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A STUDY OF FIXED POINT THEORY IN GENERALIZED *b*-METRIC SPACE

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ABSTRACT

Banach Contraction Principle (BCP), also known as Banach's Fixed Point Theorem (BFT), concerns certain contraction mappings of a complete metric space into itself. It states sufficient conditions for the existence and uniqueness of a fixed point. The theorem also provides an iterative process from which we can obtain approximations to the fixed point along with error bounds. In the study of fixed point theory, BCP has been extended and generalized in many different directions in usual metric spaces. This work deals with the introduction of b-metric space and dislocated quasi b-metric space with some examples. Besides this, we also present some fixed point results on b-metric space and dislocated quasi b-metric space.

Key Words: *b-metric space, dislocated quasi b-metric space, fixed point.*

1. INTRODUCTION AND MOTIVATION

So far a bulk numbers of generalizations of the concept of metric spaces have been done. The solutions the fixed point of some mapping play a vital role in solving several problems in pure and applied mathematics. There are bulk numbers of techniques in numerical analysis to find successive approximations to the fixed point of an approximate mapping. Over the last 60 years or so, the fixed point theory has been occupied as a very powerful tool in the study of nonlinear phenomena. Fixed point techniques have been applied in diverse areas such as Biology, Chemistry, Physics, Engineering, Game theory and Economics.

The theorems concerning the existence and properties of fixed point are known as fixed point theorems. The famous fixed point result called Banach Contraction Principle (BCP) is one of the most important results in Mathematical Analysis. It is generalized in many directions. One usual way of improving the Banach contraction principle is to replace the metric space by certain generalized metric spaces. Some problems particularly the problem of the convergence of measurable functions with respect to mesasure leads Czerwik**[2]** to a generalization of metric space and introduced the concept of *b*-metric space. Later it was studied and investigated by Bakhtin**[1]**, Czerwik**[2]** and Iqbal, Batool, Ege and Sen **[3].**Since than the concept of *b*-metric space was generalized in different directions. Several authors proved fixed-point results of single-valued and multivalued operators in *b*-metric spaces. One of such generalization is dislocated quasi *b*-metric space, generalized by Rahman and Sarwar **[7]** and Shoaib and *et.al* **[9].** Also, Kumar, Mishra and Mishra **[5]** studied common fixed point theorems in b-metric spaces. They have also established some fixed point results in this metric space. In the present paper we shall study fixed point theorems in *b*-metric spaces and dislocated quasi *b*-metric space.

2. PRELIMINARIES

In this section we define some important definitions, examples and fundamental results which are useful in the further discussion. The BCP is important as a source of existence and uniqueness theorems in different branches of sciences. This theorem provides an illustration of the unifying power of functional analytic methods.

Theorem 2.1 (Banach's fixed point theorem): [6] The point p is a fixed point of the function $f(x)$ if $f(p) = p$. Note that $f(x)$ has a root at p iff $g(x) = x - f(x)$ has a fixed point at p and $g(x)$ has a fixed point at p iff $f(x) = x - g(x)$ has a root at p.

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Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that

$$d(Tx, Ty) \leq kd(x, y)\ \forall x, y \in X \text{ and } k \in [0, 1).$$

Then, T has a unique fixed point $z$in X, and for any $x \in X$ the sequence of iterates $\{T^n(x)\}$ converges to z.

This principle was extended and improved in many ways and various fixed point theorems were obtained. Two usual ways of extending and improving the BCP are

(i) to extend the contraction condition to more general contraction conditions; and
(ii) to replace the complete metric space (X, d) by certain generalized metric spaces. The most popular iteration procedure, called Picard iteration is used in Banach fixed point theorem for the convergence towards a fixed point which is defined below:

Definition 2.2: (Picard iteration) Banach's Fixed Point Theorem (BFPT) provides us with a constructive procedure for getting better and better approximations of the fixed point. BFPT often guarantees the convergence of the scheme and uniqueness of the solution. Let (X, d) be a metric space and $T: X \to X$. Choose $x_0 \in X$ and define

$$x_1 = Tx_0,\ x_2 = Tx_1, \ldots,$$

and obtain the relation $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. Here, x_n is the n^{th} Picard iterate of x_0 in X. But in many cases, Picard iteration may not converge to the fixed point of the map, some other iterations scheme may be used in such cases. There have been a number of generalizations of the usual notion of a metric space. One such generalization is a b-metric space introduced and studied by Bakhtin**[1]** and Czerwik**[2]** in 1989. They also investigated some fixed point results in b-metric spaces.

