International Journal of Mathematical Archive-12(5), 2021, 45-49
IJMA Available online through www.ijma.info ISSN 2229 – 5046

HOMOMORPHISM OR ANTI- HOMOMORPHISM OF LEFT (α, 1) - DERIVATIONS IN PRIME RINGS

C. JAYA SUBBA REDDY[1] & SK. HASEENA*[2]

[1]Department of Mathematics,
S. V. University, Tirupati- 517502, Andhra Pradesh, India.

[2]Research Scholar, Department of Mathematics,
S. V. University, Tirupati- 517502, Andhra Pradesh, India.

(Received On: 27-04-21; Revised & Accepted On: 10-05-21)

ABSTRACT

Let R be a 2- torsion free ring and let U be a Lie ideal of R. Suppose that α, 1 are automorphisms of R. An additive mapping d: R→ R is said to be a left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation) of R if $d(xy) = \alpha(x)d(y) + yd(x)$ (resp. $d(x^2) = \alpha(x)d(x) + xd(x)$) holds for all x, y∈ R. In this paper it is established that if R admits a nonzero left (α, 1)-derivation which acts as a homomorphism or as an anti-homomorphism on I of R, then d = 0 on R. Also we prove that if G: R→ R is an additive mapping satisfying G(xy) = α(x)G(y) + yd(x) for all x, y∈ R and a left (α, 1)-derivation d of R such that G also acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then either R is commutative or d = 0 on R.

Keywords: *Prime ring, Lie ideal, (α, 1)-derivation, Jordan (α, 1)-derivation, Left (α, 1)-derivation, Jordan left (α, 1)-derivation, Generalized left (α, 1)-derivation, Generalized Jordan left (α, 1)-derivation, Homomorphism, Anti-homomorphism.*

1. INTRODUCTION

The study of left derivation was initiated by Bresar and Vukman in [6] and it was shown that if a prime ring R of characteristic different from 2 and 3 admits a nonzero Jordan left derivation then R must be commutative. Ashraf and Ali [9] introduced the concepts of generalized left derivation and generalized Jordan left derivation. Bell and Kappe [5] proved that if K is a non-zero right ideal of a prime ring R and d: R→ R a derivation of R such that d acts as a homomorphism on K, then d = 0 on R. Ashraf *et al.* [7, 8] and Jaya Subba Reddy *et al.* [10] studied Lie ideals, generalized left derivation, left (α, β)-derivation and generalized left (α, β)-derivation on prime rings R which acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then d = 0 on R. Several authors have proved commutativity theorems and some results of lie ideals with left derivation in prime rings (See. [1, 2, 3, 4]). In general every left derivation is a Jordan left derivation but the converse need not be true. Ashraf [8] proved that the converse is true in the case when the underlying ring is 2-torsion free prime ring. The purpose of this paper to study some commutative properties of left (α, 1)-derivation in prime rings. Also we prove that if d or G acts as a homomorphism and as an anti-homomorphism on R, then either commutative or d = 0 on R.

2. PRELIMINARIES

Throughout the present paper, R will denote an associative ring with center Z(R). Recall that R is prime if xRy = {0} implies that either x = 0 or y = 0. For any x, y∈ R, the symbol [x, y] stands for the commutator xy– yx. Let S be a non-empty subset of R and d: R→ R a derivation of R. If d(xy) = d(x)d(y) (resp. d(xy) = d(y)d(x)) holds for all x, y∈ S, then d is said to act as a homomorphism (resp. anti-homomorphism) on S. An additive subgroup U of R is said to be a Lie ideal of R if [u, r]∈ U for all u∈ U, r∈ R. Suppose that α, 1 are endomorphisms of R. An additive mapping d: R→ R is called a (α, 1)-derivation (resp. Jordan (α, 1)-derivation) if d(xy) = d(x)y + α(x)d(y) (resp. $d(x^2) = d(x)x + \alpha(x)d(x)$) holds for all x, y∈ R. An additive mapping d: R→ R is called a left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation)

Corresponding Author: Sk. Haseena*[2]
[2]Research Scholar, Department of Mathematics,
S. V. University, Tirupati- 517502, Andhra Pradesh, India.

if $d(xy) = \alpha(x)d(y) + yd(x)$ (resp. $d(x^2) = \alpha(x)d(x) + xd(x)$) holds for all $x, y\in R$. An additive mapping $G: R\rightarrow R$ is said to be a generalized left (α, 1)-derivation (resp. generalized Jordan left (α, 1)-derivation) if there exists a Jordan left (α, 1)-derivation $d: R\rightarrow R$ such that $G(xy) = \alpha(x)G(y) + yd(x)$ (resp. $G(x^2) = \alpha(x)G(x) + xd(x)$) holds for all $x, y\in R$. The definition of generalized right(α, 1)-derivation (resp. generalized Jordan right (α, 1)-derivation) is self-explanatory. We shall make use of the following results, all but one of which is known.

