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ANALYTICAL SOLUTION OF ABEL'S INTEGRAL EQUATION OF THE FIRST KIND BY USING q-HOMOTOPY ANALYSIS TRANSFORM METHOD

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ABSTRACT

In this paper, the analytical solution of Abel's integral equation of the first kind is investigated by using the q-homotopy analysis transform method (q-HATM). The q-HATM is a hybrid method that combines the q-homotopy analysis method (q-HAM) and the Laplace transform method (LTM). The analytical results obtained by the proposed method are in series form, indicating that the approach is simple to implement and computationally appealing.

Keywords: *Laplace transform, q-homotopy analysis method and Abel's integral equations of first kind.*

MSC (2020): *44A10, 45E10.*

1. INTRODUCTION

The Abel's integral equation of first kind is a particular case of Volterra integral equation. It's also known as a singular integral equation and Wazwaz [32] [34] presented general Abel's integral equation of first kind as

$$\int_a^{\xi} \frac{y(t)}{(\xi - t)^{\alpha}} dt = f(\xi), \; 0 < \alpha < 1. \tag{1.1}$$

where kernel $\frac{1}{(\xi - t)^{\alpha}} = K(\xi, t)$ is singular form and $K(\xi, t) \to \infty$ as $t \to \xi$.

Abel's integral equation gives the ability to describe a lot of physical and engineering phenomena accurately such as water wave [7], spectroscopic data [6], stellar wind [14], simultaneous dual relations [31] and population dynamics, chemical reaction, semiconductors, seismology, heat conduction, fluid flow, metallurgy, scattering theory [11] etc.

It can be solved without using a differential equation [1], and there are many methods for solving Abel's integral equation that are much easier to use, such as the homotopy analysis method (HAM) [12], the homotopy analysis transform method (HATM) [25], and the homotopy perturbation transform method (HPTM) [16], the Babenko's approach [5], the homotopy analysis transform method (HATM) [22], the optimal homotopy analysis transform method (OHATM) [23], the two-step Laplace decomposition algorithm (TSLDA) [13], the homotopy perturbation method HPM [15], the Taylor-collocation method [35] and solution of Abel's integral equation using Hermite Wavelet [24], Chebyshev polynomials [2][27], Normalized Bernstein Polynomials [29], Mikusinski's operator [17], etc. On the other hand, Liu and Tao [20][21] introduced a mechanical quadrature technique for approximating the solution of Abel's integral equation of the first kind, and Li M. and Zhao W. [18] used Bernstein polynomials to obtain a numerical solution of the Abel's integral equation.

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Now, in this article, we used the q-homotopy analysis transform method (q-HATM) to solve Abel's integral equation of first kind. The q-HATM method was created by combining the q-HAM and Laplace transform method, and it provides convergent series solutions. The q-HAM was introduced by E-Tavil and Huseen [9][10], where q is an embedding parameter $q \in [0,1]$ and q-HAM arising in the study by Liao HAM [19] to $q \in \left[0, \frac{1}{n}\right], n \geq 1$. The q-HATM is a very efficient method for solving many equations, including the Telegraph equation [30], Burger equation [26], coupled Burger equation [28], diffusion equation [4] [8], Abel's integral equation of second kind [3] and others.

The main goal of this paper is to find an approximate analytical solution of the Abel's integral equation of first kind.

2. PRELIMINARIES AND NOTATIONS

In this section, we give some basic definitions and notations of Laplace transform theory, which are used further in this paper.

Definition 1: The Laplace transform of a function $f(\xi),\ \xi > 0$ is defined as

$$L\{f(\xi);s\} = F(s) = \int_0^\infty e^{-s\xi} f(\xi)\, d\xi, \tag{2.1}$$

where s is real or complex number.