Now, we will begin with the definition of b-metric space.

Definition 2.3: [1], [2] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric provided that $\forall\ x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$
(ii) $d(x, y) = d(y, x)$
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$ (b-triangular inequality).

A pair (X, d) is called a b-metric space. The number s is called the coefficient of (X, d).

The following are the examples of b-metric space.

Example 2.4: [1] The set $L_p(\mathbb{R})$ where $L_p(\mathbb{R}) = \{\{x_n\} \subseteq \mathbb{R} : \sum|x_n|^p < \infty\}$ (with $0 < p < 1$) together with the function $d: L_p(\mathbb{R}) \times L_p(\mathbb{R}) \to [0, \infty)$ defined by

$$d(x, y) = \left(\sum_{i=0}^{n} |x - y|^p\right)^{1/p}$$

where $x = \{x_n\}$, $y = \{y_n\} \in L_p(\mathbb{R})$ forms a b-metric with $s = 2^{1/p}$.

Example 2.5: [1] The space $L_p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$ forms a b-metric by defining

$$d(x, y) = \left(\int_0^1 |x(t) - y(t)|^p\right)^{1/p} dt \text{ for each } x, y \in L_p[0, 1] \text{ with } s = 2^{1/p}.$$

It is clear that definition of b-metric is an extension of usual metric space. Obviously, each metric space is a b-metric space with s = 1. However, Czerwik [2] has shown that a b-metric on X need not be a metric on X.

The following example illustrates this situation.

Example 2.6: Let (X, d) be a metric space. Define $\rho(x, y) = [d(x, y)]^p$, where $p > 1$ is a real number. Then we can verify that ρ forms a b-metric with $s = 2^{p-1}$. However, if (X, d) is a metric space, then (X, ρ) is not necessarily a metric space.

For example, if $X = \mathbb{R}$ is the set of real numbers and $d(x, y) = |x - y|$ is usual Euclidean metric, then $\rho(x, y) = (x - y)^2$ is a b-metric on $\mathbb{R}$ with $s = 2$. But it is not a metric space.

It is noted that the class of b-metric spaces is larger than the class of metric spaces.

Next, we will begin with the study of Cauchy sequence, convergent sequence and completeness in b-metric space.

Definition 2.7: [1] A sequence $\{x_n\}$ in a b-metric space (X, d) is said to be

(i) b-convergent if $d(x_n, x) \to 0$ as $n \to \infty$ i.e., $\forall\ \varepsilon > 0\ \exists\ N \in \mathrm{N} : d(x_n, x) < \varepsilon\ \forall\ n \geq N$.
(ii) b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$ i.e., $\forall\ \varepsilon > 0\ \exists\ N \in \mathrm{N} : d(x_n, x_m) < \varepsilon\ \ \forall\ n, m \geq N$.
(iii) b-complete if and only if each b-Cauchy sequence in (X, d) is b-convergent.

There has been a number of generalizations of the usual notion of a metric space. One such generalization is a dislocated quasi b-metric space introduced and studied by Rahman and Sarwar in 1989.

Definition 2.8: [7] Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \rightarrow [0, \infty)$ is called a dislocated quasi b-metric provided that $\forall\, x, y, z \in X$,

(i) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$
(ii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A pair (X, d) is called a dislocated quasi b-metric space (or dqb-metric space). In the definition of dislocated quasi b-metric space if $s = 1$ then it becomes (usual) dislocated quasi metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with coefficient k and zero self distance. However, the converse is not true. The following are the examples of dislocated quasi b-metric space.

Example 2.9: [7] Let $X = \mathbb{R}^+$, for $p > 1$ and $d: X \times X \rightarrow [0, \infty)$ be defined as

$$d(x, y) = |x - y|^p + |x|^p,\ \forall\, x, y \in X.$$

Then, (X, d) is a dislocated quasi b-metric space with $s = 2^p > 1$. But it is neither b-metric nor dislocated quasi metric.

Example 2.10: [7] Let $X = \mathbb{R}$ and suppose $d(x, y) = |2x - y|^2 + |2x + y|^2$.
Then, (X, d) is a dislocated quasi b-metric space with the coefficient $s = 2$.

Definition 2.11: [10] A sequence $\{x_n\}$ in dislocated quasi b-metric space (dq b-metric space) (X, d) is said to be

(i) dqb-convergent to x if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$. In this case, x is called a dqb-limit of $\{x_n\}$ and we write $x_n \rightarrow x$.
(ii) Cauchy sequence if for $\varepsilon > 0$ there exists $N \in N$ such that $m, n \geq N$, we have $d(x_m, x_n) < \varepsilon$.
(iii) dqb-complete if every Cauchy sequence in it is a dqb-convergent.

