

LEFT AND RIGHT 2-ENGL E ELEMENTS OF DRIVATIVE OF GROUPS

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ABSTRACT

In this paper we study right and left 2-Engle elements in derivative of groups. In particular, we prove that $R_2(G')$ is a characteristic subgroup of G' . (let G be a group and G' be a derivative of G). As a consequence, in a special cases $[x, y]^{G'}$ is a abelian subgroup of G and the inverse of a right 2-Engle element of G' is a left 3-Engle element of G' .

1. INTRODUCTION:

Let G be group and G' be a derivative of G . Consider the set $R_2(G) = \{g \in G \mid [g, x, x] = 1 \text{ for all } x \in G\}$ of right 2-Engle elements of G . It is also known; see [1], that the inverse of a right 2-Engle element is a left 3-Engle element. For any two elements a and b of G we define inductively $[a, n, b]$ the n -Engle commutator of the pair (a, b) , as follows:

$$[a, 0, b] = a, [a, b] = a^{-1}b^{-1}ab \text{ and } [a, n, b] = [[a, n-1, b], b] \text{ for all } n > 0.$$

$R_2(G)$ is a characteristic subgroup of G . It is known also that $R_2(G) \subseteq L_2(G)[3]$.

2. THE RESULTS:

Definition 1: Let $[x, y]$ and $[z, t]$ be two elements of G° . We define the commentator of the pair $([x, y], [z, t])$, as follows:

$$[[x, y], [z, t]] = [x, y]^{-1} [z, t]^{-1} [x, y] [z, t].$$

And define inductively

$[x, y]_n [z, t]$ the n -Engle commentator of the pair $([x, y], [z, t])$, as follows:

$$[[x, y]_n [z, t]] = [[[x, y]_{(n-1)} [z, t]], [z, t]] \text{ for all } n > 0.$$

Definition 2: An element $[x, y]$ of G° is called a left n -Engle element if $[[z, t]_n [x, y]] = 1$ for all $[z, t] \in G^\circ$. we denote by $L_n(G^\circ)$, the set of all left n -Engle elements of G° and an element $[x, y]$ of G° is called a right n -Engle element if $[[x, y]_n [z, t]] = 1$ For all $[z, t] \in G^\circ$.

The set of all right n -Engle elements of G° , denote by $R_n(G^\circ)$.

Theorem 1: In any derivative of group G the inverse of right 2-Engle element is a left 2-Engle element. Thus $R_2(G^\circ) \subseteq L_2(G^\circ)$

Proof: Let $[x, y] \in R_2(G^\circ)$, using the definition of $R_2(G^\circ)$ we have

$$[[x, y], [x, y][z, t], [x, y][z, t]] = 1 \quad ; [z, t] \in G^\circ$$

$$1 = [[x, y], [x, y][z, t], [x, y][z, t]] = [[x, y], [z, t], [x, y][z, t]]$$

$$= [[x, y], [z, t]]^{-1} ([x, y][z, t])^{-1} [[x, y][z, t]] ([x, y][z, t])$$

$$= [[x, y], [z, t]]^{-1} [z, t]^{-1} [x, y]^{-1} [[x, y], [z, t]] [x, y][z, t]$$

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$$=[z, t]^{-1}[[x, y], [z, t]]^{-1}[x, y]^{-1}[[x, y], [z, t]][x, y][z, t]$$

$$=[z, t]^{-1}[[x, y], [z, t], [x, y]][z, t]$$

$$=[[x, y], [z, t], [x, y]]^{[z, t]}$$

$$\text{Since } [[x, y], [z, t], [x, y]]^{[z, t]} = 1$$

$$\text{So } [[x, y], [z, t], [x, y]] = 1$$

By using three subgroup lemma [3] we have $[[x, y], [z, t], [x, y]] = 1$.

Thus $[x, y] \in L_2(G^\circ)$.

Theorem 2: In any derivative of group G the inverse of a right Engle Element is a left Engle element and the inverse of a right n-Engle element is a left (n+1)-Engle element.