Definition 2: The convolution of two functions $f(\xi)$ and $g(\xi)$ is defined as

$$f(\xi) * g(\xi) = \int_0^\xi f(\xi - v)\, g(\xi)\, dv, \tag{2.2}$$

Laplace transform of convolution of two functions is given by

$$L\{f(\xi) * g(\xi)\} = L\left\{\int_0^\xi f(v)\, g(\xi - v)\, dv\right\} = F(s)G(s) \quad . \tag{2.3}$$

3. A HYBRID COMPUTATIONAL TECHNIQUE

In this part, we take a general Abel's integral equation of first kind to present the solution procedure of the proposed technique

$$y(\xi) = \int_a^\xi \frac{y(t)}{(\xi - t)^\alpha} dt,\ 0 < \alpha < 1. \tag{3.1}$$

Taking Laplace transform of Eq. (3.1), we obtain

$$L[y(\xi)] = L\left[\int_a^\xi \frac{y(t)}{(\xi - t)^\alpha} dt\right]. \tag{3.2}$$

Now, we consider a nonlinear operator as

$$N[\phi(\xi;q)] = L\left[\int_a^\xi \frac{y(t)}{(\xi - t)^\alpha} dt\right] - L[\phi(\xi;q)]. \tag{3.3}$$

We can construct a homotopy following as

$$(1 - nq) L[\phi(\xi;q) - y_0(\xi)] = \hbar q H(\xi) N[\phi(\xi;q)], \tag{3.4}$$

where, $q \in \left[0, \frac{1}{n}\right], n \geq 1$ is an embedding parameter, $H(\xi)$ is an auxiliary function which is nonzero, $\hbar$ is auxiliary parameter and has negative value at many practical situations and $\phi(\xi;q)$ is an unknown function. Clearly, the following conditions holds for $q = 0$ and $q = \frac{1}{n}$

$$\phi(\xi;0) = y_0(\xi) \text{ and } \phi\left(\xi;\frac{1}{n}\right) = y(\xi) \tag{3.5}$$

respectively.

Consequently, as q increase from 0 to $\frac{1}{n}$ the solution $\phi(\xi;q)$ transform from initial guess $y_0(\xi)$ to the solution $y(\xi)$. Function $\phi(\xi;q)$ can be written in series form with the help of Taylor's theorem about q, we have

$$\phi(\xi;q) = y_0(\xi) + \sum_{m=1}^{\infty} y_m(\xi) q^m \,, \tag{3.6}$$

where

$$y_m(\xi) = \frac{1}{\lfloor m} \frac{\partial^m \phi(\xi;q)}{\partial q^m}\Big|_{q=0} \,. \tag{3.7}$$

If $\hbar$, n and initial guess $y_0(\xi)$ are properly chosen then the series (3.6) converges at $q = \frac{1}{n}$. Then we have

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty} y_m(\xi) \left(\frac{1}{n}\right)^m \,. \tag{3.8}$$

Now, we define the vector

$$\vec{y}_m = \{y_0(\xi), y_2(\xi), \ldots, y_m(\xi)\} \,. \tag{3.9}$$

Taking m-times differentiating the zero order deformation Eq. (3.4) with respect to q and dividing by $\lfloor m$ and put $q = 0$ then we can construct the m^{th} order deformation equation

$$L\left[y_m(\xi) - k_m y_{m-1}(\xi)\right] = \hbar H(\xi) R_m\left[\vec{y}_{m-1}\right], \tag{3.10}$$

where

$$R_m\left[\vec{y}_{m-1}\right] = \frac{1}{\lfloor m-1} \frac{\partial^{m-1} N\left[\phi(\xi;q)\right]}{\partial q^{m-1}}\Big|_{q=0} \tag{3.11}$$

and

$$k_m = \begin{cases} 0, & m \le 1 \\ n, & m > 1 \end{cases}. \tag{3.12}$$

In q-HATM, we underline that it is possible to independently select initial hypotheses $y_0(\xi)$, non-zero auxiliary parameter $\hbar$ and asymmetric parameter n. The series obtained from Eq. (3.8) in a factor $\left(\frac{1}{n}\right)^m$ in the solution and should be visualized faster than obtained solution by the HATM. Note that Eq. (3.8) in q-HATM is reduced to HATM for $n = 1$. The auxiliary parameter $\hbar$ plays a very important role in controlling the convergence zone and convergence speed.

4. SOLUTION OF ABEL'S INTEGRAL EQUATION OF FIRST KIND BY *q*-HATM TECHNIQUE

In this section, we apply the q-HATM technique and check the validity with the help of some different type examples on Abel's integral equation of first kind.

