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RELATION BETWEEN PI AND ANGULAR ROTATION OF RADIUS IN A CIRCLE

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ABSTRACT

There are many infinite series to compute the value of Pi. In this paper we introduce an infinite series in term of sin and cosine function which show relation between angular rotation (θ) and Pi in a circle. For some value of θ, series show relation between Pi & nested radicals of two & three.

Keywords: *Angular rotation, Linear function, Nested radicals, Pi.*

1. INTRODUCTION

To find out this series, quarter of a circle is beaked into infinite Triangle (as shown in diagram 1), base and height of each triangle is calculated and finally area of all triangles are added.

Long before, similar method had been adopted by Archimedes & Viete.

Viète obtained his formula by comparing the areas of regular polygons with 2^n and 2^{n+1}sides inscribed in a circle. The first term in the product, √2/2, is the ratio of areas of a square and an octagon, the second term is the ratio of areas of an octagon and a hex decagon, etc

In this method relation between height & base of triangles are established in term of nested radicals of two and area of all triangles are added.

Definition1.1: Nested radicals In algebra, a nested radical is a radical expression (one containing a square root sign, cube root sign, etc.) that contains (nests) another radical expression. Examples include $\sqrt{2+\sqrt{2+..}}$

1.2 Linear function: Linear functions are those whose graph is a straight line. A linear function has the following form. y = f(x) = a + bx. A linear function has one independent variable and one dependent variable. The independent variable is x and the dependent variable is y

2. PI AND ANGULAR ROTATION (θ) IN A CIRCLE.

Theorem 1a: If $f(\theta)$ is a linear function in form of $f(\theta) = 2sin\frac{\theta}{4} + \sum_{n=1}^{\infty} 2^{n+3}\, sin^3\frac{\theta}{2^{n+3}}\, cos\frac{\theta}{2^{n+3}}$ then for different value of (θ), $f(\theta) = \pi(\frac{\theta}{360})$.

Theorem 1b: If (θ) = 240˚ & 300˚ then

$$\frac{2\pi}{3} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{3}}}\right.}\ (n)\ square\ roots$$

$$\frac{5\pi}{6} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots-\sqrt{3}}}\right.}\ (n+2)\ square\ roots$$

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Proof Theorem 1a:

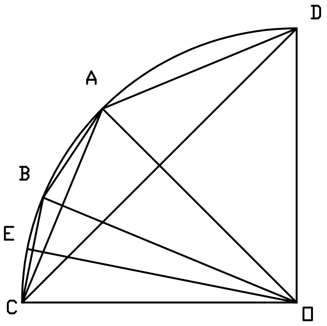

Diagram-1

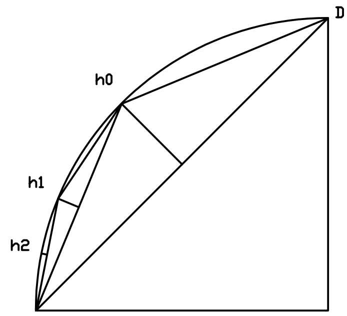

Diagram-2

In diagram 1, OCEBAD is a quarter of a circle

Line OA divide line CD into two equal parts and triangle CAD is drawn. Height of ΔCAD = h0

Line OB divide line AC into two equal parts and triangle ABC is drawn. Height of ΔABC = h1

Line OE divide line CB into two equal parts and triangle CEB is drawn. Height of ΔCEB = h2

Like the same onward infinite triangle can be drawn dividing previous arm into two equal parts.

Line OA divide quarter of circle into two equal parts (OCA & OAD)

In region OCA, after dividing line CA there is one triangle ABC & also in region OAD there will be one triangle if same procedure will adopt.

It can be written as after first division there are total two triangles in region OCA & OAD
n=1 Triangle=2

In region OCA, after dividing line BC & AB there are two triangle whose base is BC & AB and also in region OAD there will be two triangle if same procedure will adopt.

It can be written as after second division there are total four triangles in region OCA & OAD
n=2 Triangle=4

Like the same n=3 Triangle = 8

Number of triangle in term of number of division (n) can be written as

$$Number\ of\ Triangle = 2^n$$

Base & Height of triangles.