Thus $R(G^\circ) \subseteq L(G^\circ)$ and $R_{n+1}(G^\circ) \subseteq L_n(G^\circ)$

Proof: Let $[x, y]$ and $[z, t]$ be elements of G° . Usiug the fundamental commutator identities [2, 3] we obtain

$$\begin{aligned} [[x, y], {}_{n+1}[z, t]] &= [[[x, y], [z, t]], {}_n[z, t]] \\ &= [[[z, t]^{-1}, [x, y]]^{[z, t]}, {}_n[z, t]] \\ &= [[[z, t]^{-1}, [x, y]], {}_n[z, t]]^{[z, t]} \\ &= [[z, t][z, t]^{-[x, y]}, {}_n[z, t]]^{[z, t]} \\ &= [[z, t]^{-[x, y]}, {}_n[z, t]]^{[z, t]} \\ &= [[z, t]^{-[x, y]}, ({}_n[z, t]^{[x, y]})^{-[x, y]}]^{[z, t]} \\ &= [[z, t], {}_n[z, t]^{[x, y]}]^{-[x, y][z, t]} \end{aligned}$$

since $[z, t] \in R_n(G^\circ)$

$$\text{so } [[z, t], {}_n[z, t]^{[x, y]}]^{-[x, y][z, t]} = 1$$

therefore $[[x, y], {}_{n+1}[z, t]] = 1$.

Theorem 3: Let G be a group and G' be a derivative of G. The set of elements of $R_2(G')$ is a characteristic subgroup of G'.

Proof: Let $\alpha \in \text{Aut}(G)$ be an arbitrary automorphism and $[x, y] \in R_2(G')$. By definition of right 2-Engle element of G', for any $[z, t] \in G'$, we have

$$[[x, y], [z, t], [z, t]] = 1$$

So

$$\alpha[[x, y], [z, t], [z, t]] = 1$$

$$\rightarrow [\alpha[x, y], \alpha[z, t], \alpha[z, t]] = 1$$

$$\rightarrow [[\alpha(x), \alpha(y)], [\alpha(z), \alpha(t)], [\alpha(z), \alpha(t)]] = 1$$

Since $\alpha \in \text{Aut}(G)$, for any $z, t \in G$, there exists $z', t' \in G$ such that $\alpha(z) = z'$ and $\alpha(t) = t'$,

Therefore $[\alpha[x, y], [z', t'], [z', t']] = 1$.

Thus $\alpha[x, y] \in R_2(G')$.

Theorem 4: Let G be a group and G' be a derivative of G and $[x, y] \in R_2(G')$. Then $[x, y]^{G'}$ is abelian.

Proof: Let $[x, y], [z, t], [m, n]$ be elements of G' , so

$$\begin{aligned} [[x, y]^{[z,t]}, [x, y]^{[m,n]}] &= [[x, y]^{[z,t][m,n]^{-1}}, [x, y]^{[m,n]}] \\ &= [[x, y][x, y][z, t][m, n]^{-1}, [x, y]]^{[m,n]} \\ &= [[x, y], [x, y]]^{[[x,y],[z,t][m,n]^{-1}]^{[m,n]}} [[x, y], [z, t][m, n], [x, y]]^{[m,n]} \\ &= [[x, y], [z, t][m, n]^{-1}, [x, y]]^{[m,n]} \\ &= [[[z, t][m, n]^{-1}, [x, y]]^{-1}, [x, y]]^{[m,n]} \\ &= ([[[z, t][m, n]^{-1}, [x, y]], [x, y]]^{-1})^{[m,n][[z,t][m,n]^{-1}, [x,y]]^{-1}} \\ &= 1. \end{aligned}$$

The last identities holds since $R_2(G^{\circ}) \subseteq L_2(G^{\circ})$ and, therefore $[x, y]^{G'}$ is abelian.

Definition 3: Let G be a group and G' be a derivative of G . Then the verbal subgroup, $E_1(G')$, define as follows:

$$E_1(G') = \{ [x, y] \in G' : [[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]] \text{ For all } [z, t], [m, n] \in G' \}$$

Remark: Now it is shown that there is a close connection between the elements of $R_2(G')$ and verbal subgroups of G' .

Theorem 5: Let G be a group and G' be a derivative of G . Then $E_1(G') = R_2(G')$.

Proof: By definition we have

$$E_1(G') = \{ [x, y] \in G' : [[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]] \text{ For all } [z, t], [m, n] \in G' \}$$

$$R_2(G') = \{ [x, y] \in G' : [[x, y], [z, t], [z, t]] = 1 \text{ For all } [z, t] \in G' \}.$$

Now if $[x, y] \in E_1(G')$, then for all $[z, t], [m, n] \in G'$ we have

$$[[x, y][z, t], [m, n], [m, n]] = [[z, t], [m, n], [m, n]].$$

For $z = t = 1$ the identities hold. So for all $[m, n] \in G'$

$$[[x, y], [m, n], [m, n]] = 1.$$

Therefore $[x, y] \in R_2(G')$ and $E_1(G') \subseteq R_2(G')$ (I)