Example 1: Consider the following Abel's integral equation of first kind as [25]

$$\frac{4}{3}\xi^{\frac{3}{2}} = \int_0^{\xi} \frac{1}{\sqrt{\xi - t}} y(t)\, dt \tag{4.1}$$

with exact solution $y(\xi) = \xi$.

Taking Laplace transform of Eq. (4.1), we obtain

$$L\left[\frac{4}{3}\xi^{\frac{3}{2}}\right] = \frac{\sqrt{\pi}}{\Gamma(s)} L\left[y(\xi)\right]. \tag{4.2}$$

Now, we consider a nonlinear operator as

$$N\left[\phi(\xi;q)\right] = \frac{\sqrt{\pi}}{\Gamma(s)} L\left[\phi(\xi;q)\right] - L\left[\frac{4}{3}\xi^{\frac{3}{2}}\right]. \tag{4.3}$$

Now, we can construct the m^{th} order deformation equation for $H(\xi)=1$ as

$$L\left[y_m(\xi)-k_m y_{m-1}(\xi)\right]=\hbar R_m\left(\vec{y}_{m-1}\right), \tag{4.4}$$

where

$$R_m\left(\vec{y}_{m-1}\right)=\frac{\sqrt{\pi}}{\Gamma(s)}\ L\left[y_{m-1}(\xi)\right]-\left(1-\frac{k_m}{n}\right)L\left[\frac{4}{3}\xi^{\frac{3}{2}}\right] \tag{4.5}$$

and

$$k_m=\begin{cases}0, & m\le 1\\ n, & m>1\end{cases}.$$

Taking inverse Laplace transform on both sides of Eq.(4.4), we have

$$y_m(\xi)=k_m y_{m-1}(\xi)+\hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \tag{4.6}$$

Taking initial approximation $y_0(\xi)=\frac{4}{3}\xi^{\frac{3}{2}}$, we obtain

$$y_0(\xi)=\frac{4}{3}\xi^{\frac{3}{2}}\qquad , \tag{4.7}$$

$$y_1(\xi)=-\hbar\left(\frac{4}{3}\xi^{\frac{2}{3}}-\frac{\pi}{2}\xi^2\right), \tag{4.8}$$

$$y_2(\xi)=-n\hbar\left[\frac{4}{3}\xi^{\frac{2}{3}}-\frac{\pi}{2}\xi^2\right]-\hbar^2\left[\frac{\pi}{2}\xi^2-\frac{8\pi}{15}\xi^{\frac{5}{2}}\right], \tag{4.9}$$

$$y_3(\xi)=-n^2\hbar\left[\frac{4}{3}\xi^{\frac{2}{3}}-\frac{\pi}{2}\xi^2\right]-2n\hbar^2\left[\frac{\pi}{2}\xi^2-\frac{8\pi}{15}\xi^{\frac{5}{2}}\right]-\hbar^3\left[\frac{8\pi}{15}\xi^{\frac{5}{2}}-\frac{\pi^2}{6}\xi^3\right]. \tag{4.10}$$

Proceeding in this manner, we can also compute the rest of component of $y_m(\xi)$ for $m\ge 4$.The solution of given Abel's integral equation by *q*-HATM is expressed in series form as

$$y(\xi)=y_0(\xi)+\sum_{m=1}^{\infty}y_m(\xi)\left(\frac{1}{n}\right)^m \tag{4.11}$$

Now, we put the value of $y_0(\xi)$ and $y_m(\xi); m=1,2,3,4,...,24$ in Eq. (4.11) and taking $\hbar=-1$ and $n=1$, we have $y(\xi)$ up to 25^{th} terms following as