In diagram 1

$$CA^2 = (base\ of\ \Delta CAD/2)^2 + height\ of\ \Delta CAD^2$$ {CA is base of triangle after first division}

$$h1 = R - \sqrt{R^2 - (Base\ CA/2)^2}$$ {h1 is height of triangle after first division & R is radius}

In diagram 1, Base & Height of Nth triangle can be written as

$$(Base\ of\ nth\ triangle, Bn)^2 = (Base\ of\ (n-1)th\ triangle/2)^2 + (Height\ of\ (n-1)th\ triangle)^2$$

$$Bn^2 = (B\ (n-1)/2)^2 + (H\ (n-1))^2$$

Height of nth triangle(h_n) = R -$\sqrt{R^2 - (Base\ of\ \text{nth}\ triangle\ /2)^2}$

$$h_n = R - \sqrt{R^2 - (Bnt\ /2)^2}$$

B_0 & h_0= Base & Height of Δ CAD

B_0 = R√2

h_0= R (√2- 1)/√2

$Bn^2 = (B\,(n-1)/2)^2 + (H\,(n-1))^2$
$B_1{}^2 = (B0/2)^2 + (H\,0)^2$
$B_1{}^2 = R^2/2 + R^2\,(2+1-2\sqrt{2})/2$
$B_1 = R\sqrt{2-\sqrt{2}}$

h_n = R -$\sqrt{R^2-(Bnt\,/2)^2}$
h_1 = R - $\sqrt{R^2-(B1\,/2)^2}$
h_1= R - $\sqrt{R^2-R^2(2-\sqrt{2})/4}$
h_1 = R(2 -$\sqrt{2+\sqrt{2}}$) /2

$B_2{}^2 = (B1/2)^2 + (h_1)^2$
$B_2{}^2 = R^2\,(2-\sqrt{2})/4 + R^2\,(4+2+\sqrt{2}-4\sqrt{2+\sqrt{2}}\ )/4$
$B_2 =$ R$\sqrt{2-\sqrt{2+\sqrt{2}}}$

h_2 = R - $\sqrt{R^2-(B2\,/2)^2}$
h_2 = R - $\sqrt{R^2-\frac{R^2(2-\sqrt{2+\sqrt{2})}}{4}}$
h_2 = R$(2-\sqrt{2+\sqrt{2+\sqrt{2}}}\,)$ /2

Onward B3H3, B4H4………can be find out.

Base & Height of Nth triangle can be written as

$B_n = R\sqrt{\left\{2-\sqrt{2(n+1)square\ roots}\right\}}$

$h_n = R\left\{2-\sqrt{2(n+2)square\ roots}\right\}$ / 2

In diagram 1 area of quarter circle = $\frac{\pi}{4}$ R² =Area of ΔOCD + Area of Δ CAD + $\sum_{n=1}^{\infty} 2^n\ \frac{1}{2} B_n h_n$

$\frac{\pi}{4}R^2 = \frac{1}{2}$R² +$\frac{1}{2}$R²($\sqrt{2}$ -1)+ $\sum_{n=1}^{\infty} 2^n\ \frac{1}{2} B_n h_n$

$\frac{\pi}{2} = 1+\sqrt{2}$ -1+ $\sum_{n=1}^{\infty} 2^n\ B_n h_n$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n-1}\ \sqrt{\left\{2-\sqrt{2(n+1)square\ roots}\right\}} \times (2-\sqrt{2(n+2)square\ roots}\ ) \qquad (A)$$

$\left\{2-\sqrt{2(n+1)square\ roots}\right\}$ can be written as

$\left\{2-\sqrt{2(n+1)square\ roots}\right\}$ = $(2-\sqrt{2(n+2)square\ roots}\,)\times(2+\sqrt{2(n+2)square\ roots}\,)$

Example let n = 2

$\left\{2-\sqrt{2+\sqrt{2}}\right\} = (2-\sqrt{2+\ \sqrt{2+\sqrt{2}}}\,)\times(2+\sqrt{2+\sqrt{2+\sqrt{2}}}\,)$

Hence equation (A) can be written as

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n-1}\ \sqrt{(2+\sqrt{2(n+2)square\ roots}\,)\times(2-\sqrt{2(n+2)square\ roots}\ )^3} \qquad (B)$$

We know that

$\cos\frac{45}{2} = \sqrt{\frac{1+cos\,45}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$ and $\sin\frac{45}{2} = \sqrt{\frac{1-cos\,45}{2}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