MAIN RESULT

In this section, we shall study some results that characterize the fixed point theory in b-metric space and dislocated quasi b-metric space.

Lemma 3.1: [1] In a b-metric space (X, d), a b-convergent sequence has a unique limit point.

Proof: Let $\{x_n\}$ be a b-convergent sequence in a b-metric space (X, d) and suppose that $x_n \rightarrow x$ and $x_n \rightarrow z$.

Now, $d(x, z) \leq s\, [d(x, x_n) + d(x_n, z)]$
$\qquad = sd(x, x_n) + s\, d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$.
$\therefore d(x, z) \leq 0 \Rightarrow x = z$.

Hence, limit of a b-convergent sequence is unique.

Lemma 3.2: [1] Each b-convergent sequence is b-Cauchy in a b-metric space (X, d).

Proof: Let (X, d) be a b-metric space and $\{x_n\}$ be a b-convergent sequence in X. We show that $\{x_n\}$ is a b-Cauchy in X.

Now, $d(x_n, x_m) \leq s[d(x_n, x) + d(x, x_m)]$
$\qquad = sd(x_n, x) + sd(x, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$

Hence, $\{x_n\}$ is a b-Cauchy in X.

The following theorem is the Banach version of fixed point theorem in b-metric space.

Theorem 3.3: [4] Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and let $T: X \rightarrow X$ be a mapping such that $\forall\, x, y \in X$

$$d(Tx, Ty) \leq \alpha\, d(x, y),\ \alpha \in [0, 1) \text{ and } \alpha s < 1.$$

Then T has a unique fixed point in X.

Proof: Let x_0 be arbitrary in X. Define a sequence x_n in X by

$$x_0, x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, \ldots, x_{n+1} = Tx_n.$$

Consider, $d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha d(x_{n-1}, x_n)$, as ($\alpha s < 1$)

Here, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete $\exists\, z \in X$ such that $x_n \rightarrow z$. Also, every contraction mapping is continuous.

So $\lim_{n\to\infty} Tx_n = Tz$

$\Rightarrow \quad \lim_{n\to\infty} x_{n+1} = Tz$

$\Rightarrow \quad Tz = z.$

Hence, z is a fixed point of T.

Uniqueness: Let z and t be two fixed points of T. Now,

$$d(z, t) = d(Tz, Tt) \leq \alpha d(z, t).$$

Since $\alpha \in [0, 1)$, so $d(z, t) = 0$. Therefore $z = t$. Hence, T has a unique fixed point.

Our next theorem deals about Kannan type fixed point theorem in b-metric space.

Theorem 3.4: [4] Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $T: X \to X$ be a mapping for which there exists $\mu \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \mu\, [d(x, Tx) + d(y, Ty)],\ \forall\, x, y \in X.$$

Then T has a unique fixed point in X.

The following theorem deals about Chatterjea type fixed point theorem in b-metric space.

Theorem 3.5: [4] Let (X, d) be a complete b-metric space with coefficient $s \geq 1$. Let $T: X \to X$ be a mapping for which $s\mu \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \mu\, [d(x, Ty) + d(y, Tx)],\ \forall\, x, y \in X.$$

Then T has a unique fixed point in X.

The following lemma given by Rahman and Sarwar in 2016 shows the uniqueness of limit in dislocated quasi b-metric space.

Lemma 3.6: [7] Limit of convergent sequence in a dqb-metric space is unique.

Proof: Suppose that $x_n \to x$ and $x_n \to y$. Then

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = \lim_{n\to\infty} d(x_n, y) = \lim_{n\to\infty} d(y, x_n) = 0.$$

Consider,

$$0 \leq d(x, y) \leq sd(x, x_n) + sd(x_n, y) \text{ and}$$
$$0 \leq d(y, x) \leq sd(y, x_n) + sd(x_n, x).$$

Taking limit as $n, m \to \infty$, we obtain $d(x, y) = d(y, x) = 0$.

Therefore, $x = y$.

The following lemma given by Rahman and Sarwar shall be used in the later theorems.

Lemma 3.7: [7] Let (X, d) be a dqb-metric space and $\{x_n\}$ be a sequence in X such that

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \text{ for } n = 0, 1, 2, ... \text{ and } 0 \leq \alpha s < 1,\ \alpha \in [0, 1),$$

and s is defined in dqb-metric space. Then $\{x_n\}$ is a Cauchy sequence.