Conversely .If $[x, y] \in R_2(G')$, then for all $[z, t] \in G'$

$$[[x, y], [z, t], [z, t]] = 1$$

Now we have

$$\begin{aligned} [[x, y][z, t], [m, n], [m, n]] &= [[[x, y], [m, n]]^{[z,t]} [[z, t], [m, n]], [m, n]] \\ &= [[[x, y], [m, n]]^{[z,t]}, [m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n], [m, n]] \\ &= [[[x, y], [n, m]][[x, y], [m, n], [z, t]], [m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n], [m, n]] \\ &= [[x, y], [m, n], [m, n]]^{[[z,t],[m,n]][[x,y],[m,n],[z,t]]} \end{aligned}$$

$$[[x, y], [m, n], [z, t], [m, n]]^{[[z,t],[m,n]]} [[z, t], [m, n], [m, n]].$$

By using of assumption of last theorem, equal to $[[z, t], [m, n], [m, n]]$.

So $[x, y] \in E_1(G')$ and therefore $R_2(G') \subseteq E_1(G')$ (II)

Now with (I) and (II) we have the result.

Theorem 6: Let G, G' be group and derivative of group, respectively. And $[z, t] \in G', [x, y] \in R_2(G')$. Then

(i) $[x, y]^{G'}$ is abelian.

(ii) For all $r, s \in \mathbb{N}$ $[[x, y], [z, t]]^{rs} = [[x, y]^r, [z, t]^s]$.

Proof: (i) Its shown in Theorem 4.

(ii) We proceed by induction in two times.

As a first time, we proved by induction

$$[[x, y], [z, t]^n] = [[x, y]^n, [z, t]].$$

For $n=1$, being obvious.

Assume the result holds for $n-1$, i.e,

$$[[x, y], [z, t]]^{n-1} = [[x, y]^{n-1}, [z, t]].$$

Then we have

$$\begin{aligned} [[x, y]^n, [z, t]] &= [[x, y]^{n-1} [x, y], [z, t]] \\ &= [[x, y]^{n-1}, [z, t]]^{[x, y]} [[x, y], [z, t]] \\ &= [[x, y]^{n-1}, [z, t]] [[x, y]^{n-1}, [z, t], [x, y]] [[x, y], [z, t]]. \end{aligned}$$

By assumption, we have

$$\begin{aligned} [[[x, y]^{n-1}, [z, t], [x, y]]] &= [[[x, y], [z, t]]^{n-1}, [x, y]] \\ &= [[z, t], [x, y]]^{n-1} [x, y]^{-1} [[x, y], [z, t]]^{n-1} [x, y]. \end{aligned}$$

We know that $R_2(G') \subseteq L_2(G')$, so if $[x, y] \in R_2(G')$, then $[x, y] \in L_2(G')$. Therefore, for each $[z, t] \in G'$ we have $[[z, t], [x, y], [x, y]] = 1$.

So $[[z, t], [x, y]]$ commutes with $[x, y]$ and $[x, y]^{-1}$.

Therefore

$$[[[x, y]^{n-1}, [z, t], [x, y]]] = [x, y]^{-1} [[z, t], [x, y]]^{n-1} [[x, y], [z, t]]^{n-1} [x, y] = 1$$

So we have

$$\begin{aligned} [[x, y]^n, [z, t]] &= [[x, y]^{n-1}, [z, t]] [[x, y], [z, t]] \\ &= [[x, y], [z, t]]^{n-1} [[x, y], [z, t]] \\ &= [[x, y], [z, t]]^n \end{aligned}$$

As a second time, we proved by induction

$$[[x, y], [z, t]^{n-1}] = [[x, y], [z, t]]^{n-1}$$

No we have

$$\begin{aligned} [[x, y], [z, t]^n] &= [[x, y], [z, t]^{n-1} [z, t]] \\ &= [[x, y], [z, t]] [[x, y], [z, t]^{n-1}]^{[z, t]} \\ &= [[x, y], [z, t]] [[x, y], [z, t]^{n-1}] [[x, y], [z, t]^{n-1}, [z, t]] \end{aligned}$$

As a similar to the last case, proved

$$[[x, y], [z, t]^{n-1}, [z, t]] = 1$$

So

$$[[x, y], [z, t]^n] = [[x, y], [z, t]][[x, y], [z, t]]^{n-1} = [[x, y], [z, t]]^n$$

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