Lemma 1 ([3, Lemma 2]): If $U\subseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a, b\in R$ such that $aUb = \{0\}$, then $a = 0$ or $b = 0$.

Lemma 2 ([6, Lemma 4]): Let G and H are additive groups and let R be a 2-torsion free ring. Let $f: G\times G\rightarrow H$ and $g: G\times G\rightarrow R$ be biadditive mappings. Suppose that for each pair $a, b \in G$ either $f(a, b) = 0$ or $g(a, b)^2 = 0$. In this case either $f = 0$ or $g(a, b)^2 = 0$ for all $a, b\in G$ respectively.

Lemma 3 ([4, Theorem 4]): Let R be a 2-torsion free prime ring and U a Lie ideal of R. If R admits a derivation d such that $d(u)^n = 0$ for all $u\in U$, where $n \geq 1$ is a fixed integer, then $d(u) = 0$ for all $u\in U$.

Lemma 4 ([2, Lemma 1.3]): Let R be a 2-torsion free semiprime ring. If U is a commutative Lie ideal of R, then $U\subseteq Z(R)$.

3. MAIN RESULTS

Theorem 1: Let R be a prime ring and I a nonzero ideal of R, and let α, 1 be automorphisms of R. Suppose $d: R\rightarrow R$ is a (α, 1)-derivation of R.

(i) If d acts as a homomorphism on I, then $d = 0$ on R.
(ii) If d acts as an anti-homomorphism on I, then $d = 0$ on R.

Proof: (i) Let d acts as a homomorphism on I. By our hypothesis, we have

$$d(vu) = d(v)u + \alpha(v)d(u), \text{ for all } u, v\in I. \quad (1)$$

Replacing v by vr, for any $r\in R$ in the equation (1), we get

$$d(v)\alpha(r)d(u) = \alpha(v)\alpha(r)d(u),$$

and so, $(d(v)- \alpha(v))\alpha(r)d(u) = 0$, for all $v\in I$ and $r\in R$. (2)

Hence, $\alpha^{-1}(d(v)- \alpha(v))R\alpha^{-1}(d(u)) = \{0\}$, for all $v\in I$ and $r\in R$.

By the primeness of R, we conclude that either $\alpha^{-1}(d(v)- \alpha(v)) = 0$ or $\alpha^{-1}(d(u)) = 0$. Since α is an automorphism, we find that $d(v)- \alpha(v) = 0$ or $d(v) = 0$, for all $v\in I$.

For this yields that $d(v) = \alpha(v)$, for all $v\in I$. (3)

Replacing v by sv, for any $s\in R$ in the equation (3), we get

$$d(s)v + \alpha(s)d(v) = \alpha(s)\alpha(v), \text{ for all } v\in I \text{ and } s\in R.$$

Using equation (3) in the above relation, we get $d(s)v = 0$, for all $v\in I$ and $s\in R$. (4)

Hence, $d(s)I = \{0\}$, for all $s\in R$. Since the primeness of R, we have either $d(s) = 0$ or $I = 0$.

But $I \neq \{0\}$, this implies that I is central and hence R is commutative.

The last relation yields that $d(s) = 0$, i.e., $d = 0$ on R.

(ii) Let d acts as an anti-homomorphism on I. By our hypothesis, we have

$$d(uv) = d(u)v + \alpha(u)d(v), \text{ for all } u, v\in I. \quad (5)$$

Replacing u by uv in the equation (5), we get $d(uvv)= d(v)d(uv)= d(uv)v+ \alpha(uv)d(v)$

$$d(v)\alpha(u)d(v) = \alpha(u)\alpha(v)d(v), \text{ for all } u, v\in I. \quad (6)$$

Replacing u by uw in the equation (6) and using (6), we obtain

$$[\alpha(w), d(v)]\alpha(u)d(v) = 0, \text{ for all } u, v, w\in I.$$

Hence, $\alpha^{-1}([\alpha(w), d(v)])U\alpha^{-1}(d(v)) = \{0\}$, for all $v, w\in I$. Since I is a nonzero ideal of prime ring R, we obtain either $[w, \alpha^{-1}(d(v))] = 0$ or $\alpha^{-1}(d(v)) = 0$, for all $v, w\in I$.