Proof: We have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ...] \\ &\leq sd(x_n, x_{n+1}) + s^2 [d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + ...] \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \ldots \end{aligned}$$

Now using, $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$, we get

$$\begin{aligned} d(x_n, x_m) &\leq s\alpha^n d(x_0, x_1) + s^2\alpha^{n+1} d(x_0, x_1) + s^3\alpha^{n+2} d(x_0, x_1) + ... \\ &\leq [1 + s\alpha + (s\alpha)^2 + \ \ldots\ ]\, s\alpha^n d(x_0, x_1) \\ &\leq s\alpha^n \left(\frac{1}{1-s\alpha}\right) d(x_0, x_1). \end{aligned}$$

Taking limit as $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in dislocated quasi b-metric space X.

Here, Rahman and Sarwar proved the famous Banach contraction principle in complete dislocated quasi *b*-metric space.

Theorem 3.8: [7] Let (X, d) be a complete *dqb*-metric space and $T: X \to X$ be a continuous contraction with $0 \leq \alpha s < 1$, $\alpha \in [0, 1)$, where $s \geq 1$. Then T has a unique fixed point in X.

Next, Rahman and Sarwar proved the Kannan type fixed point theorem in dislocated quasi *b*-metric space.

Theorem 3.9: [7] Let (X, d) be a complete *dq b*-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a continuous self mapping for $\delta \in [0, \frac{1}{2})$ satisfying the condition

$$d(Tx, Ty) \leq \delta\,[d(x, Tx) + d(y,Ty)]\ \forall\, x, y \in X.$$

Then T has a unique fixed point in X.

The following theorem is about Chetterjea type fixed point theorem in dislocated quasi *b*-metric space given by Rahman and Sarwar.

Theorem 3.10: [7] Let (X, d) be a complete *dq b*-metric space with coefficient $s \geq 1$ and $T: X \to X$ be a continuous self mapping for $s\beta \in [0, \frac{1}{4})$ satisfying the condition

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)]\ \forall x, y \in X.$$

Then T has a unique fixed point in X.

Rahman in 2017 has proved some fixed point theorems for generalized type contractions.

Theorem 3.11: [8] Let (X, d) be a complete *dqb*- metric space with coefficient $s \geq 1$ and T be a continuous self-mapping $T: X \to X$ satisfying the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta\,[d(x, Tx) + d(y, Ty)] + \mu[d(x, Ty) + d(y, Tx)],\ \forall\, x, y \in X,$$

where $\alpha, \beta, \mu \geq 0$, with $s\alpha + (1 + s)\beta + 2(s^2 + s)\mu < 1$. Then T has a unique fixed point.

The following corollaries are deduced from the above theorem given by Rahman in 2017.

The corollary 3.12 generalize the result of Chetterjea in dislocated quasi *b*-metric space.

Corollary 3.12: [8] Let (X, d) be a complete *dqb*-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx, Ty) \leq \mu\,[d(x, Ty) + d(y, Tx)],\ \forall\, x, y \in X,$$

where $\mu \geq 0$ with $2(s^2 + s)\mu < 1$. Then T has a unique fixed point.

The corollary 3.13 generalize the result of Kannan in dislocated quasi *b*-metric space.

Corollary 3.13: [8] Let (X, d) be a complete *dqb*-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx, Ty) \leq \beta\,[d(x, Tx) + d(y, Ty)],\ \forall\, x, y \in X,$$

where $\beta \geq 0$ with $(1 + s)\beta < 1$. Then T has a unique fixed point.

The corollary 3.14 generalize the result of Hardy and Rogeres in dislocated quasi *b*-metric space.

Corollary 3.14: [8] Let (X, d) be a complete *dqb*-metric space with coefficient $s \geq 1$ and $T : X \to X$ be a continuous self-mapping satisfying the condition

$$d(Tx, Ty) \leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y,Ty) + \mu d(x, Ty) + \delta d(y, Tx),\ \forall\, x, y \in X,$$

where $\alpha, \beta, \gamma, \mu, \delta, \geq 0$ with $s\alpha + \beta + s\gamma + 2s^2\mu + 2s\delta < 1$. Then T has a unique fixed point.

CONCLUSION

In this work, we have introduced some existing properties of b-metric space as the the usual notion of a metric space. Besides this, we have studied some of the generalizations of some recent results related to fixed point theorems in *b*-metric spaces and dislocated quasi *b*-metric space. In fact, this work extends many other authors existing literature and can be used for further research work in fixed point theory in Metric space.

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