy and Aouf [21].

Now by using all the above defined terms, we are proving this inequality for the class $TS_{q,n}^*(\alpha, \beta)$ defined below

$$(1-\alpha) \frac{M_q^n f(z)}{z} + \alpha \left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} \right) < \phi(z).$$

As $\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} < \phi(z)$; proved by Abdullah Alsoboh and Maslina Darus [1].

MAIN RESULTS

Theorem 1: Let $f(z) \in TS_{q,n}^*(\alpha, \beta)$ and $\phi(z) = \frac{1+w(z)}{1-w(z)}$; $w(z)$ is a Schwarzian function, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS_{q,n}^*(\alpha, \beta)$,

$$(1-\alpha) \frac{M_q^n f(z)}{z} + \alpha \left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} \right) = \frac{1+w(z)}{1-w(z)} \tag{1.1}$$

By putting all the values in (1.1), we get

$$1 + (1-2\alpha + \alpha |2|_q) |2|_q^n a_2 z + \left[(1-2\alpha + \alpha |3|_q) |3|_q^n a_3 - \alpha |2|_q^{2n} (|2|_q - 1) a_2^2 \right] z^2 + \dots$$

$$= 1 + 2 c_1 z + 2 (c_2 + c_1^2) z^2 + \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{2c_1}{(1-2\alpha + \alpha |2|_q) |2|_q^n} \text{ and } a_3 = \frac{2c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[2|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha (|2|_q - 1)\}] c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n}.$$

So, we get

$$a_3 - \mu a_2^2 = \frac{2c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[2|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha (|2|_q - 1)\}] c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \mu \frac{4c_1^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{[2|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha (|2|_q - 1)\}]}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \right\} |c_1|^2.$$

Case-1: When $\mu \leq \frac{|2|_q^{2n}\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{2(1-2\alpha + \alpha |3|_q) |3|_q^n}$.

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{4\alpha |2|_q^{2n}(|2|_q - 1)}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \right\} |c_1|^2.$$

Subcase-1 (a): If $\mu \leq \frac{\alpha |2|_q^{2n}(|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \tag{1.2}$$

Subcase-1 (b): If $\mu \geq \frac{\alpha |2|_q^{2n}(|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \tag{1.3}$$

Case-2: When $\mu \geq \frac{|2|_q^{2n}\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{4|2|_q^{2n}\{(1-2\alpha + \alpha |2|_q)^2 + \alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} \right\} |c_1|^2.$$

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n}\{(1-2\alpha + \alpha |2|_q)^2 + \alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} \tag{1.4}$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n}\{(1-2\alpha + \alpha |2|_q)^2 + \alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \tag{1.5}$$

Combining (1.2), (1.3), (1.4) and (1.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z [1 + az]^n$$

where $a = \frac{2(1-2\alpha + \alpha |3|_q) |3|_q^n - 2|2|_q^{2n}\{\alpha |2|_q(\alpha |2|_q - 4\alpha + 4) + (1 + 4\alpha^2 - 6\alpha)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q) |3|_q^n |2|_q^{2n}}$

and $n = \frac{(1-2\alpha + \alpha |3|_q) |3|_q^n}{(1-2\alpha + \alpha |3|_q) |3|_q^n - 2|2|_q^{2n}\{\alpha |2|_q(\alpha |2|_q - 4\alpha + 4) + (1 + 4\alpha^2 - 6\alpha)\}}$.