Since α is an automorphism, we find that either $[\alpha(w), d(v)] = 0$ or $d(v) = 0$, for all $v, w \in I$.

Define for fixed $w \in R$, $I_1 = \{v \in I / d(v) = 0\}$ and $I_2 = \{v \in I / [\alpha(w), d(v)] = 0\}$. It is clearly shows that I_1 and I_2 are subgroups of I whose union is I. Hence either $I_1 = I$ or $I_2 = I$.

If $I_1 = I$. From the case (i), we have $d(v) = 0$. It shows $d = 0$ on R.

If $I_2 = I$, so we have $[\alpha(w), d(v)] = 0$, for all $v, w \in I$. (7)

Replacing w by rw, for any $r \in R$ in the equation (7), we get $[\alpha(r), d(v)]\alpha(w) = 0$, for all $v, w \in I$ and $r \in R$. So, we have $[R, \alpha^{-1}(d(v))]I = 0$, for all $v \in I$.

Since I is a nonzero ideal of prime ring R, we get $d(v) \subset Z(R)$ and it follows $d = 0$ from (i).

Hence the theorem proof is completed.

Theorem 2: Let R be a prime ring and I a nonzero ideal of R, and let α, 1 be automorphisms of R. Suppose $d: R \rightarrow R$ is a left (α, 1)- derivation of R.

(i) If d acts as a homomorphism on I, then $d = 0$ on R.
(ii) If d acts as an anti-homomorphism on I, then $d = 0$ on R.

Proof: (i) Let d acts as a homomorphism on I. By our hypothesis, we have

$$d(u)d(v) = d(uv) = \alpha(u)d(v) + vd(u), \text{ for all } u, v \in I. \quad (8)$$

Replacing u by uv in the equation (8), we get $d(uv)d(v) = \alpha(u)\alpha(v)d(v) + vd(uv)$, for all $u, v \in I$.

The application of (8) yields that $\alpha(u)d(v)d(v) = \alpha(u)\alpha(v)d(v)$, for all $u, v \in I$.

This implies that $\alpha(u)(d(v) - \alpha(v))d(v) = 0$, for all $u, v \in I$. (9)

Replacing u by ur, for any $r \in R$ in the equation (9), we get $\alpha(u)\alpha(r)(d(v) - \alpha(v))d(v) = 0$, for all $u, v \in I$, and hence, $uR\alpha^{-1}((d(v) - \alpha(v))d(v)) = \{0\}$ for all $u, v \in I$.

Since by the primeness of R, we have either $u = 0$ or $\alpha^{-1}((d(v) - \alpha(v))d(v)) = 0$. Since I is a nonzero ideal of R, we have $\alpha^{-1}((d(v) - \alpha(v))d(v)) = 0$, this yields that $(d(v) - \alpha(v))d(v) = 0$.

That is $d(v^2) = \alpha(v)d(v)$. Since d is a left (α, 1)- derivation, we find that $\alpha(v)d(v) = 0$.

Linearizing the latter relation, we have $vd(u) + ud(v) = 0$, for all $u, v \in I$.

Replacing u by vu in the above relation, we get $vud(v) = 0$ for all $u, v \in I$. (10)

Replacing u by su in the equation (10), we get $vsud(v) = 0$, for all $u, v \in I$ and $s \in R$, and hence $vRud(v) = \{0\}$. Since by the primeness of R, we have either $v = 0$ or $ud(v) = 0$, for all $u, v \in I$. Since I is a nonzero ideal of R, we have $ud(v)= 0$, for all $u, v \in I$. (11)

Replacing u by ut in the equation (11), we get $utd(v) = 0$, for all $u, v \in I$ and $t \in R$, and hence $IRd(v) = \{0\}$. By the primeness of R, we have either $I = 0$ or $d(v) = 0$, for all $u \in I$. But $I \neq \{0\}$, so we have $d(v) = 0$. From in the case of theorem 1(i), we conclude that $d = 0$ on R.

(ii) Let d acts as an anti-homomorphism on I. By our hypothesis, we have

$$d(uv) = \alpha(u)d(v) + vd(u), \text{ for all } u, v \in I. \quad (12)$$

Replacing v by uv in the equation (12), we get

$$d(uv)d(u) = d(u(uv)) = \alpha(u)d(uv) + uvd(u), \text{ for all } u, v \in I. \quad (13)$$

Now right multiplying (12) by d(u) and using the fact that d is an anti- homomorphism on I, we get

$$d(uv)d(u) = \alpha(u)d(uv) + vd(u)d(u), \text{ for all } u, v \in I. \quad (14)$$

Comparing (13) and (14), we get $uvd(u) = vd(u)d(u)$. (15)

Replacing v by rv in the equation (15), we get

$$urvd(u) = rvd(u)d(u), \text{ for all } u, v\in I \text{ and } r\in R. \quad (16)$$

Left multiplying (15) by r and comparing with (16), we obtain

$$[r, u]vd(u) = 0, \text{ for all } u, v\in I \text{ and } r\in R. \quad (17)$$

Replacing v by sv in the equation (17), we get $[r, u]svd(u) = 0$, for all $u, v\in I$ and $r, s\in R$. Hence, $[r, u]Rvd(u) = \{0\}$, for all $u, v\in I$ and $r\in R$.