$$\begin{aligned}y(\xi)=&\frac{100}{3}\xi^{\frac{3}{2}}-150\pi\xi^2+\frac{3680\pi}{3}\xi^{\frac{5}{2}}-\frac{6325\pi^2}{3}\xi^3+8096\pi^2\xi^{\frac{7}{2}}-\frac{44275\pi^3}{6}\xi^4+\frac{3076480\pi^3}{189}\xi^{\frac{9}{2}}\\&-\frac{72105\pi^4}{8}\xi^5+\frac{408595\pi^4}{2079}\xi^{\frac{11}{2}}-\frac{81719\pi^5}{18}\xi^6+\frac{114109440\pi^5}{27027}\xi^{\frac{13}{2}}-\frac{260015\pi^6}{252}\xi^7\\&+\frac{266255360\pi^6}{40541}\xi^{\frac{15}{2}}-\frac{37145\pi^7}{336}\xi^8+\frac{334721024\pi^7}{6891885}\xi^{\frac{17}{2}}-\frac{408595\pi^8}{72576}\xi^9+\frac{4922368\pi^8}{2909907}\xi^{\frac{19}{2}}\\&-\frac{4807\pi^9}{36288}\xi^{10}+\frac{14508032\pi^9}{549972423}\xi^{\frac{21}{2}}-\frac{1771\pi^{10}}{1330560}\xi^{11}+\frac{2023424\pi^{10}}{12649365729}\xi^{\frac{23}{2}}-\frac{23\pi^{11}}{4790016}\xi^{12}\\&+\frac{32768\pi^{11}}{105411381075}\xi^{\frac{25}{2}}-\frac{\pi^{12}}{249080832}\xi^{13}+\frac{16384\pi^{12}}{213458046675875}\xi^{\frac{27}{2}}\end{aligned} \tag{4.12}$$

If we take $\xi=1$ then exact solution will be $y(1)=1$ and approximate solution up to 25^{th} terms by the Eq. (4.12) will be $y(1)=1.00000545$. Here we see that if we increase number of terms in Eq. (4.12) then approximate analytical solution tends to exact solution.

Example 2: Consider the following Abel's integral equation of first kind as [25]

$$\frac{2}{105}\sqrt{\xi}\left[105-56\xi^2+48\xi^3\right]=\int_0^{\xi}\frac{1}{\sqrt{\xi-t}}y(t)\,dt \tag{4.13}$$

with the exact solution $y(\xi)=\xi^3-\xi^2+1$.

Taking Laplace transform of Eq. (4.13), we obtain

$$L\left[2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right]=\frac{\sqrt{\pi}}{\sqrt{s}}L\left[y(\xi)\right]. \tag{4.14}$$

Now, we consider a nonlinear operator as

$$N\left[\phi(\xi;q)\right]=\frac{\sqrt{\pi}}{\sqrt{s}}\,L\left[\phi(\xi;q)\right]-L\left[2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right]. \tag{4.15}$$

We can construct the m^{th} order deformation equation for $H(\xi)=1$ as

$$L\left[y_m(\xi)-k_m y_{m-1}(\xi)\right]=\hbar R_m\left(\vec{y}_{m-1}\right), \tag{4.16}$$

where

$$R_m\left(\vec{y}_{m-1}\right)=\frac{\sqrt{\pi}}{\sqrt{s}}\,L\left[y_{m-1}(\xi)\right]-\left(1-\frac{k_m}{n}\right)L\left[2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right] \tag{4.17}$$

and

$$k_m=\begin{cases}0, & m\le 1\\ n, & m>1\end{cases}.$$

Taking Inverse Laplace transform on both sides of Eq. (4.16), we have

$$y_m(\xi)=k_m y_{m-1}(\xi)+\hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \tag{4.18}$$

Taking initial approximation as $y_0(\xi)=2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}$, we obtain

$$y_0(\xi)=2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}, \tag{4.19}$$

$$y_1(\xi)=-\hbar\left(2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right)+\hbar\left(\pi\xi-\frac{\pi}{3}\xi^3+\frac{\pi}{4}\xi^4\right), \tag{4.20}$$

$$y_2(\xi)=-n\hbar\left(2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right)+\hbar(n-\hbar)\left(\pi\xi-\frac{\pi}{3}\xi^3+\frac{\pi}{4}\xi^4\right)+\hbar^2\left(\frac{4\pi}{3}\xi^{\frac{3}{2}}-\frac{32\pi}{105}\xi^{\frac{7}{2}}+\frac{64\pi}{315}\xi^{\frac{9}{2}}\right), \tag{4.21}$$