For second equation, extremal is

$$f(z) = z [1 + 2z^2]^{\frac{1}{(1-2\alpha + \alpha |3|_q) |3|_q^n}}.$$

Corollary 2: $TS_{q,n}^*(1, \beta) = TS_{q,n}^*(\beta)$; as by substituting $\alpha = 1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(|2|_q - 1) + 4}{|3|_q^n (|3|_q - 1)(|2|_q - 1)} - \frac{4\mu}{(|2|_q - 1)^2 |2|_q^{2n}}; \mu \leq \frac{(|2|_q - 1) |2|_q^{2n}}{|3|_q^n (|3|_q - 1)}; \\ \frac{2}{|3|_q^n (|3|_q - 1)}; \frac{(|2|_q - 1) |2|_q^{2n}}{|3|_q^n (|3|_q - 1)} \leq \mu \leq \frac{(|2|_q - 1) |2|_q^{2n+1}}{|3|_q^n (|3|_q - 1)}; \\ \frac{4\mu}{(|2|_q - 1)^2 |2|_q^{2n}} - \frac{2(|2|_q - 1) + 4}{|3|_q^n (|3|_q - 1)(|2|_q - 1)}; \mu \geq \frac{(|2|_q - 1) |2|_q^{2n+1}}{|3|_q^n (|3|_q - 1)}. \end{cases}$$

which is the required result given by Abdullah Alsoboh and Maslina Darus [1].

Theorem 3: Let $f(z) \in TS_{q,n}^*(\alpha, \beta, \delta)$ and $\phi(z) = \left(\frac{1+w(z)}{1-w(z)}\right)^\delta$; $w(z)$ is a Schwarzian function, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\delta^2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{|2|_q^{2n}\{(\delta^2 - \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{|2|_q^{2n}\{(\delta^2 - \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n}\{(\delta^2 + \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\delta^2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n}\{(\delta^2 + \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS_{q,n}^*(\alpha, \beta, \delta)$,

$$(1-\alpha) \frac{M_q^n f(z)}{z} + \alpha \left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} \right) = \left(\frac{1+w(z)}{1-w(z)} \right)^\delta \quad (3.1)$$

By putting all the values in (3.1), we get

$$1 + (1-2\alpha + \alpha |2|_q) |2|_q^n a_2 z + [(1 - 2\alpha + \alpha |3|_q) |3|_q^n a_3 - \alpha |2|_q^{2n} (|2|_q - 1) a_2^2] z^2 + \dots \\ = 1 + 2\delta c_1 z + 2 (\delta c_2 + \delta^2 c_1^2) z^2 + \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{2\delta c_1}{(1-2\alpha + \alpha |2|_q) |2|_q^n} \text{ and } a_3 = \frac{2\delta c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[2\delta^2 |2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}] c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n}.$$

So, we get

$$a_3 - \mu a_2^2 = \frac{2\delta c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[2\delta^2 |2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}] c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \mu \frac{4\delta^2 c_1^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{[2\delta^2 |2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}]}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \right\} |c_1|^2.$$

Case-1: When $\mu \leq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{2(1-2\alpha + \alpha |3|_q) |3|_q^n}$.

Then, $|a_3 - \mu a_2^2| \leq \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{2(\delta^2 - \delta) |2|_q^{2n} (1-2\alpha + \alpha |2|_q)^2 + 4\alpha\delta^2 |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \right\} |c_1|^2$.

Subcase-1 (a): If $\mu \leq \frac{|2|_q^{2n} \{(\delta^2 - \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{2\delta^2 \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \quad (3.2)$$

Subcase-1 (b): If $\mu \geq \frac{|2|_q^{2n} \{(\delta^2 - \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$ (3.3)

Case-2: When $\mu \geq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2(\delta^2 + \delta) |2|_q^{2n} (1-2\alpha + \alpha |2|_q)^2 + 4\alpha\delta^2 |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n} |3|_q^n} \right\} |c_1|^2$.