$\cos\frac{45}{2^{n+1}} = \frac{\sqrt{2+\sqrt{2}n+2\ square\ \ roots}}{2}$ and $\sin\frac{45}{2^{n+1}} = \frac{\sqrt{2-\sqrt{2}n+2\ square\ \ roots}}{2}$

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Equation (B) can be written as

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n-1} \times 2\cos\frac{45}{2^{n+1}} \times 8\, sin^3 \frac{45}{2^{n+1}}$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n+3} \cos\frac{45}{2^{n+1}}\; sin^3 \frac{45}{2^{n+1}}$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n+3} \sin^3 \frac{\pi}{2^{n+3}} \cos\frac{\pi}{2^{n+3}} \qquad \text{(C)}$$

Equation (A) can also be written as

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n-1} \sqrt{\left\{2 - \sqrt{2(n+1) square\ roots}\right\}} \times (2 - \sqrt{2(n+2) square\ roots}\ )$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n-1}\; 2\sin\frac{45}{2^n}\times(2\text{- }2\cos\frac{45}{2^n})$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n+1}\; (\sin\frac{45}{2^n} - \sin\frac{45}{2^n}\cos\frac{45}{2^n})$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n+1}\; (\sin\frac{\pi}{2^{n+2}} - \sin\frac{\pi}{2^{n+2}}\cos\frac{\pi}{2^{n+2}})$$

$$\frac{\pi}{2} = \sqrt{2} + \sum_{n=1}^{\infty} 2^{n+1}\; (\sin\frac{\pi}{2^{n+2}} - \frac{1}{2}\sin\frac{\pi}{2^{n+1}}) \qquad \text{(D)}$$

R.H.S of equation (D) = $\sqrt{2} + \sum_{n=1}^{\infty} 2^{n+1}\; (\sin\frac{\pi}{2^{n+2}} - \frac{1}{2}\sin\frac{\pi}{2^{n+1}})$

Expansion of equation (D) upto N tern
N=4

$$\sqrt{2} + \sum_{n=1}^{4} 2^{n+1} (\sin\frac{\pi}{2^{n+2}} - \frac{1}{2}\sin\frac{\pi}{2^{n+1}}) = \sqrt{2} + 4(\sin\frac{\pi}{8} - \frac{1}{2}\sin\frac{\pi}{4}) + 8(\sin\frac{\pi}{16} - \frac{1}{2}\sin\frac{\pi}{8}) + 16(\sin\frac{\pi}{32} - \frac{1}{2}\sin\frac{\pi}{16}) + 32(\sin\frac{\pi}{64} - \frac{1}{2}\sin\frac{\pi}{32}) = 32\sin\frac{\pi}{64}$$

RHS equation can be written as

$$\sqrt{2} + \sum_{n=1}^{N} 2^{n+1}\; (sin\frac{\pi}{2^{n+2}} - \frac{1}{2}sin\frac{\pi}{2^{n+1}}) = 2^{N+1}\, sin\frac{\pi}{2^{N+2}}$$

Suppose $\sqrt{2} = C_\theta$ & $\pi = \theta$

$$C_\theta + \sum_{n=1}^{N} 2^{n+1}\; (sin\frac{\theta}{2^{n+2}} - \frac{1}{2}sin\frac{\theta}{2^{n+1}}) = 2^{N+1}\, sin\frac{\theta}{2^{N+2}} \qquad \text{(E)}$$

Value of C_θ for different value of N
N=1

$$C_\theta + 4(sin\frac{\theta}{8} - \frac{1}{2}sin\frac{\theta}{4}) = 4\, sin\frac{\theta}{8}$$

$$C_\theta = 2\, sin\frac{\theta}{4}$$

For any value of N & θ, C_θ *will be* $2\, sin\frac{\theta}{4}$

Now Equation (E) will be

$$2\, sin\frac{\theta}{4} + \sum_{n=1}^{N} 2^{n+1}\; (sin\frac{\theta}{2^{n+2}} - \frac{1}{2}sin\frac{\theta}{2^{n+1}}) = 2^{N+1}\, sin\frac{\theta}{2^{N+2}}$$

We know that

$$\lim_{N\to\infty} 2^{N+1}\, sin\frac{\theta}{2^{N+2}} = \frac{\pi}{2} \times \frac{\theta}{180}$$