$$\begin{aligned}y_3(\xi)=&-n^2\hbar\left(2\xi^{\frac{1}{2}}-\frac{16}{15}\xi^{\frac{5}{2}}+\frac{32}{35}\xi^{\frac{7}{2}}\right)+n\hbar(n-2\hbar)\left(\pi\xi-\frac{\pi}{3}\xi^3+\frac{\pi}{4}\xi^4\right)\\&+\hbar^2(2n-\hbar)\left(\frac{4\pi}{3}\xi^{\frac{3}{2}}-\frac{32\pi}{105}\xi^{\frac{7}{2}}+\frac{64\pi}{315}\xi^{\frac{9}{2}}\right)+\hbar^3\left(\frac{\pi^2}{2}\xi^2-\frac{\pi^2}{12}\xi^4+\frac{\pi^2}{20}\xi^5\right)\end{aligned}. \tag{4.22}$$

Proceeding in this manner, we can also compute the rest of component of $y_m(\xi)$ for $m\ge 4$. The solution of given Abel's integral equation by q-HATM is expressed in series form as

$$y(\xi)=y_0(\xi)+\sum_{m=1}^{\infty}y_m(\xi)\left(\frac{1}{n}\right)^m. \tag{4.23}$$

Now, put the value of $y_0(\xi)$ and $y_m(\xi); m=1,2,3,4,...,24$ in Eq. (4.23) and taking $\hbar=-1$ and $n=1$, we have $y(\xi)$ up to 25^{th} terms following as

$$\begin{aligned}y(\xi)=&50\left(1-\frac{8}{15}\xi^2+\frac{16}{35}\xi^3\right)\xi^{\frac{1}{2}}-300\pi\left(1-\frac{1}{3}\xi^2+\frac{1}{4}\xi^3\right)\xi\\&+\frac{9200\pi}{3}\left(1-\frac{8}{35}\xi^2+\frac{16}{105}\xi^3\right)\xi^{\frac{3}{2}}-6325\pi^2\left(1-\frac{1}{6}\xi^2+\frac{1}{10}\xi^3\right)\xi^2\\&+28336\pi^2\left(1-\frac{8}{63}\xi^2+\frac{16}{231}\xi^3\right)\xi^{\frac{5}{2}}-\frac{88550\pi^3}{3}\left(1-\frac{1}{10}\xi^2+\frac{1}{20}\xi^3\right)\xi^3\\&+\frac{1538240\pi^3}{21}\left(1-\frac{8}{99}\xi^2+\frac{16}{429}\xi^3\right)\xi^{\frac{7}{2}}-\frac{360525\pi^4}{8}\left(1-\frac{1}{15}\xi^2+\frac{1}{35}\xi^3\right)\xi^4\end{aligned}$$