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n} \{(\delta^2 + \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{4\mu\delta^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\delta^2 \{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} \quad (3.4)$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n} \{(\delta^2 + \delta)(1-2\alpha + \alpha |2|_q)^2 + 2\alpha\delta^2(|2|_q - 1)\}}{2\delta^2(1-2\alpha + \alpha |3|_q) |3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{2\delta}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$ (3.5)

Combining (3.2), (3.3), (3.4) and (3.5), we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z [1 + az]^n$$

where $a = \frac{2\delta(1-2\alpha + \alpha |3|_q) |3|_q^n - 2\delta |2|_q^{2n} \{\alpha |2|_q (\alpha |2|_q - 4\alpha + 4) + (1 + 4\alpha^2 - 6\alpha)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q) |3|_q^n |2|_q^n}$

and $n = \frac{(1-2\alpha + \alpha |3|_q) |3|_q^n}{(1-2\alpha + \alpha |3|_q) |3|_q^n - |2|_q^{2n} \{\alpha |2|_q (\alpha |2|_q - 4\alpha + 4) + (1 + 4\alpha^2 - 6\alpha)\}}$.

For second equation, extremal is

$$f(z) = z [1 + 2\delta z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}}.$$

Corollary 4: $TS_{q,n}^*(\alpha, \beta, 1) = TS_{q,n}^*(\alpha, \beta)$, as by putting $\delta = 1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n} \{(1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

which is same as $TS_{q,n}^*(\alpha, \beta)$.

Corollary 5: $TS_{q,n}^*(1, \beta, 1) = TS_{q,n}^*(\beta)$, as by putting $\alpha=1$ and $\delta = 1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(|2|_q - 1) + 4}{|3|_q^n (|3|_q - 1) (|2|_q - 1)} - \frac{4\mu}{(|2|_q - 1)^2 |2|_q^{2n}}; \mu \leq \frac{(|2|_q - 1) |2|_q^{2n}}{|3|_q^n (|3|_q - 1)}; \\ \frac{2}{|3|_q^n (|3|_q - 1)}; \frac{(|2|_q - 1) |2|_q^{2n}}{|3|_q^n (|3|_q - 1)} \leq \mu \leq \frac{(|2|_q - 1) |2|_q^{2n+1}}{|3|_q^n (|3|_q - 1)}; \\ \frac{4\mu}{(|2|_q - 1)^2 |2|_q^{2n}} - \frac{2(|2|_q - 1) + 4}{|3|_q^n (|3|_q - 1) (|2|_q - 1)}; \mu \geq \frac{(|2|_q - 1) |2|_q^{2n+1}}{|3|_q^n (|3|_q - 1)}. \end{cases}$$

which is same as that of $TS_{q,n}^*(\beta)$ given by Abdullah Alsoboh and Maslina Darus (1).

Theorem 6: Let $f(z) \in TS_{q,n}^*(\alpha, \beta, A, B)$ and $\phi(z) = \frac{1+A w(z)}{1+B w(z)}$; $w(z)$ is a Schwarzian function, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\{-B(A-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)^2(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{|2|_q^{2n} \{\alpha(A-B)(|2|_q - 1) - (B+1)(1-2\alpha + \alpha |2|_q)^2\}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{|2|_q^{2n} \{\alpha(A-B)(|2|_q - 1) - (B+1)(1-2\alpha + \alpha |2|_q)^2\}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n} \{(1-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)(|2|_q - 1)\}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} + \frac{\{B(A-B)(1-2\alpha + \alpha |2|_q)^2 - \alpha(A-B)^2(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n} \{(1-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)(|2|_q - 1)\}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS_{q,n}^*(\alpha, \beta, A, B)$,

$$(1-\alpha) \frac{M_q^n f(z)}{z} + \alpha \left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)} \right) = \frac{1+A w(z)}{1+B w(z)} \tag{6.1}$$

By putting all the values in (6.1), we get

$$1 + (1-2\alpha + \alpha |2|_q) |2|_q^n a_2 z + \left[(1-2\alpha + \alpha |3|_q) |3|_q^n a_3 - \alpha |2|_q^{2n} (|2|_q - 1) a_2^2 \right] z^2 + \dots = 1 + (A-B)c_1 z + [B(B-A)c_1^2 - (B-A)c_2] z^2 + \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{(A-B)c_1}{(1-2\alpha + \alpha |2|_q) |2|_q^n} \text{ and } a_3 = \frac{(A-B)c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[(A-B)\{\alpha(A-B)(|2|_q - 1) - B(1-2\alpha + \alpha |2|_q)^2\}]c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}.$$

