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COMMON FIXED POINT THEOREMS FOR COMPATIBLE AND WEAKLY COMPATIBLE MAPPINGS IN COMPLEX VALUED METRIC SPACES

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ABSTRACT

Azam et al. [2] introduced the definition of complex-valued metric spaces and created several typical fixed point results in the sense of complex-valued metric spaces. This paper proves the existence of common fixed points for compatible and weakly compatible mappings on complex-valued metric spaces. Related observations in the literature are unified, generalized and supplemented by our findings.

Keywords: *complex-valued metric space; compatible mappings; common fixed point; rational contractions, compatible mappings*

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1. INTRODUCTION

Mathematical findings on fixed points of contraction form mappings are well known for their use in deciding the nature and uniqueness of solutions to various mathematical models. Azam *et al.* [2] presented the definition of complex valued metric spaces and came up with several fixed point results for a pair of mappings that satisfy a logical expression for contraction conditions. Verma and Pathak [8] recently established the concept of property (E:A) on a complex valued metric space in order to derive some typical fixed-point results for two pairs of weakly compatible mappings that satisfy a max form contractive condition. Ahmad et al. [1] prove some common fixed results for mappings satisfying rational expressions on a closed ball in complex valued metric spaces, while Rafiq *et al.* [6] prove some common fixed point theorems of weakly compatible mappings in complex valued metric spaces.

2. BASIC DEFINITIONS AND PRELIMINARIES

We recall a few notations and definitions that will be used in the following discussion. Azam *et al.* [2] suggested the following description recently:

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows: $z_1 \precsim z_2$ if and only if $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

Consequently, one can infer that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$, $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we write $z_1 \lnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we write $z_1 \prec z_2$ if only (iii) is satisfied. Notice that $0 \precsim z_1 \lnsim z_2 \Rightarrow |z_1| < |z_2|$, and $z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$.

Definition 2.1[2]: Let X be a nonempty set, whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d: X \times X \to \mathbb{C}$, satisfies the following conditions:

(d_1) $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

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Example 2.1[7]: Let $X = \mathbb{C}$. Define the mapping $d: X \times X \to \mathbb{C}$ by $d(x,y) = e^{ik}|x-y|$ where $k \in \mathbb{R}$ and for all $x, y \in X$. Then (X,d) is a complex valued metric space.

Definition 2.2[2]: Let (X,d) be a complex valued metric space and $B \subseteq X$.

(i) $b \in B$ is called an interior point of a set B whenever there is $0 \prec r \in \mathbb{C}$ such that $N(b,r) \subseteq B$, where $N(b,r) = \{y \in X: d(b,y) \prec r\}$
(ii) A point $x \in X$ is called a limit point of B whenever for every $0 \prec r \in \mathbb{C}$, $N(x,r) \cap (B\backslash\{X\}) \neq \phi$
(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A whereas a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family $F = \{N(x,r): x \in X, 0 \prec r$ is a sub-basis for a topology on X. We denote this complex topology by τ_c. Indeed, the topology τ_c is Hausdorff.

Definition2.3 [2]: Let (X,d) be a complex valued metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that

(i) The sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $c \in \mathbb{C}$with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$. We denote this by $\lim_n x_n$ or $x_n \to x$, as $n \to \infty$,
(ii) The sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$,
(iii) The metric space (X,d) is a complete complex valued metric space if every Cauchy sequence is convergent.

Lemma 2.1[2]: Let (X,d)be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.

Lemma 2.2 [2]: Let (X,d) be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n, m \to \infty$.

Definition 2.4[3]: If f and g are mappings from a metric space (X,d) into itself, are called compatible on X, if $\lim_{n\to\infty} d(fgx_{n,}\ gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$, for some point $x \in X$.

Definition 2.5 [4]: Let f and g be two self-maps defined on a set X, then f and g are said to be weakly compatible if they commute at coincidence point.

Lemma 2.3 [3]: Let f and g be compatible mappings from a metric space (X,d) into itself. Suppose that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$for some $x \in X$. Then $\lim_{n\to\infty} gfx_n = fx$, if f is continuous.