$$+\frac{13075040\pi^4}{189}\left(1-\frac{8}{143}\xi^2+\frac{16}{715}\xi^3\right)\xi^{\frac{9}{2}}-\frac{81719\pi^5}{3}\left(1-\frac{1}{21}\xi^2+\frac{1}{56}\xi^3\right)\xi^5$$
$$+\frac{19018240\pi^5}{693}\left(1-\frac{8}{195}\xi^2+\frac{16}{1105}\xi^3\right)\xi^{\frac{11}{2}}-\frac{260015\pi^6}{36}\left(1-\frac{1}{28}\xi^2+\frac{1}{84}\xi^3\right)\xi^6$$
$$+\frac{19018240\pi^6}{3861}\left(1-\frac{8}{255}\xi^2+\frac{16}{1615}\xi^3\right)\xi^{\frac{13}{2}}-\frac{37145\pi^7}{42}\left(1-\frac{1}{36}\xi^2+\frac{1}{120}\xi^3\right)\xi^7$$
$$+\frac{15214592\pi^7}{36855}\left(1-\frac{8}{323}\xi^2+\frac{16}{2261}\xi^3\right)\xi^{\frac{15}{2}}-\frac{408595\pi^8}{8064}\left(1-\frac{1}{45}\xi^2+\frac{1}{165}\xi^3\right)\xi^8$$
$$+\frac{223744\pi^8}{13923}\left(1-\frac{8}{399}\xi^2+\frac{16}{3059}\xi^3\right)\xi^{\frac{17}{2}}-\frac{24035\pi^9}{18144}\left(1-\frac{1}{55}\xi^2+\frac{1}{220}\xi^3\right)\xi^9$$
$$+\frac{94208\pi^9}{340119}\left(1-\frac{8}{483}\xi^2+\frac{16}{4025}\xi^3\right)\xi^{\frac{19}{2}}-\frac{253\pi^{10}}{17280}\left(1-\frac{1}{66}\xi^2+\frac{1}{286}\xi^3\right)\xi^{10}$$
$$+\frac{94208\pi^{10}}{49997493}\left(1-\frac{8}{575}\xi^2+\frac{16}{5175}\xi^3\right)\xi^{\frac{21}{2}}-\frac{1538240\pi^3}{399168}\left(1-\frac{1}{78}\xi^2+\frac{1}{364}\xi^3\right)\xi^{11}$$
$$+\frac{16384\pi^{11}}{4216455243}\left(1-\frac{8}{675}\xi^2+\frac{16}{6525}\xi^3\right)\xi^{\frac{23}{2}}-\frac{1538240\pi^3}{19160064}\left(1-\frac{1}{91}\xi^2+\frac{1}{455}\xi^3\right)\xi^{12}$$
$$+\frac{8192\pi^{12}}{7905853580625}\left(1-\frac{8}{783}\xi^2+\frac{16}{8091}\xi^3\right)\xi^{\frac{25}{2}} \qquad (4.24)$$

If we take $\xi=1$ then exact solution will be $y(1)=1$ and approximate solution up to 25^{th} terms by the Eq. (4.24) will be $y(1)=0.99998122$. Here we see that if we increase number of terms in Eq. (4.24) then approximate analytical solution tends to exact solution.

Example 3: In this example, the following Abel's integral equation of first kind is considered as [33]

$$\frac{\pi}{2}\xi=\int_0^{\xi}\frac{1}{\sqrt{\xi-t}}y(t)dt \qquad (4.25)$$

with the exact solution $y(\xi)=\sqrt{\xi}$.

Taking Laplace transform of Eq. (4.25), we obtain

$$L\left[\frac{\pi}{2}\xi\right]=\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}L\left[y(\xi)\right]. \qquad (4.26)$$

Now, we consider a nonlinear operator as

$$N\left[\phi(\xi;q)\right]=\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}L\left[\phi(\xi;q)\right]-L\left[\frac{\pi}{2}\xi\right]. \qquad (4.27)$$

We can construct the m^{th} order deformation equation for $H(\xi)=1$ as

$$L\left[y_m(\xi)-k_m y_{m-1}(\xi)\right]=\hbar R_m\left(\vec{y}_{m-1}\right), \qquad (4.28)$$

where

$$R_m\left(\vec{y}_{m-1}\right)=\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}L\left[y_{m-1}(\xi)\right]-\left(1-\frac{k_m}{n}\right)L\left[\frac{\pi}{2}\xi\right] \qquad (4.29)$$

and

$$k_m=\begin{cases}0, & m\le 1\\ n, & m>1\end{cases}.$$

Taking inverse Laplace transform on both sides of Eq. (4.28), we have

$$y_m(\xi) = k_m y_{m-1}(\xi) + \hbar L^{-1}\left[R_m\left(\vec{y}_{m-1}\right)\right]. \tag{4.30}$$

Taking initial approximation $y_0(\xi) = \frac{\pi}{2}\xi$, we obtain

$$y_0(\xi) = \frac{\pi}{2}\xi \quad , \tag{4.31}$$

$$y_1(\xi) = -\hbar\frac{\pi}{2}\xi + \hbar\left(\frac{2\pi}{3}\xi^{\frac{3}{2}}\right), \tag{4.32}$$

$$y_2(\xi) = -n\hbar\frac{\pi}{2}\xi + \hbar(n-\hbar)\frac{2\pi}{3}\xi^{\frac{3}{2}} + \hbar^2\frac{\pi^2}{4}\xi^2 \tag{4.33}$$