So, we get

$$a_3 - \mu a_2^2 = \frac{(A-B)c_2}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \frac{[(A-B)\{\alpha(A-B)(|2|_q - 1) - B(1-2\alpha + \alpha |2|_q)^2\}]c_1^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \mu \frac{(A-B)^2 c_1^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \left| \frac{[(A-B)\{\alpha(A-B)(|2|_q - 1) - B(1-2\alpha + \alpha |2|_q)^2\}]}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \right| - \frac{(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \right\} |c_1|^2.$$

Case-1: When $\mu \leq \frac{|2|_q^{2n} \{\alpha(A-B)(|2|_q - 1) - B(1-2\alpha + \alpha |2|_q)^2\}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}$.

Then, $|a_3 - \mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{\alpha(A-B)^2 (|2|_q - 1) - (A-B)(B+1)(1-2\alpha + \alpha |2|_q)^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} \right\} |c_1|^2.$

Subcase-1 (a): If $\mu \leq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\{-B(A-B)(1-2\alpha+\alpha|2|_q)^2 + \alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} \quad (6.2)$$

Subcase-1 (b): If $\mu \geq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-(B+1)(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \quad (6.3)$$

Case-2: When $\mu \geq \frac{|2|_q^{2n}\{\alpha(A-B)(|2|_q-1)-B(1-2\alpha+\alpha|2|_q)^2\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} + \left\{ \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{\alpha(A-B)^2(|2|_q-1)+(A-B)(1-B)(1-2\alpha+\alpha|2|_q)^2}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} \right\} |c_1|^2.$$

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\mu(A-B)^2}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} + \frac{\{B(A-B)(1-2\alpha+\alpha|2|_q)^2-\alpha(A-B)^2(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} \quad (6.4)$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n}\{(1-B)(1-2\alpha+\alpha|2|_q)^2+\alpha(A-B)(|2|_q-1)\}}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}$

$$\text{Then, } |a_3 - \mu a_2^2| \leq \frac{(A-B)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \quad (6.5)$$

Combining (6.2), (6.3), (6.4) and (6.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z [1 + az]^n$$

$$\text{where } a = \frac{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n - 2|2|_q^{2n}\{-B\alpha|2|_q(\alpha|2|_q-4\alpha+3)-B(1+4\alpha^2-5\alpha)+\alpha A(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)|3|_q^n|2|_q^{2n}}$$

$$\text{and } n = \frac{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n}{(A-B)(1-2\alpha+\alpha|3|_q)|3|_q^n - 2|2|_q^{2n}\{-B\alpha|2|_q(\alpha|2|_q-4\alpha+3)-B(1+4\alpha^2-5\alpha)+\alpha A(|2|_q-1)\}}.$$

For second equation, extremal is

$$f(z) = z [1 + (A-B)z^2]^{\frac{1}{(1-2\alpha+\alpha|3|_q)|3|_q^n}}.$$

Corollary 7: $TS_{q,n}^*(\alpha, \beta, 1, -1) = TS_{q,n}^*(\alpha, \beta)$, as by putting $A=1$ and $B=-1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n} - \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}}; \mu \leq \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{2}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \frac{\alpha|2|_q^{2n}(|2|_q-1)}{(1-2\alpha+\alpha|3|_q)|3|_q^n} \leq \mu \leq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}; \\ \frac{4\mu}{(1-2\alpha+\alpha|2|_q)^2|2|_q^{2n}} - \frac{2\{(1-2\alpha+\alpha|2|_q)^2+2\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)(1-2\alpha+\alpha|2|_q)^2|3|_q^n}; \mu \geq \frac{|2|_q^{2n}\{(1-2\alpha+\alpha|2|_q)^2+\alpha(|2|_q-1)\}}{(1-2\alpha+\alpha|3|_q)|3|_q^n}. \end{cases}$$

which is same as $TS_{q,n}^*(\alpha, \beta)$.