Equations (C) & (D) can be written as

$$2\, sin\frac{\theta}{4} + \sum_{n=1}^{\infty} 2^{n+3}\; sin^3\frac{\theta}{2^{n+3}}\, cos\frac{\theta}{2^{n+3}} = \frac{\pi}{2} \times \frac{\theta}{180}$$

$$2\, sin\frac{\theta}{4} + \sum_{n=1}^{\infty} 2^{n+1}\; (sin\frac{\theta}{2^{n+2}} - \frac{1}{2}sin\frac{\theta}{2^{n+1}}) = \frac{\pi}{2} \times \frac{\theta}{180}$$

Theorem 1b: If *(θ)* = 240˚ & 300˚ then

$$\frac{2\pi}{3} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{3}}}\right.}\ \textit{(n) square roots}$$

$$\frac{5\pi}{6} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots-\sqrt{3}}}\right.}\ \textit{(n+2) square roots}$$

Proof:

We know that

$$\lim_{N\to\infty} 2^{N+1} \sin\frac{\theta}{2^{N+2}} = \frac{\pi}{2}\times\frac{\theta}{180}$$

When, θ= 240˚

$$\lim_{N\to\infty} 2^{N+1} \sin\frac{240}{2^{N+2}} = \frac{\pi}{2}\times\frac{240}{180}$$

$$\lim_{N\to\infty} 2^{N} \sin\frac{240}{4\times 2^{N}} = \frac{\pi}{3}$$

$$\lim_{N\to\infty} 2^{N} \sin\frac{60}{2^{N}} = \frac{\pi}{3} \qquad \text{(F)}$$

$$\sin\frac{60}{2} = \sqrt{\frac{1-\cos 60}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2 60}}{2}} = \frac{1}{2}$$

$$\sin\frac{60}{2^2} = \sqrt{\frac{1-\cos\frac{60}{2}}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{60}{2}}}{2}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$\sin\frac{60}{2^3} = \sqrt{\frac{1-\cos\frac{60}{2^2}}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{60}{2^2}}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{3}}}}{2}$$

$$\sin\frac{60}{2^4} = \sqrt{\frac{1-\cos\frac{60}{2^3}}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{60}{2^3}}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}}{2}$$

Hence $\sin\frac{60}{2^n}$ can be written as

$$\sin\frac{60}{2^n} = \frac{\sqrt{\left\{2-\sqrt{2+\sqrt{2+\cdots\ldots+\sqrt{3}\ (n)\ square\ \ roots}}\right.}}{2}$$

Put the value of $\sin\frac{60}{2^n}$ in equation (F)

$$\frac{2\pi}{3} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{3}}}\right.}\ \textit{(n) square roots}$$

When, θ= 300˚

$$\lim_{N\to\infty} 2^{N+1} \sin\frac{300}{2^{N+2}} = \frac{\pi}{2}\times\frac{300}{180}$$

$$\lim_{N\to\infty} 2^{N+1} \sin\frac{300}{4\times 2^{N}} = \frac{5\pi}{6}$$

$$\lim_{N\to\infty} 2^{N+1} \sin\frac{75}{2^{N}} = \frac{5\pi}{6} \qquad \text{(G)}$$

We know that cos75˚= $\frac{\sqrt{2-\sqrt{3}}}{2}$

$$\sin\frac{75}{2} = \sqrt{\frac{1-\cos 75}{2}} = \frac{\sqrt{2-\sqrt{2-\sqrt{3}}}}{2}$$

$$\sin\frac{75}{2^2} = \sqrt{\frac{1-\cos\frac{75}{2}}{2}} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{75}{2}}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{3}}}}}{2}$$

$$\sin\frac{75}{2^3} = \sqrt{\frac{1-\sqrt{1-\sin^2\frac{75}{2^2}}}{2}} = \frac{\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{3}}}}}}{2}$$

Hence $\sin\frac{75}{2^n}$ can be written as

$$\sin\frac{75}{2^n} = \frac{\sqrt{\left\{2-\sqrt{2+\sqrt{2+\cdots\ldots-\sqrt{3}\ (n+2)\ square\ \ roots}}\right.}}{2}$$

Put the value of $\sin\frac{75}{2^n}$ in equation (G)

$$\frac{5\pi}{6} = lim_{n\to\infty}\ 2^n \sqrt{\left\{2-\sqrt{2+\sqrt{2+\ldots-\sqrt{3}}}\right.}\ (n+2)\ square\ roots$$

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