$$y_3(\xi) = -n^2\hbar\frac{\pi}{2}\xi + n\hbar(n-2\hbar)\frac{2\pi}{3}\xi^{\frac{3}{2}} + \hbar^2(2n-\hbar)\frac{\pi^2}{4}\xi^2 + \hbar^3\frac{4\pi^2}{15}\xi^{\frac{5}{2}} \tag{4.34}$$

$$y_4(\xi) = -n^3\hbar\frac{\pi}{2}\xi + n^2\hbar(n-3\hbar)\frac{2\pi}{3}\xi^{\frac{3}{2}} + 3n\hbar^2(n-\hbar)\frac{\pi^2}{4}\xi^2 + \hbar^3(3n-\hbar)\frac{4\pi^2}{15}\xi^{\frac{5}{2}} + \hbar^4\frac{\pi^3}{12}\xi^3 \tag{4.35}$$

Proceeding in this manner, we can also compute the rest of component of $y_m(\xi)$ for $m \geq 5$. The solution of given Abel's integral equation by q-HATM is expressed in series form as

$$y(\xi) = y_0(\xi) + \sum_{m=1}^{\infty} y_m(\xi)\left(\frac{1}{n}\right)^m. \tag{4.36}$$

Now, we put the value of $y_0(\xi)$ and $y_m(\xi); m = 1,2,3,4,\ldots,24$ in Eq. (4.36) and taking $\hbar = -1$ and $n = 1$, we have $y(\xi)$ up to 25^{th} terms following as

$$\begin{aligned} y(\xi) = {} & \frac{25}{2}\xi - 200\pi\xi^{\frac{3}{2}} + 575\pi^2\xi^2 - \frac{10120\pi^2}{3}\xi^{\frac{5}{2}} + \frac{8855\pi^3}{2}\xi^3 - \frac{40480\pi^3}{3}\xi^{\frac{7}{2}} + \frac{120175\pi^4}{4}\xi^4 \\ & - \frac{384560\pi^4}{21}\xi^{\frac{9}{2}} + \frac{408595\pi^5}{48}\xi^5 - \frac{1901824\pi^5}{189}\xi^{\frac{11}{2}} + \frac{111435\pi^6}{36}\xi^6 - \frac{9509120\pi^6}{3861}\xi^{\frac{13}{2}} \\ & + \frac{260015\pi^7}{504}\xi^7 - \frac{7607296\pi^7}{27027}\xi^{\frac{15}{2}} + \frac{81719\pi^8}{2016}\xi^8 - \frac{559360\pi^8}{36855}\xi^{\frac{17}{2}} + \frac{24035\pi^9}{16128}\xi^9 \\ & - \frac{235520\pi^9}{626535}\xi^{\frac{19}{2}} + \frac{253\pi^{10}}{6912}\xi^{10} - \frac{47104\pi^{10}}{11904165}\xi^{\frac{21}{2}} + \frac{23\pi^{11}}{145852}\xi^{11} - \frac{40960\pi^{11}}{2749862115}\xi^{\frac{23}{2}} \\ & + \frac{\pi^{12}}{958002900}\xi^{12} - \frac{4096\pi^{12}}{316234143225}\xi^{\frac{25}{2}} + \frac{\pi^{13}}{12454041600}\xi^{13} \end{aligned} \tag{4.37}$$

If we take $\xi = 1$ then exact solution will be $y(1) = 1$ and approximate solution up to 25^{th} terms by the Eq. (4.36) will be $y(1) = 1.00000739$. Here we see that if we increase number of terms in Eq. (4.36) then approximate analytical solution tends to exact solution.

5. CONCLUSION

The Abel's integral equation of first kind is solved using the method of q-HATM in this article. This method provided high efficiency and accuracy over a vast area of convergence. The results of the Abel's integral equations are obtained as the series form which are calculated very easily. The q-HATM is more powerful method than other methods. This method provides the freedom to select the values for the auxiliary linear operator L, the auxiliary function $H(\xi)$ and the initial function $y_0(\xi)$. The obtained solutions with auxiliary parameter $\hbar$ and asymptotic parameter n provide an easy way to adjust and control the convergence region and rate of convergence of the derived series solution.

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