Corollary 8: $TS_{q,n}^*(1, \beta, 1, -1) = TS_{q,n}^*(\beta)$, as by putting $\alpha=1, A=1$ and $B=-1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)} - \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}}; \mu \leq \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)}; \\ \frac{2}{|3|_q^n(|3|_q-1)}; \frac{(|2|_q-1)|2|_q^{2n}}{|3|_q^n(|3|_q-1)} \leq \mu \leq \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}; \\ \frac{4\mu}{(|2|_q-1)^2|2|_q^{2n}} - \frac{2(|2|_q-1)+4}{|3|_q^n(|3|_q-1)(|2|_q-1)}; \mu \geq \frac{(|2|_q-1)|2|_q^{2n+1}}{|3|_q^n(|3|_q-1)}. \end{cases}$$

which is same as given by Abdullah Alsoboh and Maslina Darus [1].

Theorem 9: Let $f(z) \in TS_{q,n}^*(\alpha, \beta, A, B, \delta)$ and $\phi(z) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\delta$; $w(z)$ is a Schwarzian function, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B)\right\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta^2(A-B)^2(|2|_q - 1)}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n} - \frac{\mu\delta^2(A-B)^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}}; \\ \mu \leq \frac{|2|_q^{2n}[\alpha\delta(A-B)(|2|_q - 1) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}(1-2\alpha + \alpha|2|_q)^2]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n}; \\ \frac{\delta(A-B)}{(1-2\alpha + \alpha|3|_q)|3|_q^n}; \frac{|2|_q^{2n}[\alpha\delta(A-B)(|2|_q - 1) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}(1-2\alpha + \alpha|2|_q)^2]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n} \leq \\ \mu \leq \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1\right\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta(A-B)(|2|_q - 1)\right]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n}; \\ \frac{\mu\delta^2(A-B)^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}} - \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B)\right\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta^2(A-B)^2(|2|_q - 1)}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n}; \\ \mu \geq \frac{|2|_q^{2n}\left[\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) + 1\right\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta(A-B)(|2|_q - 1)\right]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n}. \end{cases}$$

The result is sharp.

Proof: By definition of $TS_{q,n}^*(\alpha, \beta, A, B, \delta)$

$$(1-\alpha)\frac{M_q^n f(z)}{z} + \alpha\left(\frac{z D_q \{M_q^n f(z)\}}{M_q^n f(z)}\right) = \left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\delta \tag{9.1}$$

By putting all the values in (9.1), we get

$$1 + (1-2\alpha + \alpha|2|_q)|2|_q^n a_2 z + \left[(1-2\alpha + \alpha|3|_q)|3|_q^n a_3 - \alpha|2|_q^{2n}(|2|_q - 1)a_2^2\right]z^2 + \dots \\ = 1 + \delta(A-B)c_1 z + [\delta(A-B)c_2 + \frac{\delta}{2}\{\delta(A-B)^2 - (A^2 - B^2)\}c_1^2]z^2 + \dots$$

By comparing the coefficients, we get

$$a_2 = \frac{\delta(A-B)c_1}{(1-2\alpha + \alpha|2|_q)|2|_q^n} \text{ and} \\ a_3 = \frac{\delta(A-B)c_2}{(1-2\alpha + \alpha|3|_q)|3|_q^n} + \frac{[\alpha\delta^2(A-B)^2(|2|_q - 1) + \frac{\delta}{2}\{\delta(A-B)^2 - (A^2 - B^2)\}(1-2\alpha + \alpha|2|_q)^2]c_1^2}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n}.$$