Let $I_a= \{u\in I/ vd(u) = 0$, for all $v\in I\}$ and $I_b= \{u\in I/ [r, u] = 0$, for all $r\in R\}$. It is clearly shows that I_a and I_b are subgroups of I whose union is I. Since by the primeness of R, we have either $vd(u) = 0$ or $[r, u] = 0$, for all $u, v\in I$ and $r\in R$.

If $[r, u] = 0$, replacing u by su, we get $[r, s]u = 0$, for all $u\in I$ and $r, s\in R$.

This implies that $[r, s]I = \{0\}$. Since I is a nonzero ideal of prime ring R, we have either $u = 0$ or $[r, s] = 0$. But $I \neq \{0\}$, so we have $[r, s] = 0$, for all $r, s\in R$. i.e., R is commutative. i.e., d is a (α, 1)-derivation which acts as an anti-homomorphism on I.

Hence by Theorem 1(ii), we have $d = 0$ on R.

If $vd(u) = 0$, for all $u, v\in I$. (18)

In the above case we get the result (i) from (11). In the same technique by using we get the required result from (18).

Hence the theorem proof is completed.

Theorem 3: Let R be a prime ring and let I be a nonzero ideal of R. Suppose that α, 1 are automorphisms of R and $G: R\to R$ is a generalized left (α, 1)-derivation of R with associated left (α, 1)- derivation d.

(i) If G acts as a homomorphism on I, then either R is commutative or d= 0 on R.
(ii) If G acts as an anti-homomorphism on I, then either R is commutative or d= 0 on R.

Proof: (i) Let G acts as a homomorphism on I. By our hypothesis, we have

$$G(uv) = G(u)G(v) = \alpha(u)G(v) + vd(u), \text{ for all } u, v\in I. \quad (19)$$

Replacing v by vw in the equation (19), we get

$$G(uvw) = G(u(vw)) = \alpha(u)G(vw) + vwd(u), \text{ for all } u, v, w\in I. \quad (20)$$

On the other hand, we have

$$G(uvw) = G((uv)w)) = G(uv)G(w) = \alpha(u)G(v)G(w) + vd(u)G(w). \quad (21)$$

Comparing (21) and (20) and using (19), we get

$$vwd(u) = vd(u)G(w), \text{ for all } u, v, w\in I. \quad (22)$$

This implies that $v\{wd(u) - d(u)G(w)\} = 0$, for all $u, v, w\in I$.

Replacing v by vr, for any $r\in R$, we find that $vr\{wd(u) - d(u)G(w)\} = \{0\}$, for all $u, v, w\in I$.

Hence $IR\{wd(u) - d(u)G(w)\} = \{0\}$ for all $u, v, w\in I$. Since I is a nonzero ideal of prime ring R, this yields that

$$wd(u) = d(u)G(w), \text{ for all } u, w\in I. \quad (23)$$

Replacing u by uv in the equation (23), we get

$$w\alpha(u)d(v) + wvd(u) = \alpha(u)d(v)G(w) + vd(u)G(w), \text{ for all } u, v, w\in I. \quad (24)$$

Using (23) in (24), we have $[w, v]d(u) + [w, \alpha(u)]d(v) = 0$, for all $u, v, w\in I$. (25)

Hence in particular, we find that $[v, \alpha(u)]d(v) = 0$, for all $u, v\in I$. (26)

Replacing u by ru in the equation (26), for any $r\in R$ and using (26), we get

$$[v, \alpha(r)]\alpha(u)d(v) = 0, \text{ for all } u, v\in I.$$

The above relation implies that $\alpha^{-1}([v, \alpha(r)])u\alpha^{-1}(d(v)) = 0$ for all $u, v\in I$ and $r\in R$.

This can be rewritten as $\alpha^{-1}([v, \alpha(r)])IR\alpha^{-1}(d(v)) = \{0\}$, for all $v\in I$ and $r\in R$. Since by the primeness of R, we have either $\alpha^{-1}([v, \alpha(r)])I = \{0\}$ or $\alpha^{-1}(d(v)) = 0$.