Definition 2.6 [8]: The 'max' function for the partial order $\precsim$ is defined as follows:
(1) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$.
(2) $z_1 \precsim \max\{z_2, z_3\} \Rightarrow z_1 \precsim z_2$ or $z_1 \precsim z_3$.
(3) $\max\{z_1, z_2\} = z_2 \Leftrightarrow z_1 \precsim z_2$ or $|z_1| \leq |z_2|$

3. Main results: The key consequence of this section is a generalization of Azam's [2] and Nashine's [5] findings.

Theorem 3.1: Let (X,d) be a complete complex valued metric space and mappings $f, g, S, T: X \to X$ satisfying:
(3.1)$S \subset g, T \subset f$;
(3.2) $d(Sx,Ty) \precsim \alpha.max\left\{d(fx,gy), \frac{d(fx\ ,Sx)\ d(gy,Ty)}{d(fx,Ty)+d(gy,Sx)\ +d(fx,gy)}\right\}$
for all $x, y \in X$ such that $x \neq y$, $d(fx,Ty) + d(gy,Sx) + d(fx,gy) \neq 0$, where α is nonnegative real with $\alpha < 1$.

Suppose that one of S or f is continuous, pair (S,f) is compatible and (T,g) is weak compatible.Then f, g, S and T have a unique common fixed point in X.

Proof: Suppose x_0 be an arbitrary point in X. We define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2};\ n = 0,1,2 \ldots .$$

Then,

$$\begin{aligned} d(y_{2n,}y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\precsim \alpha.max\left\{d(fx_{2n}, gx_{2n+1}), \frac{d(fx_{2n},\ Sx_{2n})\,d(gx_{2n+1},\ Tx_{2n+1})}{d(fx_{2n,}Tx_{2n+1}) + d(gx_{2n+1,}Sx_{2n}) + d(fx_{2n,}\ gx_{2n+1})}\right\} \\ &\precsim \alpha.\max\left\{d(y_{2n-1,}y_{2n}), \frac{d(y_{2n-1,}y_{2n})d(y_{2n,}y_{2n+1})}{d(y_{2n-1,}y_{2n+1}) + d(y_{2n,}y_{2n}) + d(y_{2n-1,}y_{2n})}\right\} \end{aligned}$$

$$\precsim \alpha.\max\left\{d(y_{2n-1,}y_{2n}),\frac{d(y_{2n-1,}y_{2n})d(y_{2n,}y_{2n+1})}{d(y_{2n,}y_{2n+1})}\right\}$$
$$\precsim \alpha.d(y_{2n-1,}y_{2n})$$
$$d(y_{2n,}y_{2n+1}) \precsim \alpha.\, d(y_{2n-1,}\, y_{2n}).$$

Similarly, we can show that
$$d(y_{2n+1}, y_{2n+2}) \precsim \alpha.\, d(y_{2n}, y_{2n+1}).$$

If $\alpha < 1$, then
$$|d(y_{2n+1}, y_{2n+2})| \precsim \alpha\left|d(y_{2n,}y_{2n+1})\right| \precsim \cdots \precsim \alpha^{2n+1}|d(y_0, y_1)|$$

So that for any $m > n$,
$$|d(y_{2n}, y_{2m})| \precsim |d(y_{2n,}y_{2n+1}) + d\,(y_{2n+1,}\, y_{2n+2}) + \ldots + d\,(y_{2m-1,}\, y_{2m})|$$
$$\precsim (\alpha^{2n} + \alpha^{2n+1} + \ldots + \alpha^{2m-1})\left|d(y_{0,}y_1)\right|$$
$$\precsim \frac{\alpha^{2n}}{1-\alpha}\left|d(y_{0,}y_1)\right| \to 0, \text{ as } m, n \to \infty.$$

Hence $\{y_{2n}\}$ is a Cauchy sequence and since X is complete, sequence $\{y_{2n}\}$ converges to some point t in X and its subsequences $Sx_{2n,}\, Tx_{2n+1,}\, fx_{2n+2,}$ and gx_{2n+1} of sequence $\{y_{2n}\}$ also converges to point t.

Suppose that f is continuous and since the mappings S and f are compatible on X. Then by lemma (2.3), we have
f^2x_{2n} and $Sfx_{2n} \to ft$ as $n \to \infty$.