So, we get

$$a_3 - \mu a_2^2 = \frac{\delta(A-B)c_2}{(1-2\alpha + \alpha|3|_q)|3|_q^n} + \\ \frac{\left[\frac{\delta}{2}\{\delta(A-B)^2 - (A^2 - B^2)\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta^2(A-B)^2(|2|_q - 1)\right]c_1^2}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n} - \mu \frac{\delta^2(A-B)^2 c_1^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}}.$$

Applying mode on both sides and using $|c_2| \leq 1 - |c_1|^2$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha + \alpha|3|_q)|3|_q^n} \\ + \left\{ \left| \frac{\left[\frac{\delta}{2}\{\delta(A-B)^2 - (A^2 - B^2)\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta^2(A-B)^2(|2|_q - 1)\right]}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n} - \frac{\mu\delta^2(A-B)^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}} \right| - \frac{\delta(A-B)}{(1-2\alpha + \alpha|3|_q)|3|_q^n} \right\} |c_1|^2.$$

Case-1: When $\mu \leq \frac{|2|_q^{2n}[\alpha\delta^2(A-B)^2(|2|_q - 1) + \frac{\delta}{2}\{\delta(A-B)^2 - (A^2 - B^2)\}(1-2\alpha + \alpha|2|_q)^2]}{\delta^2(A-B)^2(1-2\alpha + \alpha|3|_q)|3|_q^n}$.

Then, $|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha + \alpha|3|_q)|3|_q^n} + \left\{ \frac{\left[\alpha\delta^2(A-B)^2(|2|_q - 1) + \delta(A-B)\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}(1-2\alpha + \alpha|2|_q)^2\right]}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n} - \frac{\mu\delta^2(A-B)^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}} \right\} |c_1|^2.$

Subcase-1 (a): If $\mu \leq \frac{|2|_q^{2n}[\alpha\delta(A-B)(|2|_q - 1) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}(1-2\alpha + \alpha|2|_q)^2]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)\left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B)\right\}(1-2\alpha + \alpha|2|_q)^2 + \alpha\delta^2(A-B)^2(|2|_q - 1)}{(1-2\alpha + \alpha|3|_q)(1-2\alpha + \alpha|2|_q)^2|3|_q^n} - \frac{\mu\delta^2(A-B)^2}{(1-2\alpha + \alpha|2|_q)^2|2|_q^{2n}} \tag{9.2}$$

Subcase-1 (b): If $\mu \geq \frac{|2|_q^{2n}[\alpha\delta(A-B)(|2|_q - 1) + \left\{\frac{\delta}{2}(A-B) - \frac{1}{2}(A+B) - 1\right\}(1-2\alpha + \alpha|2|_q)^2]}{\delta(A-B)(1-2\alpha + \alpha|3|_q)|3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha + \alpha|3|_q)|3|_q^n}$ \tag{9.3}

Case-2: When $\mu \geq \frac{|2|_q^{2n} \{ \alpha \delta^2 (A-B)^2 (|2|_q - 1) + \frac{\delta}{2} \{ \delta (A-B)^2 - (A^2 - B^2) \} (1-2\alpha + \alpha |2|_q)^2 \}}{\delta^2 (A-B)^2 (1-2\alpha + \alpha |3|_q) |3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} + \left\{ \frac{\mu \delta^2 (A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{\alpha \delta^2 (A-B)^2 (|2|_q - 1) + \delta(A-B) \left\{ \frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) + 1 \right\} (1-2\alpha + \alpha |2|_q)^2}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} \right\} |c_1|^2$.

Subcase-2 (a): If $\mu \geq \frac{|2|_q^{2n} \left\{ \left[\frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) + 1 \right] (1-2\alpha + \alpha |2|_q)^2 + \alpha \delta (A-B) (|2|_q - 1) \right\}}{\delta(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}$

By using $|c_1| \leq 1$, we get

$$|a_3 - \mu a_2^2| \leq \frac{\mu \delta^2 (A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{\delta(A-B) \left\{ \frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) \right\} (1-2\alpha + \alpha |2|_q)^2 + \alpha \delta^2 (A-B)^2 (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} \tag{9.4}$$

Subcase-2 (b): If $\mu \leq \frac{|2|_q^{2n} \left\{ \left[\frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) + 1 \right] (1-2\alpha + \alpha |2|_q)^2 + \alpha \delta (A-B) (|2|_q - 1) \right\}}{\delta(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}$

Then, $|a_3 - \mu a_2^2| \leq \frac{\delta(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}$ (9.5)

Combining (9.2), (9.3), (9.4) and (9.5) we get the required result.