Now define $I_\alpha = \{v\in I/ \alpha^{-1}([v, \alpha(r)])I = \{0\}$, for all $r\in R\}$ and $I_\beta = \{v\in I/ \alpha^{-1}(d(v)) = 0\}$. It is clearly shows that I_α and I_β are subgroups of I whose union is I. Since by the primeness of R, we have either $I_\alpha = I$ or $I_\beta = I$.

If $I_\alpha = I$, then we have $\alpha^{-1}([v, r'])I = \{0\}$, for all $v\in I$ and $\alpha(r) = r'\in R$. Since I is a nonzero ideal of prime ring R and α is an automorphism, this yields that $[v, r'] = 0$. This implies that I is central and hence R is commutative.

If $I_\beta = I$, then $\alpha^{-1}(d(v)) = 0$, for all $v\in I$. Since α is an automorphism and from the case of theorem 1(i), it follows that d = 0 on R.

(ii) Let G acts as an anti-homomorphism on I. By our hypothesis, we have

$$G(uv) = G(v)G(u) = \alpha(u)G(v) + vd(u), \text{ for all } u, v \in I. \quad (27)$$

Replacing v by uv in the equation (27), we get

$$G(u^2v) = G(uv)G(u) = \alpha(u)G(uv) + uvd(u), \text{ for all } u, v\in I. \quad (28)$$

Now using (27) in (28), we get

$$\alpha(u)G(v)G(u) + vd(u)G(u) = \alpha(u)G(uv) + uvd(u). \quad (29)$$

Again using (27) in (29), we get $uvd(u) = vd(u)G(u)$, for all $u, v\in I$. (30)

Replacing v by rv, for any $r\in R$ in the equation (30), we get

$$urvd(u) = rvd(u)G(u), \text{ for all } u, v\in I. \quad (31)$$

Left multiplying (30) by r, we get $ruvd(u) = rvd(u)G(u)$. (32)

Comparing (31) and (32), we get $[u, r]vd(u) = 0$, for all $u, v\in I$ and $r\in R$.
This relation can be written as $[u, r]I(d(u)) = \{0\}$. i.e., $[u, r]IRd(u) = \{0\}$, for all $u\in I$ and $r\in R$. This implies that for each fixed $u\in I$ either $[u, r]I = \{0\}$ or $d(u) = 0$.

Now using similar techniques as above, we get the required result.

Hence the theorem proof is completed.

5. CONCLUSION

In this study, we have introduced the (α, 1)-derivation (resp. Jordan (α, 1)-derivation), Left (α, 1)-derivation (resp. Jordan left (α, 1)-derivation), Generalized left (α, 1)-derivation (resp. Generalized Jordan left (α, 1)-derivation) and also established that if R admits a nonzero left (α, 1)-derivation acts as a homomorphism or as an anti-homomorphism on a nonzero ideal I of R, then either R is commutative or d = 0 on R.

REFERENCES

1. Posner.B.C, Derivations in prime rings, Proc.Amer.Math. 8 (1957), 1093- 1100.
2. Herstein.I.N, Topics in ring theory, Univ.of Chicago Press, Chicago 1969.
3. Bergen.J, Herstein.I.N and Kerr.J.W, Lie ideals and derivations of prime rings, J.Algebra, 71 (1981), 259–267.
4. Cairini.L and Giambruno.A, Lie ideals and nil derivations, Boll.Un.Mat.Ital.,6 (1985), 497–503.
5. Bell.H.E and Kappe.L.C, Rings in which derivations satisfy certain algebraic conditions, Acta Math.Hungar., 53 (1989), 339–346.
6. Bresar and Vukman.J, On left derivations and related mappings, Proc.Amer.Math.Soc, 110 (1) (1990), 7–16.
7. Ashraf.M, Rehman.N and Quadri.M.A, On Lie ideals and (σ, τ)-Jordan derivations on prime rings, Tamkang J.Math., 32 (2001), 247–252.
8. Ashraf.M, On left (θ, φ)-derivations of prime rings, Arch.Math.(Brno), 41(2) (2005), 157–166.
9. Ashraf.M and Ali.S, On generalized left (α, β) - derivations in rings, Bull.Iranian Math.Soc., 38(4) (2012), 893- 905.
10. Jaya Subba Reddy.C, Vasantha Kumar.S and Madhusudhan Reddy.K, Lie Ideals with Left Generalized Jordan Derivations in Prime rings, Global J.Pure & Appl.Math., 12(3) (2016), 116- 118.