Consider
$$d(Sfx_{2n},Tx_{2n+1}) \precsim \alpha.\max\left\{d(f^2x_{2n}, gx_{2n+1}),\frac{d(f^2x_{2n},\ Sfx_{2n})\, d(gx_{2n+1,}Tx_{2n+1})}{d(f^2x_{2n,}Tx_{2n+1}) + d(gx_{2n+1,}Sfx_{2n}) + d(f^2\ x_{2n},\, gx_{2sn+1})}\right\}$$

Letting $n \to \infty$, we get
$$d(ft, t) \precsim \alpha\,.\max\left\{d(ft, t),\frac{d(ft, ft)d(t, t)}{d(ft, t) + d(t, ft) + d(ft, t)}\right\}$$
$$(1-\alpha)d(ft, t)) \precsim 0$$

This yields $d(ft, t) \precsim 0$, so that $ft = t$.
$$d(St, Tx_{2n+1}) \precsim \alpha\,.\max\left\{d(ft, gx_{2n+1}),\frac{d(ft, St)\ d(gx_{2n+1,}Tx_{2n+1})}{d(ft, Tx_{2n+1}) +\ d(gx_{2n+1,}St) + d(ft,\ gx_{2n+1})}\right\}$$
Letting $n \to \infty$, we get
$$d(St, t) \precsim \alpha\,.\max\left\{d(t, t),\frac{d(t,\ St)d(t, t)}{d(t, t) + d(t, t) + d(t, t)}\right\}$$
$d(St, t) \precsim 0$ so that $St = t$.

Now since $S \subset g$ and there exists another point u in X, such that $t = St = gu$.

Consider
$$d(t, Tu) = d(St, Tu)$$
$$\precsim \alpha.\max\left\{d(ft, gu),\frac{d(ft, St)d(gu, Tu)}{d(ft, Tu) +\ d(gu, St) + d(ft, gu)}\right\}$$
$$\precsim \alpha.\max\left\{d(t, t),\frac{d(t, t)d(t, Tu)}{d(t, Tu) + d(t, t) + d(t, t)}\right\}$$
$d(t, Tu) \precsim 0$ so that $Tu = t$.

Since T and g weakly are compatible on X and $Tu = gu = t$ and $d(gTu, Tgu) = 0$
implies that $gt = gTu = Tgu = Tt$.

Consider
$$d(t, gt) = d(St, Tt)$$
$$\precsim \alpha.\max\left\{d(ft, gt),\frac{d(ft, St)d(gt, Tt)}{d(ft, Tt) + d(gt, St) + d(ft, gt)}\right\}$$
$(1-\alpha)d(t, gt) \precsim 0$ so that $gt = t$.

Hence $ft = gt = St = Tt = t$.

Thus t is a common fixed point of f, g, S and T.

Similarly, we can show that t is a common fixed point of f, g, S and T, when S is continuous.

Now, we prove the uniqueness of t.

Suppose that $w \neq t$ be another common fixed point of f, g, S and T.
Then

$$d(t,w) = d(St,Tw)$$
$$\precsim \alpha.max\left\{d(ft,gw), \frac{d(ft,St)d(gw,Tw)}{d(ft,Tw)+d(gw,St)+d(ft,gw)}\right\}$$
$$\precsim \alpha.max\left\{d(t,w), \frac{d(t,t)\,d(w,w)}{d(t,w)+d(w,t)+d(t,w)}\right\}$$
$$d(t,w) \precsim \alpha\, d(t,w)$$
$(1-\alpha)d(t,w) \precsim 0$, which is a contradiction. Hence $t = w$.

Therefore, t is unique common fixed point of f, g, S and T.

By setting $f = g = I$, we get the following corollary:

Corollary3.2: Let (X, d) be a complete complex valued metric space and mappings $S, T: X \to X$ satisfying $S \subset T$ and $d(Sx,Ty) \precsim \alpha.max\left\{d(x,y), \frac{d(x,Sx)\,d(y,Ty)}{d(x,Ty)+d(y,Sx)+d(x,y)}\right\}$ for all $x, y \in X$ such that $x \neq y$, $d(x,Ty) + d(y,Sx) + d(x,y) \neq 0$. where $\alpha$is non-negative real with $\alpha < 1$. If pair (S,T) is weakly compatible. Then S and T have a unique common fixed point in X.

4. CONCLUSION

In complex valued metric spaces, compatibility, weak compatibility continuity and the max function have been used to prove certain fixed point results. Our findings generalize and reinforce previous findings in complex valued metric spaces by [1, 2, 5, 6].

COMPETING INTERESTS

The authors declare that they have no competing interests.

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