Extremal: For first and third equations, extremal is

$$f(z) = z [1 + az]^n$$

where $a = \frac{\delta^2 (A-B)^2 (1-2\alpha + \alpha |3|_q) |3|_q^n - 2|2|_q^{2n} \left\{ \left[\frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) \right] (1-2\alpha + \alpha |2|_q)^2 + \alpha \delta (|2|_q - 1) (A-B) \right\}}{\delta(A-B)(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q) |3|_q^n |2|_q^{2n}}$

and $n = \frac{\delta^2 (A-B)^2 (1-2\alpha + \alpha |3|_q) |3|_q^n}{\delta^2 (A-B)^2 (1-2\alpha + \alpha |3|_q) |3|_q^n - 2|2|_q^{2n} \left\{ \left[\frac{\delta}{2} (A-B) - \frac{1}{2} (A+B) \right] (1-2\alpha + \alpha |2|_q)^2 + \alpha \delta (|2|_q - 1) (A-B) \right\}}$.

For second equation, extremal is

$$f(z) = z [1 + \delta(A-B)z]^{\frac{1}{(1-2\alpha + \alpha |3|_q) |3|_q^n}}$$

Corollary 10: $TS_{q,n}^*(\alpha, \beta, A, B, 1) = TS_{q,n}^*(\alpha, \beta, A, B)$, as by putting $\delta=1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\{-B(A-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)^2(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{|2|_q^{2n} \{ \alpha(A-B)(|2|_q - 1) - (B+1)(1-2\alpha + \alpha |2|_q)^2 \}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{(A-B)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{|2|_q^{2n} \{ \alpha(A-B)(|2|_q - 1) - (B+1)(1-2\alpha + \alpha |2|_q)^2 \}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n} \{ (1-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)(|2|_q - 1) \}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{\mu(A-B)^2}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} + \frac{\{B(A-B)(1-2\alpha + \alpha |2|_q)^2 - \alpha(A-B)^2(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n} \{ (1-B)(1-2\alpha + \alpha |2|_q)^2 + \alpha(A-B)(|2|_q - 1) \}}{(A-B)(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

same as $TS_{q,n}^*(\alpha, \beta, A, B)$.

Corollary 11: $TS_{q,n}^*(\alpha, \beta, 1, -1, 1) = TS_{q,n}^*(\alpha, \beta)$, as by putting $A=1, B=-1$ and $\delta=1$, the result becomes

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n} - \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}}; \mu \leq \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{2}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \frac{\alpha |2|_q^{2n} (|2|_q - 1)}{(1-2\alpha + \alpha |3|_q) |3|_q^n} \leq \mu \leq \frac{|2|_q^{2n} \{ (1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1) \}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}; \\ \frac{4\mu}{(1-2\alpha + \alpha |2|_q)^2 |2|_q^{2n}} - \frac{2\{(1-2\alpha + \alpha |2|_q)^2 + 2\alpha(|2|_q - 1)\}}{(1-2\alpha + \alpha |3|_q)(1-2\alpha + \alpha |2|_q)^2 |3|_q^n}; \mu \geq \frac{|2|_q^{2n} \{ (1-2\alpha + \alpha |2|_q)^2 + \alpha (|2|_q - 1) \}}{(1-2\alpha + \alpha |3|_q) |3|_q^n}. \end{cases}$$

Same as $TS_{q,n}^*(\alpha, \beta)$.

REFERENCES

1. Alsoboh, A. and Darus, M., On Fekete–Szegő problem associated with q-derivative operator, *IOP Conference series: journal of Physics: Conference series* 1212(2019) 012003.
2. Aral, A. and Gupta, V., Generalized q-Baskakov operators. *Mathematica Slovaca*, 61(4), (2011) 619-634.
3. Aral, A; Gupta, V. and Agarwal, P., Applications of q-calculus in operator theory (New York: Springer) (2013).
4. Bieberbach, L., Über die Koeffizientem derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskrsises vermitteln, *S. – B. Preuss. Akad. Wiss.* (1916), 940-955.
5. Branges, L.D., A proof of Bieberbach Conjecture, *Acta. Math.*, 154 (1985), 137-152.
6. Duren, P.L., Coefficient of univalent functions, *Bull. Amer. Math. Soc.*, 83, (1977), 891- 911.

7. Fekete, M. and Szegő, G. Eine Bemerkung uber ungerade schlichte funktionen, *J. London Math. Soc.*, 8, (1933), 85-89.
8. Garabedian, P.R. and Schiffer, M., A Proof for the Bieberbach Conjecture for the fourth coefficient, *Arch. Rational Mech. Anal.*, 4, (1955), 427-465.
9. Jackson F. H., On q -definite integrals Q. J., *Pure Appl. Math.*, 41, (1910), 193-203.
10. Jackson F. H. XI.ion q -functions and a certain difference operator, *Earth and Environmental Science Trans R Soc. Edin*, 46(2), (1909), 253-281.
11. Keogh, F.R. and Merkes, E.P., A coefficient inequality for certain classes of analytic function, *Proc. Of Amer. Math. Soc.*, 20, (1989), 8-12.
12. Koebe, P., Uber Die uniformisierungs beliebig analytischer Kurven, *Nach. Ges. Wiss. Gottingen*, (1907), 633-669.
13. Lindelof, E. Memoire sur certaines inegalities dans la theorie des fonctions monogenes et sur quelques proprietes nouvelles de ces fonctions dans la voisinage d'un point singulier essentiel, *Acta Soc. Sci. Fenn.*, 23, (1909), 481-519.
14. Miller, S.S., Mocanu, P.T. and Reade, M.O., All convex functions are univalent and starlike, *Proc. of Amer. Math. Soc.*, 37, (1973), 553-554.
15. Minda, D. and Ma, W., A unified treatment of some special classes of univalent functions, *In proceedings of the conference on complex analysis*, Z. Li, F. Ren, I. Yang and S. Zhang (Eds), *Int. Press* (1994), 157-169.
16. Mohammed, A and Darus, M., A generalized operator involving the q -hypergeometric function. *Matematiski vesnik*, 65(4), (2013), 454-465.
17. Nevanlinna, R., Uber die Eigenschaften einer analytischen funktion in der umgebung einer singularen stele order Linte, *Acta Soc. Sci. Fenn.*, 50, (1922), 1-46.
18. Pederson, R. A proof for the Bieberbach conjecture for the sixth coefficient, *Arch. Rational Mech. Anal.*, 31, (1968-69), 331-351.
19. Pederson, R. and Schiffer, M., A proof for the Bieberbach conjecture for the fifth coefficient, *Arch. Rational Mech. Anal.*, 45, (1972), 161-193.
20. Rathore, G. S.; Singh, G. and Komawat, L., Coefficient Inequality of a significant class of analytic functions, *Journal of Rajasthan Academy of Physical Sciences*, 19, (2020), 267-274.
21. Seoudy, M. and Aouf, M., Coefficient estimates of new classes of q -starlike and q -convex functions of complex order., *J. Math. Inequal*, 10(1), (2016) 135-145.

Source of support: Nil, Conflict of interest: None Declared.

[Copy right © 2021. This is an Open Access article distributed under the terms of the International Journal of Mathematical Archive (IJMA), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.]